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A PRECONDITIONER FOR THE FETI-DP FORMULATION WITH MORTAR METHODS IN TWO DIMENSIONS*

HYEA HYUN KIM† AND CHANG-OCK LEE†

Abstract. In this paper, we consider a dual-primal FETI (FETI-DP) method for elliptic problems on nonmatching grids. The FETI-DP method is a domain decomposition method that uses Lagrange multipliers to match solutions continuously across subdomain boundaries in the sense of dual-primal variables. We use the mortar matching condition as the continuity constraints for the FETI-DP formulation. We construct a preconditioner for the FETI-DP operator and show that the condition number of the preconditioned FETI-DP operator is bounded by

$$C \max_{i=1,\dots,N}\{(1+\log(H_i/h_i))^2\},$$

where H_i and h_i are sizes of domain and mesh for each subdomain, respectively, and C is a constant independent of H_i's and h_i's. We allow jumps of coefficients of elliptic problems across subdomain boundaries. Numerical results are included.

Key words. FETI-DP, nonmatching grids, mortar methods, preconditioner

AMS subject classifications. 65N30, 65N55

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1. Introduction.

This paper is concerned with preconditioners for an iterative method for the parallel solution of symmetric, positive definite systems of linear equations that arise from elliptic boundary value problems discretized by finite elements on nonconforming meshes. Nonconforming discretizations are important for multiphysics simulations, contact-impact problems, the generation of meshes and partitions aligned with jumps in diffusion coefficients, *hp*-adaptive methods, and special discretizations in the neighborhood of singularities (corners or joints).

Of the many methods for nonmatching meshes, including [3] and [13], we consider the mortar method [1, 2, 17, 18]. In mortar methods, orthogonality relations between the jumps in the traces across subdomain interfaces are satisfied using a discrete Lagrange multiplier space. The sparse linear systems that arise in mortar methods are similar to the systems solved by an iterative substructuring method with Lagrange multipliers developed for conforming discretizations (see [6, 8, 12, 14] for an introduction).

Recently the dual-primal FETI (FETI-DP) method introduced by Farhat, Lesoinne, and Pierson [7] has been applied to mortar finite elements methods [4, 5, 15]. The primary contribution of our work is using the framework of parallel subspace correction methods [19] to better formulate the FETI-DP preconditioner for the mortar matching condition.

The FETI-DP method enforces the continuity of the solution at cross points directly in the formulation of the dual problem: the degrees of freedom (d.o.f.) at a cross point remain common to all subdomains sharing the cross point and the continuity of the remaining d.o.f. on the interfaces are enforced by Lagrange multipliers [10]. The

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[†] Division of Applied Mathematics, KAIST, Daejeon 305-701, Korea (mashy@amath.kaist.ac.kr, colee@amath.kaist.ac.kr).

d.o.f. are then eliminated and the resulting dual problem for the Lagrange multipliers is solved by preconditioned conjugate gradients (CGs).

For FETI-DP methods on nonmatching grids, Dryja and Widlund [4] proposed a preconditioner, the so-called Dirichlet preconditioner, which gives a condition number bound $C(1+\log(H/h))^2$ with the Neumann–Dirichlet ordering of substructures, where H and h denote the maximum diameter of subdomains and minimum size of meshes of all subdomains, respectively. In general cases, that is, without considering ordered substructures, they obtained $C(1+\log(H/h))^4$ for the condition number bound. Moreover, in [5], they proposed a different preconditioner, which is similar to the one in [9], and proved the condition number bound $C(1+\log(H/h))^2$. However, the constant C in the condition number bound depends on the ratio of meshes between neighboring subdomains. This restriction is impractical when the coefficients of elliptic problems are highly discontinuous between subdomains (see Wohlmuth [18]).

In this paper, we formulate an FETI-DP operator in a different way from that of Dryja and Widlund [4, 5] and propose a Neumann–Dirichlet preconditioner which gives the condition number bound $C(1+\log(H/h))^2$ with the constant C not depending on the ratio of meshes between neighboring subdomains. The proposed preconditioner is easy to implement and the operator from the nodal values on the interfaces of subdomains to the Lagrange multiplier space requires only the nodal values on the slave side. Hence, the cost for multiplying the operator to a vector is reduced by half compared with preconditioners developed elsewhere (see, e.g., [4, 5]). For the elliptic problems with heterogeneous coefficients, with careful choices of slave and master sides according to the magnitude of coefficients, the preconditioner gives the same condition number bound, which does not depend on the coefficients.

In the mortar matching condition, we consider a standard Lagrange multiplier space introduced by [2]. In the condition number analysis, we use the continuity of the mortar projection operator in an $H_{00}^{1/2}$-norm. Hence, our result can be extended to Lagrange multiplier spaces with this property. A few such Lagrange multiplier spaces are developed by Wohlmuth [17, 18].

This paper is organized as follows. In section 2, we introduce finite element spaces and norms and, in section 3, we derive the FETI-DP operator using the mortar matching condition as continuity constraints and propose a preconditioner. In section 4, we analyze the condition number bound of the preconditioned FETI-DP operator. Numerical results are provided in section 5. In the numerical tests, we compare the proposed preconditioner with that of Dryja and Widlund [5] for solving elliptic problems with highly discontinuous coefficients on noncomparable meshes.

2. Finite element spaces and norms.

2.1. A model problem and Sobolev spaces. Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ and $L^2(\Omega)$ be the space of square integrable functions defined in Ω equipped with the norm $\|\cdot\|_{0,\Omega}$:

$$\|v\|_{0,\Omega}^2 := \int_\Omega v^2\,dx.$$

The space $H^1(\Omega)$ is the set of functions, which are square integrable up to the first weak derivatives, and the norm is given by

$$\|v\|_{1,\Omega} := \left(\int_\Omega \nabla v\cdot\nabla v\,dx + \frac{1}{d_\Omega^2}\int_\Omega v^2\,dx\right)^{1/2},$$

where d_Ω denotes the diameter of Ω.

We consider an FETI-DP method on nonmatching grids for the following elliptic problem: For $f \in L^2(\Omega)$, find $u \in H^1(\Omega)$ such that

$$\begin{aligned} -\nabla \cdot (A(x)\nabla u(x)) + \beta(x)u(x) &= f(x) \ \text{ in } \Omega, \\ u(x) &= 0 \ \text{ on } \Gamma_D, \\ \boldsymbol{n} \cdot (A(x)\nabla u(x)) &= 0 \ \text{ on } \Gamma_N. \end{aligned} \tag{2.1}$$

Here, $A(x) = (\alpha_{ij}(x))$ for $i, j = 1, 2$ and $\boldsymbol{n}$ is the outward unit vector normal to Γ_N. We assume that $\alpha_{ij}(x), \beta(x) \in L^\infty(\Omega)$, $A(x)$ is uniformly elliptic, $\beta(x) \geq 0$ for all $x \in \Omega$, and $|\Gamma_D| \neq 0$, where $|\Gamma_D|$ denotes the measure of Γ_D.

Let Ω be partitioned into nonoverlapping polygonal subdomains $\{\Omega_i\}_{i=1}^N$. We assume that the partition is geometrically conforming, which means that the subdomains intersect with neighboring subdomains on the whole of an edge or at a vertex. Ω_i^h denotes a quasi-uniform triangulation of the subdomain Ω_i. The quasi uniformity means that there exist constants γ and σ such that $\gamma h_i \leq d_\tau \leq \sigma\rho_\tau$ for all $\tau \in \Omega_i^h$, where ρ_τ is the diameter of the circle inscribed in τ, d_τ is the diameter of τ, and $h_i = \max_{\tau \in \Omega_i^h} d_\tau$. We note that the meshes need not match across the subdomain interfaces.

For each subdomain Ω_i, we introduce a finite element space

$$X_i := \{v \in H_D^1(\Omega_i) \ : \ v|_\tau \in P_1(\tau), \ \tau \in \Omega_i^h\},$$

where $H_D^1(\Omega_i) := \{v \in H^1(\Omega_i) \ : \ v = 0 \text{ on } \Gamma_D \cap \partial\Omega_i\}$ and $P_1(\tau)$ is a set of polynomials of degree ≤ 1 in τ. For $(u_i, v_i) \in X_i \times X_i$, define a bilinear form

$$a_i(u_i, v_i) := \int_{\Omega_i} A(x)\nabla u_i \cdot \nabla v_i \, dx + \int_{\Omega_i} \beta(x) u_i v_i dx.$$

To get the FETI-DP formulation, we need a finite element space in Ω as follows:

$$X := \left\{ v \in \prod_{i=1}^N X_i \ : \ v \text{ is continuous at subdomain vertices} \right\}.$$

By restricting the space X_i on the boundary of the subdomain Ω_i, we define

$$W_i := X_i|_{\partial\Omega_i} \ \forall i = 1, \dots, N.$$

Then we let

$$W := \left\{ w \in \prod_{i=1}^N W_i \ : \ w \text{ is continuous at subdomain vertices} \right\}. \tag{2.2}$$

In this paper, we will use the same notation for finite element functions and the corresponding vectors of nodal values. For example, w_i is used to denote a finite element function or the vector of nodal values of that function. The same applies to the notation for function spaces such as W_i, X, W, etc.

We define S^i as the Schur complement matrix obtained from the bilinear form $a_i(\cdot, \cdot)$ over the finite elements X_i (see page 50 in [11]). Using this operator, a seminorm is defined for $w_i \in W_i$:

$$|w_i|_{S^i}^2 := \langle S^i w_i, w_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the l^2-inner product of vectors. For $w \in W$, since w is continuous at subdomain vertices, by summing up these seminorms, we define a norm

$$\|w\|_W^2 := \sum_{i=1}^{N} |w_i|_{S^i}^2, \quad w_i = w|_{\partial\Omega_i}. \tag{2.3}$$

Moreover, we define a subspace of W

$$W_r := \{w \in W \ : \ w \text{ vanishes at subdomain vertices}\}. \tag{2.4}$$

Now, we introduce Sobolev spaces defined on the boundaries of subdomains. The space $H^{1/2}(\partial\Omega_i)$ is the trace space of $H^1(\Omega_i)$ equipped with the norm

$$\|w_i\|_{1/2,\partial\Omega_i}^2 := |w_i|_{1/2,\partial\Omega_i}^2 + \frac{1}{d_{\Omega_i}} \|w_i\|_{0,\partial\Omega_i}^2,$$

where

$$|w_i|_{1/2,\partial\Omega_i}^2 := \int_{\partial\Omega_i} \int_{\partial\Omega_i} \frac{|w_i(x) - w_i(y)|^2}{|x-y|^2} \, ds(x) \, ds(y).$$

For any $\Gamma_{ij} \in \partial\Omega_i$, $H_{00}^{1/2}(\Gamma_{ij})$ is the set of functions in $L^2(\Gamma_{ij})$ such that the zero extension of the function into $\partial\Omega_i$ is contained in $H^{1/2}(\partial\Omega_i)$. For $v \in H_{00}^{1/2}(\Gamma_{ij})$, let

$$|v|_{H_{00}^{1/2}(\Gamma_{ij})}^2 := |v|_{1/2,\Gamma_{ij}}^2 + \int_{\Gamma_{ij}} \frac{v^2(x)}{\operatorname{dist}(x, \partial\Gamma_{ij})} \, ds,$$

and the norm is given by

$$\|v\|_{H_{00}^{1/2}(\Gamma_{ij})} := \left(|v|_{H_{00}^{1/2}(\Gamma_{ij})}^2 + \frac{1}{d_{\Omega_i}} \|v\|_{0,\Gamma_{ij}}^2 \right)^{1/2}.$$

From section 4.1 in [19], for $v \in H_{00}^{1/2}(\Gamma_{ij})$ we have the following relation:

$$C_1 \|\widetilde{v}\|_{1/2,\partial\Omega_i} \le \|v\|_{H_{00}^{1/2}(\Gamma_{ij})} \le C_2 \|\widetilde{v}\|_{1/2,\partial\Omega_i}, \tag{2.5}$$

where the constants C_1 and C_2 are independent of d_{Ω_i} and $\widetilde{v}$ denotes the zero extension of v into $\partial\Omega_i$.

2.2. Mortar matching conditions. We note that the space X is not contained in $H^1(\Omega)$. To approximate the solution of the problem (2.1) in X, we use the mortar matching condition. More precisely, we construct the Lagrange multiplier space as follows.

First, let $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$. For Γ_{ij} such that $|\Gamma_{ij}| \neq 0$, we distinguish $\Omega_i^h|_{\Gamma_{ij}}$ and $\Omega_j^h|_{\Gamma_{ij}}$, as in Figure 1. We assume that both sides have more than three nodal points including end points. Then we choose one as a slave side and the other as a master side and define

$$m_i := \{j \ : \ |\Gamma_{ij}| \neq 0, \ \Omega_j^h|_{\Gamma_{ij}} \text{ is a master side of } \Gamma_{ij}\},$$
$$s_i := \{j \ : \ |\Gamma_{ij}| \neq 0, \ \Omega_j^h|_{\Gamma_{ij}} \text{ is a slave side of } \Gamma_{ij}\}.$$

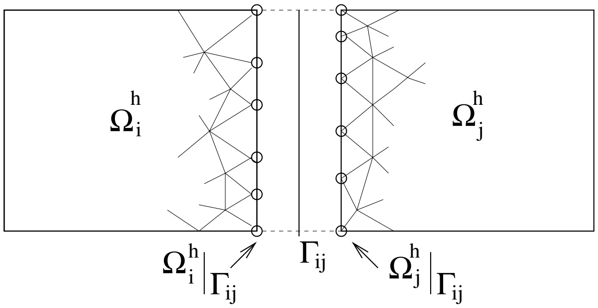

FIG. 1. *Master and slave sides of* Γ_{ij}.

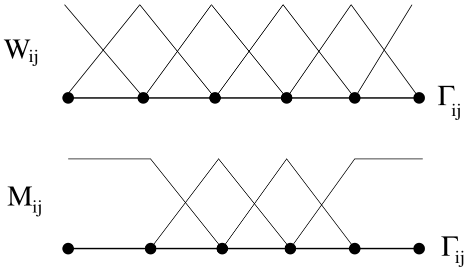

FIG. 2. *Basis functions for* W_{ij} *and* M_{ij}.

For $j \in m_i$, $\Omega_i^h|_{\Gamma_{ij}}$ is the slave side of Γ_{ij} and from the finite elements on the slave side, we get

$$W_{ij} := \{v|_{\Gamma_{ij}} \;:\; v \in X_i\} \;\; \forall\, j \in m_i.$$

Next, let

$$\{\phi_0^{ij},\, \phi_1^{ij}, \dots, \phi_{N_{ij}}^{ij},\, \phi_{N_{ij}+1}^{ij}\}$$

be nodal basis functions for W_{ij}. Moreover, we assume that the basis functions are sequentially ordered according to the location of nodes on Γ_{ij}. Let (see Figure 2)

$$M_{ij} := \operatorname{span}\{\phi_0^{ij} + \phi_1^{ij},\, \phi_2^{ij}, \dots, \phi_{N_{ij}-1}^{ij}, \phi_{N_{ij}}^{ij} + \phi_{N_{ij}+1}^{ij}\}.$$

Then we take the Lagrange multiplier space

$$M := \prod_{i=1}^{N} \prod_{j \in m_i} M_{ij}.$$

Bernardi, Maday, and Patera [2] first introduced this type of Lagrange multiplier space. They imposed the following mortar matching condition on X, i.e., $v \in X$ satisfies

$$\int_{\Gamma_{ij}} (v_i - v_j)\lambda_{ij}\, ds = 0 \;\; \forall\, \lambda_{ij} \in M_{ij},\; i = 1, \dots, N,\; j \in m_i. \tag{2.6}$$

In our FETI-DP formulation, we use (2.6) as continuity constraints and define a bilinear form $b(\cdot,\cdot) : W \times M \to \mathbb{R}$ as

$$b(w,\mu) := \sum_{i=1}^{N} \sum_{j \in m_i} \int_{\Gamma_{ij}} (w_i - w_j)\, \mu_{ij}\, ds \quad \forall (w,\mu) \in W \times M. \tag{2.7}$$

For $|\partial\Omega_i \cap \partial\Omega_j| \neq 0$, we denote $\partial\Omega_i \cap \partial\Omega_j$ as Γ_{ij} if $\Omega_i^h|_{\Gamma_{ij}}$ is a slave side and as Γ_{ji} otherwise. We assume that $\Omega_i^h|_{\Gamma_{ij}}$ is the slave side and $\Omega_j^h|_{\Gamma_{ij}}$ is the master side of Γ_{ij}. Denote the basis for M_{ij} by $\{\xi_k^{ij}\}_{k=1}^{N_{ij}}$ and let $\{\phi_k^{ji}\}_{k=0}^{N_{ji}+1}$ be the basis functions for $W_j|_{\Gamma_{ij}}$. Define matrices B_i^{ij} and B_j^{ij} with entries

$$\left(B_i^{ij}\right)_{lk} = \int_{\Gamma_{ij}} \xi_l^{ij} \phi_k^{ij}\, ds,\ l = 1, \dots, N_{ij},\ k = 0, \dots, N_{ij}+1,$$

$$\left(B_j^{ij}\right)_{lk} = -\int_{\Gamma_{ij}} \xi_l^{ij} \phi_k^{ji}\, ds,\ l = 1, \dots, N_{ij},\ k = 0, \dots, N_{ji}+1.$$

Then we rewrite (2.6) as

$$B_i^{ij} w_i^{ij} + B_j^{ij} w_j^{ij} = 0,$$

where $w_i^{ij} = v_i|_{\Gamma_{ij}}$ and $w_j^{ij} = v_j|_{\Gamma_{ij}}$.

Now define $E_{ij} : M_{ij} \to M$, an extension operator from M_{ij} to M by zero, and $R_{ij}^l : W_l \to W_l|_{\Gamma_{ij}}$ for $l = i, j$, a restriction operator. Let

$$B_i = \sum_{j \in m_i} E_{ij} B_i^{ij} R_{ij}^i + \sum_{j \in s_i} E_{ji} B_i^{ji} R_{ji}^i.$$

Then the mortar matching condition (2.6) becomes

$$\sum_{i=1}^{N} B_i w_i = 0,$$

where $w_i = v_i|_{\partial\Omega_i}$.

Define

$$W_{ij}^0 := \{v \in W_{ij}\ :\ v = 0 \text{ at the end points of } \Gamma_{ij}\}$$

and let

$$W^0 = \prod_{i=1}^{N} \prod_{j \in m_i} W_{ij}^0.$$

For $w_{ij} \in W_{ij}^0$, we define $\widetilde{w}_{ij} \in W_i$ by the zero extension of w_{ij} into $\partial\Omega_i$. Let $\widetilde{w}_i = \sum_{j \in m_i} \widetilde{w}_{ij}$ and $\widetilde{w} = (\widetilde{w}_1, \dots, \widetilde{w}_N)$. Since $\widetilde{w}$ is continuous at subdomain vertices, $\widetilde{w} \in W$. Hence for $w \in W^0$, we define a norm by

$$\|w\|_{W^0} := \|\widetilde{w}\|_W. \tag{2.8}$$

Let $\langle \cdot, \cdot \rangle_m$ be a duality pairing between M and W^0 such that

$$\langle \lambda, w \rangle_m := \sum_{i=1}^{N} \sum_{j \in m_i} \int_{\Gamma_{ij}} \lambda_{ij} w_{ij}\, ds\ \ \forall\, (\lambda, w) \in M \times W^0. \tag{2.9}$$

Using this, we define a dual norm on M by

$$\|\lambda\|_{(W^0)'} := \max_{w \in W^0 \setminus \{0\}} \frac{\langle \lambda, w \rangle_m}{\|w\|_{W^0}}. \tag{2.10}$$

3. FETI-DP formulation.

3.1. FETI-DP operator.

In this section, we construct the FETI-DP operator for the problem (2.1) with the mortar matching condition as constraints. The derivation of the FETI-DP equation for the Lagrange multipliers follows [10]. However, the FETI-DP operator with mortar matching condition is new. Dryja and Widlund [4, 5] eliminate unknowns on both interior and vertex nodal points and impose a mortar matching condition over W_r in (2.4). Hence, the resulting solution u does not satisfy the mortar matching condition (2.6). We eliminate only interior nodal points and impose the mortar matching condition on the function over W in (2.2).

For $w_i \in W_i$ we write

$$w_i = \begin{pmatrix} w_r^i \\ w_c^i \end{pmatrix},$$

where r and c stand for the nodal values on the edges and vertices. From now on, we use the subscript symbol r and c to represent the d.o.f. corresponding to nodes on the edges and at the vertices, respectively.

Define W_c as the set of vectors which have d.o.f. corresponding to the union of subdomain vertices, that is, global corner points. For $w = (w_1, \ldots, w_N) \in W$, since w is continuous at subdomain vertices, there exists $w_c \in W_c$ such that $L_c^i w_c = w_c^i$ for all $i = 1, \ldots, N$, where the matrix L_c^i consists of 0 and 1 and restricts the value of w_c on the vertices of subdomain Ω_i. Hence, for $w = (w_1, \ldots, w_N) \in W$, we write

$$w_i = \begin{pmatrix} w_r^i \\ L_c^i w_c \end{pmatrix} \ \forall i \ \text{ for some } w_c \in W_c.$$

Recall that S^i is the Schur complement matrix obtained from the bilinear form $a_i(\cdot,\cdot)$ and let g^i be the Schur complement forcing vector obtained from $\int_{\Omega_i} f v_i \, dx$. The matrix S^i and vector g^i are ordered in the following way:

$$S^i = \begin{pmatrix} S_{rr}^i & S_{rc}^i \\ S_{cr}^i & S_{cc}^i \end{pmatrix}, \quad g^i = \begin{pmatrix} g_r^i \\ g_c^i \end{pmatrix}.$$

Let $B_{i,r}$ and $B_{i,c}$ be matrices that consist of the columns of B_i corresponding to the nodal points on the edges and at the vertices, respectively.

Then the problem (2.1) becomes the following: Find $(w_r, w_c, \lambda) \in W_r \times W_c \times M$ such that

$$S_{rr} w_r + S_{rc} w_c + B_r^t \lambda = g_r, \tag{3.1}$$

$$S_{cr} w_r + S_{cc} w_c + B_c^t \lambda = g_c, \tag{3.2}$$

$$B_r w_r + B_c w_c = 0, \tag{3.3}$$

where

$$S_{rr} = \operatorname{diag}_{i=1,\ldots,N}\left(S_{rr}^i\right),$$

$$S_{rc} = \begin{pmatrix} S_{rc}^1 L_c^1 \\ \vdots \\ S_{rc}^N L_c^N \end{pmatrix},$$

$$\begin{aligned} S_{cr} &= S_{rc}^t, \\ S_{cc} &= \sum_{i=1}^{N} (L_c^i)^t S_{cc}^i L_c^i, \\ B_r &= (B_{1,r}, \dots, B_{N,r}), \quad B_c = \sum_{i=1}^{N} B_{i,c} L_c^i, \\ g_r &= \begin{pmatrix} g_r^1 \\ \vdots \\ g_r^N \end{pmatrix}, \; g_c = \sum_{i=1}^{N} (L_c^i)^t \, g_c^i, \; w_r = \begin{pmatrix} w_r^1 \\ \vdots \\ w_r^N \end{pmatrix}. \end{aligned}$$

Since S_{rr} is invertible, we solve (3.1) for w_r to get

$$w_r = S_{rr}^{-1} \left(g_r - S_{rc} w_c - B_r^t \lambda \right).$$

After substituting w_r into (3.3) and (3.2), we obtain

$$\begin{aligned} B_r S_{rr}^{-1} B_r^t \lambda + \left(B_r S_{rr}^{-1} S_{rc} - B_c \right) w_c &= B_r S_{rr}^{-1} g_r, \\ \left(S_{cr} S_{rr}^{-1} B_r^t - B_c^t \right) \lambda - \left(S_{cc} - S_{cr} S_{rr}^{-1} S_{rc} \right) w_c &= - \left(g_c - S_{cr} S_{rr}^{-1} g_r \right). \end{aligned}$$

Let

$$\begin{aligned} F_{I_{rr}} &= B_r S_{rr}^{-1} B_r^t, \\ F_{I_{rc}} &= B_r S_{rr}^{-1} S_{rc} - B_c, \\ F_{I_{cr}} &= S_{cr} S_{rr}^{-1} B_r^t - B_c^t \left(= F_{I_{rc}}^t \right), \\ F_{I_{cc}} &= S_{cc} - S_{cr} S_{rr}^{-1} S_{rc}, \\ d_r &= B_r S_{rr}^{-1} g_r, \\ d_c &= g_c - S_{cr} S_{rr}^{-1} g_r. \end{aligned} \tag{3.4}$$

Then (λ, w_c) satisfies

$$\begin{pmatrix} F_{I_{rr}} & F_{I_{rc}} \\ F_{I_{cr}} & -F_{I_{cc}} \end{pmatrix} \begin{pmatrix} \lambda \\ w_c \end{pmatrix} = \begin{pmatrix} d_r \\ -d_c \end{pmatrix}.$$

Eliminating w_c in the above equation, we obtain

$$\left(F_{I_{rr}} + F_{I_{rc}} F_{I_{cc}}^{-1} F_{I_{cr}} \right) \lambda = d_r - F_{I_{rc}} F_{I_{cc}}^{-1} d_c.$$

Here, $F_{DP} = F_{I_{rr}} + F_{I_{rc}} F_{I_{cc}}^{-1} F_{I_{cr}}$ is called the FETI-DP operator for the problem (2.1).

3.2. Preconditioner. From now on, we will propose $\widehat{F}_{DP}^{-1}$, a preconditioner for F_{DP}, which is derived from the dual norm on the Lagrange multiplier space M in the following sense:

$$\langle \widehat{F}_{DP} \lambda, \lambda \rangle = \| \lambda \|_{(W^0)'}^2. \tag{3.5}$$

We will derive the matrix form of $\widehat{F}_{DP}$ from the above relation. Define $E_{ij}^i : W_{ij}^0 \to W_i$ as the extension operator by 0. Then we get

$$\widetilde{w}_{ij} = E_{ij}^i w_{ij} \quad \text{for} \quad w_{ij} \in W_{ij}^0.$$

Define $R_{ij} : W^0 \to W_{ij}^0$ to be a restriction operator. For $w \in W^0$, we let $w_{ij} = R_{ij}w$. Hence by (2.8) and (2.3), we get

$$\|w\|_{W^0}^2 = \sum_{i=1}^N \left\langle S^i \left(\sum_{j\in m_i} E_{ij}^i R_{ij} w \right), \sum_{j \in m_i} E_{ij}^i R_{ij} w \right\rangle.$$

Let $E^i = \sum_{j\in m_i} E_{ij}^i R_{ij}$; then we have

$$\|w\|_{W^0}^2 = \langle \widehat{S} w, w\rangle \quad \text{with} \quad \widehat{S} = \sum_{i=1}^N (E^i)^t S^i E^i. \tag{3.6}$$

Recall that

$$(B_i^{ij})_{lk} = \int_{\Gamma_{ij}} \xi_l^{ij} \phi_k^{ij}\, ds, \; l = 1, \dots, N_{ij}, \; k = 0, 1, \dots, N_{ij}+1,$$

and take $(B_{i,r}^{ij})_{lk} = (B_i^{ij})_{lk}$ for $l, k = 1, \dots, N_{ij}$. Since $w_{ij} \in W_{ij}^0$, we have

$$\lambda_{ij}^t B_{i,r}^{ij} w_{ij} = \int_{\Gamma_{ij}} \lambda_{ij} w_{ij}\, ds.$$

Let

$$\widehat{B} = \mathrm{diag}_{i=1,\dots,N} \left(\mathrm{diag}_{j\in m_i} \left(B_{i,r}^{ij} \right) \right).$$

Then, for $(w, \lambda) \in W^0 \times M$, we get

$$\lambda^t \widehat{B} w = \sum_{i=1}^N \sum_{j\in m_i} \int_{\Gamma_{ij}} \lambda_{ij} w_{ij}\, ds, \tag{3.7}$$

where $\lambda_{ij} = \lambda|_{\Gamma_{ij}}$ and $w_{ij} = w|_{\Gamma_{ij}}$.

From the definition of the dual norm (2.10), (2.9), (3.7), and (3.6), we obtain

$$\|\lambda\|_{(W^0)'}^2 = \max_{w\in W^0\setminus\{0\}} \frac{\langle \lambda, \widehat{B} w\rangle^2}{\langle \widehat{S} w, w\rangle}.$$

Since $\widehat{S}$ is symmetric and positive definite on W^0, in the above equation the maximum occurs when $\widehat{B}^t \lambda = \widehat{S} w$. Therefore, we have

$$\|\lambda\|_{(W^0)'}^2 = \langle \widehat{B} \widehat{S}^{-1} \widehat{B}^t \lambda, \lambda\rangle$$

and let $\widehat{F}_{DP} = \widehat{B}\widehat{S}^{-1}\widehat{B}^t$. Then we take $\widehat{F}_{DP}^{-1} = (\widehat{B}\widehat{S}^{-1}\widehat{B}^t)^{-1}$ as a preconditioner for F_{DP} and call it a Neumann–Dirichlet preconditioner.

Note that $\widehat{F}_{DP}^{-1} = (\widehat{B}^t)^{-1} \widehat{S} \widehat{B}^{-1}$ is easy to implement due to the block diagonal structure of $\widehat{B}$ and $\widehat{B}^t = \widehat{B}$. Therefore, we have

$$\widehat{F}_{DP}^{-1} = \sum_{i=1}^N \left(\sum_{j\in m_i} R_{ij}^t (B_{i,r}^{ij})^{-1} (E_{ij}^i)^t \right) S^i \left(\sum_{j \in m_i} E_{ij}^i (B_{i,r}^{ij})^{-1} R_{ij} \right) \tag{3.8}$$

so that the work can be done in parallel in each subdomain. Let

$$\widehat{B}_i = \sum_{j \in m_i} R_{ij}^t (B_{i,r}^{ij})^{-1} (E_{ij}^i)^t.$$

Moreover, from the operator $\widehat{B}_i$, we can see that the preconditioner $\widehat{F}_{DP}^{-1}$ is different from the preconditioners in [4, 5, 8, 9, 10]. Only on the slave sides of interfaces are the function values transferred between the spaces W_i and M. Hence, the cost needed to compute $\widehat{B}_i w_i$ and $\widehat{B}_i^t \lambda$ is reduced by half compared with other FETI (-DP) preconditioners.

4. Condition number estimation for the preconditioned FETI-DP operator. The following well-known result is given when $a_i(u,v) = \int_{\Omega_i} \nabla u \cdot \nabla v \, dx$ (see Theorem 4.1.3 in [11]). With slight modification, we can obtain a similar result for a general case.

LEMMA 4.1. *For* $w_i \in W_i$, *we have*

$$C_1 |w_i|_{1/2,\partial\Omega_i}^2 \le \langle S^i w_i, w_i \rangle \le C_2 \|w_i\|_{1/2,\partial\Omega_i}^2,$$

where C_1 *and* C_2 *are constants depending on* $A(x)$ *and* $\beta(x)$, *but independent of* H_i *and* h_i.

In the following, we obtain a formula that is useful for analyzing the condition number bound and the result is the same as Lemma 4.3 of Mandel and Tezaur [10]. However, in our formulation, the continuity constraints are imposed on $w \in W$, that is, the d.o.f. on edges and global corners (see (3.3)). The proof can be done similarly as in Lemma 37 of Tezaur [16].

LEMMA 4.2. *For* $\lambda \in M$, *we have*

$$\max_{w \in W \setminus \{0\}} \frac{b(w,\lambda)^2}{\|w\|_W^2} = \langle F_{DP}\lambda, \lambda \rangle.$$

Now, we estimate the lower bound of the condition number for the operator $\widehat{F}_{DP}^{-1} F_{DP}$.

LEMMA 4.3. *For any* $\lambda \in M$, *we have*

$$\max_{w \in W \setminus \{0\}} \frac{b(w,\lambda)^2}{\|w\|_W^2} \ge \|\lambda\|_{(W^0)'}.$$

Proof. For $w \in W^0$, let $\widetilde{w}_i = \sum_{j \in m_i} \widetilde{w}_{ij}$ and $\widetilde{w} = (\widetilde{w}_1, \dots, \widetilde{w}_N)$. Then we have $\widetilde{w} \in W$. Hence it follows that

$$\max_{w \in W \setminus \{0\}} \frac{b(w,\lambda)^2}{\|w\|_W^2} \ge \max_{w \in W^0 \setminus \{0\}} \frac{b(\widetilde{w},\lambda)^2}{\|\widetilde{w}\|_W^2}. \tag{4.1}$$

Since $\widetilde{w}_j = 0$ on Γ_{ij} for $j \in m_i$, we have

$$b(\widetilde{w},\lambda) = \sum_{i=1}^N \sum_{j \in m_i} \int_{\Gamma_{ij}} w_{ij} \lambda_{ij} \, ds = \langle \lambda, w \rangle_m. \tag{4.2}$$

Combining (4.2), (2.8), and (2.10), we obtain

$$\max_{w \in W^0 \setminus \{0\}} \frac{b(\widetilde{w},\lambda)^2}{\|\widetilde{w}\|_W^2} = \max_{w \in W^0 \setminus \{0\}} \frac{\langle \lambda, w \rangle_m^2}{\|w\|_{W^0}^2} = \|\lambda\|_{(W^0)'}^2. \tag{4.3}$$

From (4.1) and (4.3), we complete the proof. ∎

To estimate the upper bound for $\langle F_{DP}\lambda, \lambda\rangle$, we need the following estimate for $\|w_i - w_j\|^2_{H^{1/2}_{00}(\Gamma_{ij})}$.

LEMMA 4.4. *For* $w \in W$, *let* $w_i = w|_{\partial\Omega_i}$ *and* $w_j = w|_{\partial\Omega_j}$. *Then we have*

$$\|w_i - w_j\|^2_{H^{1/2}_{00}(\Gamma_{ij})} \le C \max_{l\in\{i,j\}} \left\{ \left(1 + \log \frac{H_l}{h_l}\right)^2 \right\} \left(|w_i|^2_{1/2,\partial\Omega_i} + |w_j|^2_{1/2,\partial\Omega_j} \right),$$

where C *is a constant independent of* h_i*'s and* H_i*'s.*

Proof. Let $I^H w$ be a linear function on Γ_{ij} that has the same value as w at the end points of Γ_{ij}. From Lemma 5.1 in [10], we have

$$|w_l - I^H w_l|_{H^{1/2}_{00}(\Gamma_{ij})} \le C \left(1 + \log \frac{H_l}{h_l}\right) |w_l|_{1/2,\partial\Omega_l} \text{ for } l = i, j.$$

Using the above bound and the equivalence of $|\cdot|_{H^{1/2}_{00}(\Gamma_{ij})}$ and $\|\cdot\|_{H^{1/2}_{00}(\Gamma_{ij})}$, the result follows. ∎

DEFINITION 4.5. *We define a projection* $\pi_{ij} : H^{1/2}_{00}(\Gamma_{ij}) \to W^0_{ij}$ *for* $v \in H^{1/2}_{00}(\Gamma_{ij})$ *by*

$$\int_{\Gamma_{ij}} (v - \pi_{ij} v)\lambda_{ij}\, ds = 0 \quad \forall \lambda_{ij} \in M_{ij}.$$

From Lemma 2.2 in [1], π_{ij} is a continuous operator on $H^{1/2}_{00}(\Gamma_{ij})$, i.e., there exists a constant C such that

$$\|\pi_{ij} v\|_{H^{1/2}_{00}(\Gamma_{ij})} \le C \|v\|_{H^{1/2}_{00}(\Gamma_{ij})} \quad \forall v \in H^{1/2}_{00}(\Gamma_{ij}). \tag{4.4}$$

We note that the constant C is independent of H_i's and h_i's.

Now, we estimate the upper bound for the operator $\widehat{F}^{-1}_{DP} F_{DP}$.

LEMMA 4.6. *For* $\lambda \in M$, *we have*

$$\max_{w\in W\setminus\{0\}} \frac{b(w,\lambda)^2}{\|w\|^2_W} \le C \max_{i=1,\dots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \|\lambda\|^2_{(W^0)'},$$

where C *is a constant depending on* $A(x)$ *and* $\beta(x)$, *but independent of* h_i*'s and* H_i*'s.*

Proof. From the definitions of $b(w, \lambda)$ in (2.7) and π_{ij}, we have

$$b(w,\lambda)^2 = \left(\sum_{i=1}^{N} \sum_{j\in m_i} \int_{\Gamma_{ij}} \pi_{ij}(w_i - w_j)\lambda_{ij}\, ds \right)^2.$$

We let $z \in W^0$ such that $z|_{\Gamma_{ij}} = \pi_{ij}(w_i - w_j)$. Then the above equation is the duality pairing between λ and z. Hence, using the definition of dual norm on λ, we get

$$b(w,\lambda)^2 \le \|\lambda\|^2_{(W^0)'} \|z\|^2_{W^0}. \tag{4.5}$$

Let $\widetilde{z} = (\widetilde{z}_1, \dots, \widetilde{z}_N) \in W$ be the zero extension of z. Then from (2.8), (2.3), Lemma 4.1, (2.5), (4.4), and Lemma 4.4,

$$\begin{aligned}
\|z\|_{W^0}^2 &= \sum_{i=1}^{N} \langle S^i \widetilde{z}_i, \widetilde{z}_i \rangle \\
&\le C \sum_{i=1}^{N} \|\widetilde{z}_i\|_{1/2,\partial\Omega_i}^2 \\
&\le C \sum_{i=1}^{N} \sum_{j \in m_i} \|\pi_{ij}(w_i - w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})}^2 \\
&\le C \sum_{i=1}^{N} \sum_{j \in m_i} \|w_i - w_j\|_{H_{00}^{1/2}(\Gamma_{ij})}^2 \\
&\le C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \sum_{i=1}^{N} |w_i|_{1/2,\partial\Omega_i}^2 \\
&\le C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \|w\|_W^2.
\end{aligned} \tag{4.6}$$

Here, C denotes a generic constant independent of h_i's and H_i's, which may vary from occurrence to occurrence. Combining (4.5) and (4.6), we complete the proof. □

Since the preconditioner $\widehat{F}_{DP}^{-1}$ follows from the dual norm of $\lambda \in M$ (see (3.5)), combining Lemmas 4.2, 4.3, and 4.6, we obtain the following estimate.

THEOREM 4.7. *For $\lambda \in M$, we have*

$$\langle \widehat{F}_{DP}\lambda, \lambda \rangle \le \langle F_{DP}\lambda, \lambda \rangle \le C \max_{i=1,\dots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \langle \widehat{F}_{DP}\lambda, \lambda \rangle,$$

where C is a constant depending on $A(x)$ and $\beta(x)$, but independent of H_i's and h_i's.

COROLLARY 4.8. *We have the following condition number estimate:*

$$\kappa\left(\widehat{F}_{DP}^{-1} F_{DP}\right) \le C \max_{i=1,\dots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\},$$

where C is a constant depending on $A(x)$ and $\beta(x)$, but independent of H_i's and h_i's.

Remark 4.1. On each Γ_{ij}, the choices of master and slave sides are arbitrary.

Remark 4.2. In Corollary 4.8, the condition number depends on $A(x)$ and $\beta(x)$. Now we consider the problem

$$\begin{aligned}
-\nabla \cdot (\alpha(x) \nabla u(x)) &= f(x) \text{ in } \Omega, \\
u &= 0 \text{ on } \partial\Omega,
\end{aligned}$$

where $\alpha(x)$ is a piecewise constant and has jumps across the subdomain boundaries, i.e., $\alpha(x) = \rho_i$ for all $x \in \Omega_i$ for some constant $\rho_i > 0$. On Γ_{ij}, we choose $\Omega_i^h|_{\Gamma_{ij}}$ as the slave side if $\rho_i \le \rho_j$. Otherwise, we choose $\Omega_i^h|_{\Gamma_{ij}}$ as the master side. Then we have

$$C_1 \rho_i |w_i|_{1/2,\partial\Omega_i}^2 \le \langle S^i w_i, w_i \rangle \le C_2 \rho_i \|w_i\|_{1/2,\partial\Omega_i}^2,$$

where C_1 and C_2 are constants independent of ρ_i's, h_i's, and H_i's. Following the proof of Lemma 4.6 and using the above inequalities instead of Lemma 4.1, we obtain

$$\begin{aligned}
\|z\|^2_{W^0} &\le C \sum_{i=1}^{N} \sum_{j \in m_i} \rho_i \|w_i - w_j\|^2_{H_{00}^{1/2}(\Gamma_{ij})} \\
&\le C \sum_{i=1}^{N} \sum_{j \in m_i} \left\{ \max_{l \in \{i,j\}} \left\{ \left(1 + \log \frac{H_l}{h_l}\right)^2 \right\} \right. \\
&\qquad\qquad \left. \times \left(\rho_i |w_i|^2_{1/2,\partial\Omega_i} + \rho_i |w_j|^2_{1/2,\partial\Omega_j} \right) \right\} \\
&\le C \sum_{i=1}^{N} \sum_{j \in m_i} \left\{ \max_{l \in \{i,j\}} \left\{ \left(1 + \log \frac{H_l}{h_l}\right)^2 \right\} \right. \\
&\qquad\qquad \left. \times \left(\langle S^i w_i, w_i\rangle + \frac{\rho_i}{\rho_j} \langle S^j w_j, w_j\rangle \right) \right\},
\end{aligned}$$

where C is a generic constant independent of ρ_i's, H_i's, and h_i's. Since $\rho_i \le \rho_j$, we can see that the constant C in Lemma 4.6 is bounded independently of the coefficients. Hence, the condition number bound is independent of ρ_i's.

5. Numerical results.

In this section, we provide numerical tests for the FETI-DP formulation developed in this paper. Let $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ and consider the following model problem:

$$\begin{aligned} -\nabla \cdot (\alpha(x,y)\nabla u) &= f \quad \text{in } \ \Omega, \\ u &= 0 \quad \text{on } \ \partial\Omega. \end{aligned} \tag{5.1}$$

We compare the proposed preconditioner (3.8) with the preconditioner of Dryja and Widlund [5] for the cases when $\alpha(x,y) = 1$ and mesh sizes are comparable between neighboring subdomains, and when $\alpha(x,y)$ are highly discontinuous across subdomain interfaces and mesh sizes are not comparable. In the following, we use the notation $\widehat{F}_{KL}^{-1}$ for the preconditioner (3.8) and use $\widehat{F}_{DW}^{-1}$ for Dryja and Widlund's. The preconditioner $\widehat{F}_{DW}^{-1}$ is

$$\widehat{F}_{DW}^{-1} = (B_r \widetilde{B}_r^t)^{-1} \widetilde{B}_r S_{rr} \widetilde{B}_r^t (\widetilde{B}_r B_r^t)^{-1},$$

where $\widetilde{B}_r$ is the scaled matrix of B_r divided by the mesh parameters of slave and master sides (see (3.13) in [5]).

First, we compare these two preconditioners for the same problem with nonmatching discretizations. We take $\alpha(x,y) = 1$ and the exact solution $u(x,y) = y(1-y)\sin \pi x$. The CG iteration continues until the relative residual norm is less than 10^{-6}. We use n to denote the number of nodes on edges, including end points, and use N to denote the number of subdomains. In this problem, we use the same n for all subdomains, divide Ω into rectangular subdomains, as in Figure 3, and denote each subdomain by Ω_{ij}.

To make nonmatching grids across subdomain interfaces, we generate triangulations in each subdomain in the following way: For each subdomain, we have chosen n random quasi-uniform nodes on each horizontal and vertical edge. Using these

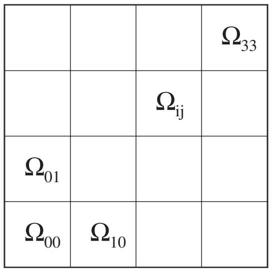

FIG. 3. *Partition of subdomains when $N = 4 \times 4$.*

TABLE 1

Comparison between the proposed preconditioner F_{KL}^{-1} and the Dryja–Widlund preconditioner F_{DW}^{-1} on nonmatching grids when n increases with $N = 4 \times 4$: Iter (number of CG iteration), Cond (condition number of the preconditioned FETI-DP operator).

$n-1$	L^2-error	H^1-error	$\widehat{F}_{KL}^{-1}$		$\widehat{F}_{DW}^{-1}$	
			Iter	Cond	Iter	Cond
4	5.0850e-4	6.0126e-2	10	3.07	7	1.94
8	1.2865e-4	3.0128e-2	13	5.67	8	2.68
16	3.2235e-5	1.5072e-2	15	7.68	10	3.69
32	8.0627e-6	7.5374e-3	16	9.99	10	4.80
64	2.0163e-6	3.7688e-3	17	12.6	11	6.14

nodes, we generate nonuniform structured grids in each subdomain. Since we use the same n for all subdomains, the sizes of meshes between neighboring subdomains are comparable.

In Table 1, we divide Ω into $N = 4 \times 4$ subdomains (see Figure 3), increase the number of nodes n, and compute L^2- and H^1-errors, the number of CG iterations and condition numbers for those preconditioners. For the H^1-error, we compute the broken H^1-norm of errors over all subdomains. Table 2 shows the numerical results when we fix $n - 1 = 4$ and increase the number of subdomains N. For the cases $N = 8 \times 8$, 16×16, and 32×32, we divide Ω into subdomains in the same manner as $N = 4 \times 4$. Here, we used the FETI-DP formulation developed in this paper. From Tables 1 and 2, we can see that our FETI-DP formulation gives $O(h^2)$ and $O(h)$ convergences for L^2- and H^1-errors, respectively. Furthermore, we can see that both preconditioners seem to give the $\log^2$-growth of the condition number bound and that the CG iteration of $\widehat{F}_{DW}^{-1}$ is smaller than $\widehat{F}_{KL}^{-1}$.

Now, we consider the problem (5.1) when $\alpha(x, y)$ is highly discontinuous across subdomain interfaces and the mesh sizes between subdomains are not comparable. In this situation, we will compare two preconditioners $\widehat{F}_{KL}^{-1}$ and $\widehat{F}_{DW}^{-1}$.

We consider the cases of $N = 2 \times 2$, 4×4, 8×8 subdomains. For each subdomain Ω_{ij}, we choose the coefficient $\alpha(x, y)$ in the following way:

$$\alpha(x,y) = \begin{cases} 1 & \text{if both } i \text{ and } j \text{ are even,} \\ 250 & \text{if } i \text{ is odd and } j \text{ is even,} \\ 5000 & \text{if } i \text{ is even and } j \text{ is odd,} \\ 10 & \text{if both } i \text{ and } j \text{ are odd,} \end{cases}$$

and denote them by ρ_{ij}. In addition, we consider the exact solution $u(x, y)$, which

Table 2

Comparison between the proposed preconditioner F_{KL}^{-1} *and the Dryja–Widlund preconditioner* F_{DW}^{-1} *on nonmatching grids when* N *increases with* $n-1=4$*: Iter (number of CG iteration), Cond (condition number of the preconditioned FETI-DP operator).*

$N \times N$	L^2-error	H^1-error	$\hat{F}_{KL}^{-1}$		$\hat{F}_{DW}^{-1}$	
			Iter	Cond	Iter	Cond
4×4	5.0850e-4	6.0126e-2	10	3.07	7	1.94
8×8	1.1744e-4	2.9900e-2	11	3.22	8	2.13
16×16	2.9743e-5	1.4980e-2	12	3.39	8	2.11
32×32	7.4318e-6	7.4917e-3	12	3.51	8	2.10

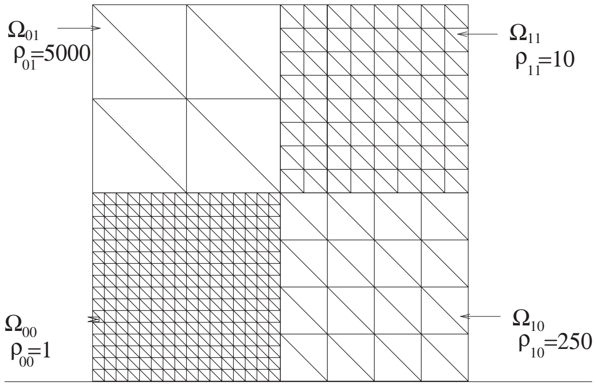

Fig. 4. *Triangulations for the case* $N = 2\times 2$ *and* $\max(H_{ij}/h_{ij}) = 16$.

belongs to $H^1(\Omega)$, according to the partition of the domain:

$$u(x,y) = \begin{cases} p_1(x,y)\sin(\pi x)\sin(\pi y)/\alpha(x,y) & \text{when } N = 2\times 2, \\ p_2(x,y)\sin(2\pi x)\sin(2\pi y)/\alpha(x,y) & \text{when } N = 4\times 4, \\ \sin(8\pi x)\sin(8\pi y)/\alpha(x,y) & \text{when } N = 8\times 8, \end{cases}$$

where

$$p_1(x,y) = (x-1/2)(y-1/2),$$
$$p_2(x,y) = (x-1/4)(x-3/4)(y-1/4)(y-3/4).$$

Following [18, section 1.5.3], we have chosen a different mesh size in each subdomain according to the ratio of coefficients between neighboring subdomains, that is,

$$\frac{h_{ij}}{h_{kl}} \simeq \sqrt[4]{\frac{\rho_{ij}}{\rho_{kl}}},$$

where h_{ij} is the mesh size of the subdomain Ω_{ij}, and we use H_{ij} to denote the size of the subdomain Ω_{ij}. Using the mesh sizes of these ratios, we divide each subdomain into uniform meshes. When $N = 2\times 2$ and $\max(H_{ij}/h_{ij}) = 16$, we

TABLE 3

Comparison between the proposed preconditioner $\hat{F}_{KL}^{-1}$ and the Dryja–Widlund preconditioner $\hat{F}_{DW}^{-1}$ for the problem of highly discontinuous coefficients: Iter (number of GC iteration).

N	$\max(H_{ij}/h_{ij})$	L^2-error	H^1-error	$\hat{F}_{KL}^{-1}$	$\hat{F}_{DW}^{-1}$
				Iter	Iter
2×2	16	3.0571e-5	7.6362e-3	3	17
	32	7.8276e-6	3.8249e-3	3	26
	64	1.9747e-6	1.9133e-3	4	39
	128	4.9571e-7	9.5675e-4	4	50
	256	1.2421e-7	4.7839e-4	4	60
4×4	16	2.1574e-6	1.0939e-3	4	75
	32	5.4460e-7	5.4805e-4	4	81
	64	1.3799e-7	2.7415e-4	4	111
	128	3.4810e-8	1.3709e-4	4	130
8×8	16	1.0262e-3	8.8753e-1	3	113
	32	2.4870e-4	4.4462e-1	4	136
	64	6.4579e-5	2.2240e-1	4	168

obtain triangulations as in Figure 4 and the triangulations are not comparable between neighboring subdomains.

In section 1.5.3 of [18], it was shown that a good approximation of the solution is obtained when the slave side is chosen to give a Lagrange multiplier space of higher dimension. Hence, choosing the subdomain with smaller h_{ij} (smaller ρ_{ij}) as the slave side, we can approximate the exact solution more accurately. This observation coincides with the choices of master and slave sides in Remark 4.2.

Table 3 shows L^2- and H^1-errors and CG iterations with $\widehat{F}_{KL}^{-1}$ and $\widehat{F}_{DW}^{-1}$ as preconditioners. In CG iteration, we use the same stopping criterion 10^{-6} as before. Increasing $\max(H_{ij}/h_{ij})$, we observe the $O(h^2)$ and $O(h)$ convergences of L^2- and H^1-errors, respectively, for all cases of N. Furthermore, we see that the CG iteration of $\widehat{F}_{KL}^{-1}$ is much smaller than $\widehat{F}_{DW}^{-1}$. Since, the condition number bound of the preconditioner $\widehat{F}_{DW}^{-1}$ depends on the ratio of meshes between neighboring subdomains, the preconditioner works inefficiently for these problems with noncomparable meshes.

From our numerical results, we conclude that our formulation gives the correct approximation of the model problem with nonmatching grids. For the case of continuous coefficients and comparable meshes between subdomain interfaces, the preconditioner $\widehat{F}_{DW}^{-1}$ by Dryja and Widlund gives a smaller number of iterations than our preconditioner $\widehat{F}_{KL}^{-1}$. However, our preconditioner $\widehat{F}_{KL}^{-1}$ turns out to be much more efficient than $\widehat{F}_{DW}^{-1}$ for the problem of highly discontinuous coefficients on noncomparable meshes.

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