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Preconditioners for the Dual-Primal FETI Methods on Nonmatching Grids: Numerical Study

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Abstract—The FETI-DP method is a substructuring method that uses Lagrange multipliers to match the continuity condition on the subdomain boundaries. For the FETI-DP method on nonmatching grids, two different formulations are known with respect to how to employ the mortar matching condition. Keeping step with the developments of the FETI-DP methods, a variety of preconditioners for the FETI-DP operator have been developed. However, there has not been any numerical study for the FETI-DP method, which compares those preconditioners on nonmatching grids while there have been a few papers for numerical study on the comparison of FETI preconditioners. Therefore, we present the numerical study of four different preconditioners for two-dimensional elliptic problems. The numerical results confirm the superiority of the preconditioner by Kim and Lee [1] for noncomparably nonmatching grids, while the superiority of the preconditioner by Dryja and Widlund [2] is confirmed for matching grids and comparably nonmatching grids. © 2006 Elsevier Ltd. All rights reserved.

Keywords—FETI-DP, Nonmatching grids, Mortar matching condition, Preconditioner.

1. INTRODUCTION

The finite-element tearing and interconnecting (FETI) method is one of the substructuring methods, which was first introduced by Farhat and Roux [3]. The main idea is to match the continuity condition across subdomain boundaries by Lagrange multipliers. By eliminating primal variables of subdomains, an operator for the Lagrange multipliers is obtained.

In [4], Farhat *et al.* introduced a different substructuring method called the dual-primal FETI (FETI-DP) method. In the FETI-DP method, the continuity condition across the subdomain boundaries is matched by primal variables at corners and dual variables (Lagrange multipliers)

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on edges. Mandel and Tezaur [5] showed its optimal condition number bound,

$$\kappa \le C\left(1+\log\left(\frac{H}{h}\right)\right)^2 \tag{1.1}$$

with Dirichlet preconditioner for both second- and fourth-order elliptic problems in 2-D, where H and h denote the sizes of subdomain and mesh, respectively. Furthermore, Klawonn *et al.* [6] obtained the same bound by employing a new preconditioner for 3-D elliptic problems with heterogeneous coefficients.

The original FETI-DP methods were designed on matching grids. Recently, the FETI-DP methods on nonmatching grids were developed. For the FETI-DP formulation on nonmatching grids, the mortar matching condition is employed to match the continuity condition across the subdomain boundaries. Dryja and Widlund [2,7] imposed the mortar matching condition after eliminating unknowns on both interior and vertex nodal points. Furthermore, to obtain the stability of the mortar projection operator under $H^{-1/2}$-norm, they imposed a restriction that $h_{\delta_{m(i)}}$ and $h_{\gamma_{m(j)}}$, the sizes of meshes on the nonmortar side and the mortar side, respectively, are comparable. Kim and Lee [1] formulated the FETI-DP operator in a different way by imposing the mortar matching condition after eliminating unknowns on interior nodal points only. Then they proposed a Neumann-Dirichlet preconditioner which gives the optimal condition number bound $(1+\log(H/h))^2$ without the restriction that $h_{\delta_{m(i)}} \sim h_{\gamma_{m(j)}}$. The proposed preconditioner is easy to implement and the operator from the nodal values on the interface of subdomains to the Lagrange multiplier space requires only the nodal values on the nonmortar side. Hence, the cost for multiplying the operator to a vector is reduced by half compared with preconditioners developed in other papers (see [2,7]).

In this paper, we compare four kinds of preconditioners: the Dirichlet preconditioner [8], and the preconditioner by Klawonn and Widlund [9], which are developed originally for matching grids, and the preconditioner by Dryja and Widlund [2] and the preconditioner by Kim and Lee [1], which are developed for the FETI-DP operator on nonmatching grids. The numerical results show that the preconditioner by Dryja and Widlund works the most efficiently on matching grids and comparably nonmatching grids. On the other hand, the numerical results for noncomparably nonmatching grids confirm the superiority of the preconditioner by Kim and Lee. Furthermore, we showed heuristically that the preconditioner by Kim and Lee is the limit form of the preconditioner by Klawonn and Widlund.

This paper is organized as follows. The FETI-DP formulation developed by Kim and Lee is described in Section 2, and four preconditioners are introduced in Section 3. In Section 4, we provide the comparison based on numerical results, and the conclusion is given in Section 5.

2. FETI-DP FORMULATION

In this section, we introduce a FETI-DP formulation developed by Kim and Lee [1].

2.1. A Model Problem and Finite-Element Formulation

In this paper, we consider the FETI-DP method on nonmatching grids for the following elliptic problem:

$$\begin{aligned} -\nabla\cdot(A(x)\nabla u(x)) &= f(x), && \text{in } \Omega,\\ u(x) &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $A(x) = (\alpha_{ij}(x))$ for $i,j = 1,2$. We assume that $\alpha_{ij}(x) \in L^\infty(\Omega)$, $f(x) \in L^2(\Omega)$, and $A(x)$ is uniformly elliptic for all $x \in \Omega$. We also assume that the domain Ω is decomposed into a finite number of nonoverlapping bounded subdomains, i.e., $\bar{\Omega} = \bigcup_{i=1}^N \bar{\Omega}_i$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \ne j$ and $|\Omega_i| < \infty$ for all i. Moreover, we assume that this partition is geometrically conforming, which means that the subdomains intersect neighboring subdomains on a whole edge or at a vertex.

Then we triangulate each subdomain Ω_i independently so that the meshes need not match across the subdomain boundaries.

We write problem (2.1) in a variational form as follows: for $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v)_\Omega, \qquad \forall\, v \in H_0^1(\Omega), \tag{2.2}$$

where

$$a(u,v) := \int_\Omega A\nabla u \cdot \nabla v\, dx,$$
$$(f,v)_\Omega := \int_\Omega f v\, dx.$$

Here,

$$H_0^1(\Omega) = \left\{ v \in L^2(\Omega) : \int_\Omega \nabla v \cdot \nabla v\, dx + \int_\Omega v^2\, dx < \infty,\ v = 0 \text{ on } \partial\Omega \right\}.$$

We let Ω_i^h be a quasi-uniform triangulation of the subdomain Ω_i. That is, there exist positive constants γ and σ such that $\gamma h_i \le h_\tau \le \sigma \rho_\tau$ for all $\tau \in \Omega_i^h$, where $h_\tau = |\tau|$, ρ_τ is the diameter of the circle inscribed in τ and $h_i = \max_{\tau \in \Omega_i^h} |\tau|$. For each subdomain Ω_i, we define a finite-element space

$$X_i := \left\{ v \in H_D^1(\Omega_i) : v|_\tau \in P_1(\tau),\ \tau \in \Omega_i^h \right\},$$

where $H_D^1(\Omega_i) := \{v \in H^1(\Omega_i) : v = 0 \text{ on } \partial\Omega \cap \partial\Omega_i\}$ and $P_1(\tau)$ is the set of polynomials of degree ≤ 1 in τ. Then we define finite-element spaces as follows:

$$X := \left\{ v \in \prod_{i=1}^N X_i : v \text{ is continuous at subdomain vertices} \right\},$$
$$W_i := X_i|_{\partial\Omega_i}, \qquad \forall\, i = 1, \dots, N,$$
$$W := \left\{ w \in \prod_{i=1}^N W_i : w \text{ is continuous at subdomain vertices} \right\}.$$

Now, we approximate the solution of problem (2.2) in X. To do this, on nonmatching grids we construct a Lagrange multiplier space. Let $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$. On Γ_{ij} we distinguish $\Omega_j|_{\Gamma_{ij}}$ and $\Omega_i|_{\Gamma_{ij}}$ as in Figure 1, and then we choose one as a mortar side and the other as a nonmortar side. We define

$$m_i := \left\{ j : |\Gamma_{ij}| \ne 0,\ \Omega_j|_{\Gamma_{ij}} \text{ is a mortar side of } \Gamma_{ij} \right\},$$
$$s_i := \left\{ j : |\Gamma_{ij}| \ne 0,\ \Omega_j|_{\Gamma_{ji}} \text{ is a nonmortar side of } \Gamma_{ji} \right\},$$

and

$$W_{ij} := \{ v|_{\Gamma_{ij}} : v \in X_i \}, \qquad \text{for } i = 1, \dots, N \quad \text{and} \quad j \in m_i.$$

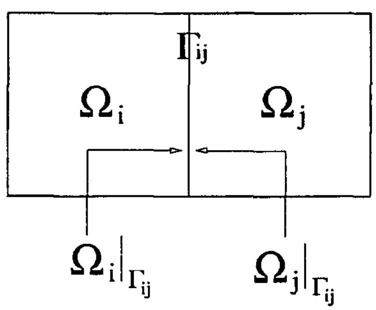

Figure 1. Mortar and nonmortar sides of Γ_{ij}.

Next, we let

$$\left\{\phi_0^{ij}, \phi_1^{ij}, \ldots, \phi_{N_{ij}}^{ij}, \phi_{N_{ij}+1}^{ij}\right\}$$

be the nodal basis functions for W_{ij}. We assume that the basis functions are sequentially ordered according to the location of nodes on Γ_{ij}. We define

$$\begin{aligned} \xi_1^{ij} &:= \phi_0^{ij} + \phi_1^{ij}, \\ \xi_k^{ij} &:= \phi_k^{ij}, \qquad \text{for } k = 2, \ldots, N_{ij} - 1, \\ \xi_{N_{ij}}^{ij} &:= \phi_{N_{ij}}^{ij} + \phi_{N_{ij}+1}^{ij}, \end{aligned}$$

and

$$M_{ij} := \text{span}\left\{\xi_1^{ij}, \ldots, \xi_{N_{ij}}^{ij}\right\}.$$

Then we take the Lagrange multiplier space as

$$M := \prod_{i=1}^{N} \prod_{j \in m_i} M_{ij}.$$

With this Lagrange multiplier space, we impose mortar matching condition on $v = (v_1, \ldots, v_N) \in X$

$$\int_{\Gamma_{ij}} (v_i - v_j)\lambda_{ij}\, ds = 0, \qquad \forall\, \lambda_{ij} \in M_{ij}, \quad i = 1, \ldots, N, \quad j \in m_i. \tag{2.3}$$

We define

$$\begin{aligned} V &:= \{v \in X : v \text{ satisfies } (2.3)\}, \\ a_i(u, v) &:= \int_{\Omega_i} A\nabla u \cdot \nabla v\, dx, \\ f_i(v) &:= (f, v)_{\Omega_i}. \end{aligned}$$

Then we consider a variational problem: find $u \in V$ such that

$$\sum_{i=1}^{N} a_i(u, v) = \sum_{i=1}^{N} (f, v)_{\Omega_i}, \qquad \forall\, v \in V. \tag{2.4}$$

In the sequel, we use the boldface character to represent the vector of which entries are the nodal values of a function. Similarly, we use the boldface character to represent the set of vectors corresponding to a function space.

2.2. FETI-DP Formulation

In this section, we construct the FETI-DP operator for problem (2.2) with the mortar matching condition as constraints. The discrete problem (2.4) can be written as the following equivalent minimization problem with constraints: find $u \in V$ such that

$$\sum_{i=1}^{N} \left(\frac{1}{2}a_i(u, u) - f_i(u)\right) = \min_{v \in V} \sum_{i=1}^{N} \left(\frac{1}{2}a_i(v, v) - f_i(v)\right). \tag{2.5}$$

We introduce a matrix B_i to implement the mortar matching condition (2.3). For $|\partial\Omega_i \cap \partial\Omega_j| \neq 0$, we denote $\partial\Omega_i \cap \partial\Omega_j$ as Γ_{ij} if $\Omega_i|_{\Gamma_{ij}}$ is a nonmortar side and as Γ_{ji}, otherwise. Then we let $\mathbf{W}_l^{ij}$ be the set of vectors that correspond to the nodal values for the functions in $\mathbf{W}_l$ restricted on Γ_{ij}.

We assume that $\Omega_i|_{\Gamma_{ij}}$ is the nonmortar side and $\Omega_j|_{\Gamma_{ij}}$ is the mortar side of Γ_{ij}. We define matrices $B_l^{ij} : \mathbf{W}_l^{ij} \to \mathbf{M}_{ij}$ for $l = i, j$ by

$$\left(B_i^{ij}\right)_{lk} = \int_{\Gamma_{ij}} \xi_l^{ij} \phi_k^{ij}\, ds, \qquad \text{for } l = 1, \dots, N_{ij}, \quad \text{and} \quad k = 0, \dots, N_{ij} + 1,$$

$$\left(B_j^{ij}\right)_{lk} = -\int_{\Gamma_{ij}} \xi_l^{ij} \phi_k^{ji}\, ds, \qquad \text{for } l = 1, \dots, N_{ij}, \quad \text{and} \quad k = 0, \dots, N_{ji} + 1.$$

For $\mathbf{w}_i^{ij} \in \mathbf{W}_i^{ij}$ and $\mathbf{w}_j^{ij} \in \mathbf{W}_j^{ij}$, we rewrite the mortar matching condition (2.3) as

$$B_i^{ij}\mathbf{w}_i^{ij} + B_j^{ij}\mathbf{w}_j{}^{ij} = 0.$$

Now, we define $E_{ij} : \mathbf{M}_{ij} \to \mathbf{M}$, an extension operator from $\mathbf{M}_{ij}$ to $\mathbf{M}$ by zero, and $R_{ij}^l : \mathbf{W}_l \to \mathbf{W}_l^{ij}$ for $l = i, j$, a restriction operator. Let

$$B_i = \sum_{j \in m_i} E_{ij} B_i^{ij} R_{ij}^i + \sum_{j \in s_i} E_{ji} B_i^{ji} R_{ji}^i.$$

Then the mortar matching condition (2.3) becomes

$$\sum_{i=1}^{N} B_i \mathbf{w}_i = 0, \tag{2.6}$$

where $\mathbf{w}_i \in \mathbf{W}_i$.

In the sequel, we use the subscript symbols r and c to represent the degrees of freedom corresponding to nodes on the edges and at vertices, respectively. For $\mathbf{w}_i \in \mathbf{W}_i$, we may write

$$\mathbf{w}_i = \begin{pmatrix} \mathbf{w}_r^i \\ \mathbf{w}_c^i \end{pmatrix}.$$

We denote $\mathbf{W}_c$ as the set of vectors which have degrees of freedom corresponding to the union of subdomain vertices, that is, global corner points. We define the matrix L_c^i which consists of 0 and 1 and restricts the value of $\mathbf{w}_c \in \mathbf{W}_c$ on the vertices of subdomain Ω_i. Therefore, for $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_N) \in \mathbf{W}$, there exists $\mathbf{w}_c \in \mathbf{W}_c$ satisfying $L_c^i \mathbf{w}_c = \mathbf{w}_c^i$ for all $i = 1, \dots, N$. Hence, for $\mathbf{w} \in \mathbf{W}$, the coefficient vector can be written as

$$\mathbf{w} = \begin{pmatrix} \mathbf{w}_{r,c}^1 \\ \vdots \\ \mathbf{w}_{r,c}^N \end{pmatrix}, \qquad \text{where } \mathbf{w}_{r,c}^i = \begin{pmatrix} \mathbf{w}_r^i \\ L_c^i \mathbf{w}_c \end{pmatrix}, \quad \text{for some } \mathbf{w}_c \in \mathbf{W}_c.$$

Let A^i be the stiffness matrix induced from the bilinear form $a_i(\cdot,\cdot)$, S^i the Schur complement matrix from A^i, and $\mathbf{g}^i$ the Schur complement forcing vector induced from $f_i(v)$.

Now, we eliminate interior variables in (2.5). Then problem (2.5) becomes: find $\mathbf{z} \in \mathbf{W}$ satisfying

$$\frac{1}{2}\mathbf{z}^t S \mathbf{z} - \mathbf{z}^t \mathbf{g} = \min_{\mathbf{w} \in \mathbf{W}} \left(\frac{1}{2} \mathbf{w}^t S \mathbf{w} - \mathbf{w}^t \mathbf{g} \right), \qquad \text{subject to } \sum_{i=1}^{N} B_i \mathbf{w}_i = 0, \tag{2.7}$$

where

$$S = \operatorname{diag}_{i=1,\dots,N}\left(S^i\right), \qquad S^i = \begin{pmatrix} S_{rr}^i & S_{rc}^i \\ S_{cr}^i & S_{cc}^i \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} \mathbf{g}^1 \\ \vdots \\ \mathbf{g}^N \end{pmatrix}.$$

Let $B_{i,r}$ be the columns of B_i of which entries are multiplied by the nodal values on the edges, and let $B_{i,c}$ be the columns of B_i, that are multiplied by the nodal values at the vertices. Then (2.7) can be written as follows: find $\mathbf{z} \times \boldsymbol{\lambda} \in \mathbf{W} \times \mathbf{M}$ such that

$$S_{rr}\mathbf{z}_r + S_{rc}\mathbf{z}_c + B_r^t\lambda = \mathbf{g}_r, \tag{2.8a}$$

$$S_{cr}\mathbf{z}_r + S_{cc}\mathbf{z}_c + B_c^t\lambda = \mathbf{g}_c, \tag{2.8b}$$

$$B_r\mathbf{z}_r + B_c\mathbf{z}_c = 0, \tag{2.8c}$$

where

$$S_{rr} = \mathrm{diag}_{i=1,\dots,N}\left(S_{rr}^i\right),$$

$$S_{rc} = \begin{pmatrix} S_{rc}^1 L_c^1 \\ \vdots \\ S_{rc}^N L_c^N \end{pmatrix}, \qquad S_{cr} = S_{rc}^t,$$

$$S_{cc} = \sum_{i=1}^{N} \left(L_c^i\right)^t S_{cc}^i L_c^i,$$

$$B_r = \left(B_{1,r}, \dots, B_{N,r}\right), \qquad B_c = \sum_{i=1}^{N} B_{i,c} L_c^i,$$

$$\mathbf{g}_r = \begin{pmatrix} \mathbf{g}_r^1 \\ \vdots \\ \mathbf{g}_r^N \end{pmatrix}, \qquad \mathbf{g}_c = \sum_{i=1}^{N} \left(L_c^i\right)^t \mathbf{g}_c^i, \qquad \mathbf{z}_r = \begin{pmatrix} \mathbf{z}_r^1 \\ \vdots \\ \mathbf{z}_r^N \end{pmatrix}.$$

Solving (2.8a) for $\mathbf{z}_r$, we get

$$\mathbf{z}_r = S_{rr}^{-1}\left(\mathbf{g}_r - S_{rc}\mathbf{z}_c - B_r^t\lambda\right).$$

By substituting $\mathbf{z}_r$ into (2.8b) and (2.8c), we obtain

$$B_r S_{rr}^{-1} B_r^t \lambda + \left(B_r S_{rr}^{-1} S_{rc} - B_c\right)\mathbf{z}_c = B_r S_{rr}^{-1}\mathbf{g}_r,$$

$$\left(S_{cr} S_{rr}^{-1} B_r^t - B_c^t\right)\lambda - \left(S_{cc} - S_{cr} S_{rr}^{-1} S_{rc}\right)\mathbf{z}_c = -\left(\mathbf{g}_c - S_{cr} S_{rr}^{-1}\mathbf{g}_r\right).$$

Let

$$F_{rr} = B_r S_{rr}^{-1} B_r^t,$$

$$F_{rc} = B_r S_{rr}^{-1} S_{rc} - B_c,$$

$$F_{cr} = S_{cr} S_{rr}^{-1} B_r^t - B_c^t \left(= F_{rc}^t\right),$$

$$F_{cc} = S_{cc} - S_{cr} S_{rr}^{-1} S_{rc},$$

$$d_r = B_r S_{rr}^{-1}\mathbf{g}_r,$$

$$d_c = \mathbf{g}_c - S_{cr} S_{rr}^{-1}\mathbf{g}_r.$$

Then $(\boldsymbol{\lambda}, \mathbf{z_c})$ satisfies

$$\begin{pmatrix} F_{rr} & F_{rc} \\ F_{cr} & -F_{cc} \end{pmatrix}\begin{pmatrix} \lambda \\ \mathbf{z}_c \end{pmatrix} = \begin{pmatrix} d_r \\ -d_c \end{pmatrix}.$$

Eliminating $\mathbf{z_c}$ in the above equation, we obtain

$$\left(F_{rr} + F_{rc} F_{cc}^{-1} F_{cr}\right)\lambda = d_r - F_{rc} F_{cc}^{-1} d_c.$$

Here, $F_{\mathrm{DP}} = F_{rr} + F_{rc} F_{cc}^{-1} F_{cr}$ is called the FETI-DP operator.

3. PRECONDITIONERS FOR THE FETI-DP OPERATOR

In this section, we introduce four preconditioners that will be applied to the FETI-DP operator formulated in the previous section. In the first two sections, we consider the preconditioners developed for matching grids, where the continuity condition across the interface Γ_{ij} is given by

$$\mathbf{u}_r^i|_{\Gamma_{ij}} - \mathbf{u}_r^j|_{\Gamma_{ij}} = 0, \qquad \text{for } \mathbf{u}_r \in \mathbf{X}. \tag{3.1}$$

Then, this continuity condition induces a Boolean matrix $\tilde{B}_r$ satisfying

$$\tilde{B}_r \mathbf{u}_r = 0.$$

We use this notation $\tilde{B}_r$ to introduce the preconditioners in Section 3.1 and 3.2. The next two sections deal with the preconditioners developed for the FETI-DP preconditioners on nonmatching grids.

3.1. The Dirichlet Preconditioner

The Dirichlet preconditioner was first designed for the FETI operator on matching grids by Farhat *et al.* [8]. In [8], it has been shown numerically that the condition number of the FETI operator with the Dirichlet preconditioner is bounded by $C(1+\log(H/h))^2$ when it is applied to the second-order elliptic problems like Poisson problem, plane stress problem, and plain strain problem. Here, H/h is the ratio of the subdomain and the mesh size. Mandel and Tezaur [10] proved that the condition number is bounded by $C(1+\log(H/h))^m$ with $m \le 3$ for the second-order elliptic problems in 2-D and 3-D both.

Furthermore, Mandel and Tezaur [5] obtained $C(1+\log(H/h))^2$ for FETI-DP operator with the Dirichlet preconditioner of the form

$$\hat{F}_D^{-1} := \tilde{B}_r S_{rr} \tilde{B}_r^t,$$

for the second- and fourth-order elliptic problems in 2-D. The numerical results are provided by Farhat *et al.* [4].

3.2. The Preconditioner by Klawonn and Widlund

Klawonn and Widlund [9] designed a preconditioner for the FETI operator with matching grids, working on second-order elliptic problems with jumps of coefficients. We apply the preconditioner to the FETI-DP operator by eliminating the corner effects.

We let ρ_i be the constant coefficient depending on the subdomain Ω_i and $\partial\Omega_{i,r}^h$ the set of nodes on $\partial\Omega_i$ excluding vertices. We also denote N_x as the set of indices of the subdomains which have x on its boundary. The weighted counting function $\mu_i(x)$ which is associated with the individual $\partial\Omega_i$ is defined as

$$\mu_i(x) = \sum_{j\in N_x} \rho_j^\gamma(x), \qquad \text{for } x \in \partial\Omega_{i,r}^h, \quad \text{with } \gamma \in \left[\frac{1}{2}, \infty\right).$$

The diagonal matrix $D_{i,r}$ is composed of the diagonal entry $\rho_i^\gamma(x)/\mu_i(x)$ corresponding to the point $x \in \partial\Omega_{i,r}^h$, and the matrix D_r is defined as

$$D_r := \operatorname{diag}_{i=1,\dots,N}(D_{i,r}). \tag{3.2}$$

Then the preconditioner is of the form

$$\hat{F}_{\mathrm{KW}}^{-1} := \left(\tilde{B}_r D_r^{-1} \tilde{B}_r^t\right)^{-1} \tilde{B}_r D_r^{-1} S_{rr} D_r^{-1} \tilde{B}_r^t \left(\tilde{B}_r D_r^{-1} \tilde{B}_r^t\right)^{-1}. \tag{3.3}$$

Stefanica and Klawonn provided the numerical results in [11] applying $\hat{F}_D^{-1}$ and $\hat{F}_{\mathrm{KW}}^{-1}$ to the FETI operator for the two-dimensional elliptic problems and showed that $\hat{F}_{\mathrm{KW}}^{-1}$ is superior to $\hat{F}_D^{-1}$ both on matching grids and nonmatching grids.

3.3. The Preconditioner by Dryja and Widlund

Dryja and Widlund [2] formulated the FETI-DP operator on nonmatching grids by employing the mortar matching condition. For $|\partial\Omega_i \cap \partial\Omega_j| \neq 0$, we denote the mortar and nonmortar edges of $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ by $\gamma_{m(j)}$ and $\delta_{m(i)}$ if $\partial\Omega_j$ is the mortar edge and $\partial\Omega_i$ is the nonmortar edge, respectively. For convenience, we denote the matrix $B_{i,r}|_{\Gamma_{ij}}$ by $B_{\delta_{m(i)}}$ when Ω_i is the nonmortar side, so that $B_{j,r}|_{\Gamma_{ij}}$ is denoted by $B_{\gamma_{m(j)}}$. Then, we define a scaling matrix $D_{\delta_{m(i)}}$ given by

$$D_{\delta_{m(i)}} \begin{pmatrix} u_i|_{\delta_{m(i)}} \\ u_j|_{\gamma_{m(j)}} \end{pmatrix} \equiv \begin{pmatrix} \dfrac{1}{h_{\delta_{m(i)}}} B_{\delta_{m(i)}} & \dfrac{1}{h_{\gamma_{m(j)}}} B_{\gamma_{m(j)}} \end{pmatrix} \begin{pmatrix} u_i|_{\delta_{m(i)}} \\ u_j|_{\gamma_{m(j)}} \end{pmatrix}.$$

Here, $h_{\delta_{m(i)}}$ and $h_{\gamma_{m(j)}}$ are the mesh parameters of $\delta_{m(i)}$ and $\gamma_{m(j)}$, respectively. If we define $\hat{B} \equiv \operatorname{diag}_{i=1,\dots,N}\{D_{\delta_{m(i)}}\}$, the preconditioner by Driya and Widlund is of the form

$$\hat{F}_{\text{DW}}^{-1} := \left(B_r \hat{B}^t\right)^{-1} \hat{B} S_{rr} \hat{B}^t \left(B_r \hat{B}^t\right)^{-1},$$

where B_r is the mortar matching matrix defined on edges. It was proven that the condition number of the FETI-DP method with this preconditioner is bounded by $C \max_{i=1,\dots,N}(1 + \log(H_i/h_i))^2$ where H_i and h_i are the subdomain size and mesh size of Ω_i, respectively. In proving this optimal condition number estimate, it was assumed that the sizes of meshes on the nonmortar side and mortar side are comparable. To the best of our knowledge, no numerical results have been reported yet for this FETI-DP operator with $\hat{F}_{DW}^{-1}$.

For the above three preconditioners, we remark the following: if the exact matching condition is used, $\hat{F}_D^{-1}$, $\hat{F}_{KW}^{-1}$, and $\hat{F}_{DW}^{-1}$ are identical up to constant for the FETI-DP operator with 2-D elliptic problems of which coefficients do not permit jumps across the subdomain boundaries. In fact, $\tilde{B}_r \tilde{B}_r^t = 2I$ and $D_r = (1/2)I$. Moreover, under the assumption that $h_{\delta_{m(i)}}$ and $h_{\gamma_{m(j)}}$ are comparable, $\hat{F}_{KW}^{-1}$ and $\hat{F}_{DW}^{-1}$ are identical up to a constant factor even on nonmatching grids if the coefficients of the problems do not allow jumps across the interfaces.

3.4. The Preconditioner by Kim and Lee

Kim and Lee [1] developed the FETI-DP method on nonmatching grids through the different approach from Dryja and Widlund [2]. They also designed a new preconditioner, the so-called Neumann-Dirichlet preconditioner, and proved its optimal condition number bound estimate. To introduce this preconditioner, we first define vector spaces as follows: for $|\partial\Omega_i \cap \partial\Omega_j| \neq 0$,

$$\begin{aligned} W_i^0 &:= \{v \in W_i : v = 0 \text{ at the corner points of } \Omega_i\}, \\ W_{ij}^0 &:= \{v \in W_{ij} : v = 0 \text{ at the end points of } \Gamma_{ij}\}, \\ W^0 &:= \prod_{i=1}^{N} \prod_{j \in m_i} W_{ij}^0. \end{aligned}$$

Then the preconditioner is of the form,

$$\hat{F}_{\text{KL}}^{-1} := \sum_{i=1}^{N} \left(\sum_{j \in m_i} E_{ij}^i {B_{i,r}^{ij}}^{-1} R_{ij} \right)^t S \sum_{j \in m_i} E_{ij}^i {B_{i,r}^{ij}}^{-1} R_{ij},$$

where $E_{ij}^i : W_{ij}^0 \to W_i^0$ is the extension operator by 0, $B_{i,r}^{ij} : W_{ij}^0 \to M_{ij}$ is the mortar matching matrix on the nonmortar edge of Γ_{ij}, and $R_{ij} : M \to M_{ij}$ is a restriction operator. In their FETI-DP formulation, the choice of mortar side and nonmortar side is arbitrary, and noncomparably nonmatching grids are permitted. In addition, the preconditioned FETI-DP method permits jumps of coefficients with careful choice of the nonmortar side, and then the condition number bound is independent of the coefficients. The numerical results have been provided for a two-dimensional Poisson problem in [1].

4. COMPARISON OF PRECONDITIONERS BASED ON NUMERICAL RESULTS

In this section, we provide numerical tests to compare various preconditioners introduced in the previous section for the FETI-DP method on nonmatching grids. We consider the following problem on the domain $\Omega = [0,1] \times [0,1]$:

$$\begin{aligned} -\nabla \cdot (\alpha(x,y)\nabla u(x,y)) &= f(x,y), && \text{in } \Omega, \\ u &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where $\alpha(x,y)$ is a piecewise constant function with jumps across the subdomain boundaries.

We employ piecewise bilinear finite elements for the triangulations on each subdomain. Since the induced linear system is symmetric and positive definite, we use the conjugate gradient (CG) algorithm to solve it. The stopping criterion of CG is $\|r_k\|/\|r_0\| \le 10^{-8}$, where r_k is the residual at k^{th} iteration of CG and $\|r_k\|$ is the Euclidean norm of the vector r_k.

We perform the numerical experiments on both matching grids and nonmatching grids, and the results of these experiments are provided in Section 4.1 for matching grids and in Section 4.2 for nonmatching grids.

4.1. Performances on Matching Grids

The experiments using matching grids are performed for both cases that the preconditioners take the Boolean matrix $\tilde{B}_r$ which implements the continuity condition (3.1) and the matrix B_r which implements the mortar matching condition (2.6). For these cases, we consider the elliptic problem (4.1) with $\alpha(x,y) = 1$ and the exact solution $u_{\text{exact}}(x,y) = \sin(\pi x)y(1-y)$.

Table 1 shows the numerical results of the case that we use the continuity condition $\tilde{B}_r\mathbf{u}_r = 0$. Here, N, Iter., Error, and Cond. denote the number of subdomains, the number of CG iterations, the relative L^2 error, i.e., $\|\hat{u}_h - u_{\text{exact}}\|_0/\|u_{\text{exact}}\|_0$, and the condition number of the preconditioned FETI-DP operators, respectively. We do not test all preconditioners because the preconditioners $\hat{F}_D^{-1}$, $\hat{F}_{\text{DW}}^{-1}$, and $\hat{F}_{\text{KW}}^{-1}$ with the Boolean matrix $\tilde{B}_r$ are identical up to a constant factor on matching grids. Hence, we just compare $\hat{F}_{\text{KL}}^{-1}$ and $\hat{F}_D^{-1}$. We observe that the ratio of relative errors, $\|\hat{u}_h - u_{\text{exact}}\|_0/\|\hat{u}_{2h} - u_{\text{exact}}\|_0$, approaches 0.25 as the mesh size reduces by half in the test of the FETI-DP operator without a preconditioner. In addition, in the cases with $\hat{F}_{\text{KL}}^{-1}$ and $\hat{F}_D^{-1}$,

Table 1. Results on matching grids with the exact matching condition.

		No Preconditioner		$\hat{F}_{\text{KL}}^{-1}$		$\hat{F}_D^{-1}$	
N	$\frac{H}{h}$	Iter.	Error (Factor)	Iter.	Cond.	Iter.	Cond.
4×4	4	9	3.23e − 3	15	4.63	6	2.10
	8	15	8.05e − 4 (0.249)	17	7.56	7	3.05
	16	21	2.01e − 4 (0.250)	18	11.35	8	3.82
	32	32	5.03e − 5 (0.250)	20	15.46	9	5.16
	64	42	1.26e − 5 (0.250)	22	21.84	10	6.54
	128	60	3.14e − 6 (0.249)	25	27.07	11	8.63
8×8	4	17	8.05e − 4	17	5.13	10	2.72
	8	25	2.01e − 4 (0.250)	21	8.49	12	4.31
	16	36	5.03e − 5 (0.250)	24	12.36	13	5.75
	32	52	1.26e − 5 (0.250)	27	18.07	15	6.94
	64	73	3.14e − 6 (0.249)	29	24.69	17	8.82
16×16	4	19	2.01e − 4	18	5.21	10	2.87
	8	29	5.03e − 5 (0.250)	22	8.54	12	4.29
	16	44	1.26e − 5 (0.250)	25	12.52	14	6.30
	32	66	3.14e − 6 (0.249)	29	17.46	16	6.96

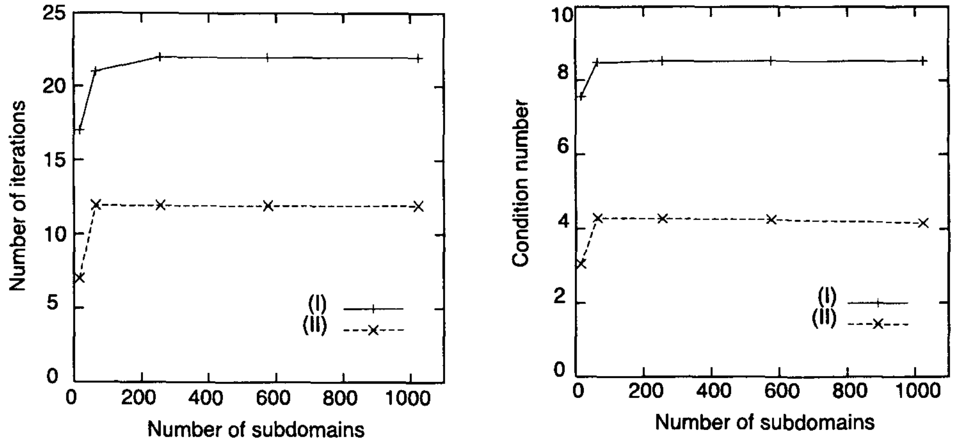

Figure 2. Numerical scalability on matching grids: $H/h = 8$, (I) $= \hat{F}_{\mathrm{KL}}^{-1}$, (II) $= \hat{F}_{D}^{-1}$ with the exact matching condition or $\hat{F}_{\mathrm{DW}}^{-1}$ with the mortar matching condition.

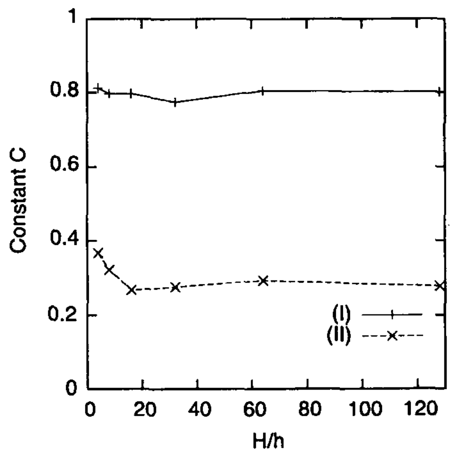

Figure 3. Estimation of $C \approx \kappa/(1+\log(H/h))^2$ on matching grids: $N = 4 \times 4$, (I) $= \hat{F}_{\mathrm{KL}}^{-1}$, (II) $= \hat{F}_{D}^{-1}$ with the exact matching condition or $\hat{F}_{\mathrm{DW}}^{-1}$ with the mortar matching condition.

we get the relative errors of the same level as without a preconditioner. Therefore, we assume that we solve the problem up to the truncation level with the above stopping criterion of CG.

We observe in Table 1 that the CG iteration numbers of the FETI-DP operator with $\hat{F}_{\mathrm{KL}}^{-1}$ and $\hat{F}_{D}^{-1}$ are much smaller than those without a preconditioner. In the comparison between $\hat{F}_{\mathrm{KL}}^{-1}$ and $\hat{F}_{D}^{-1}$, we observe that the CG iteration number with $\hat{F}_{D}^{-1}$ is smaller than that with $\hat{F}_{\mathrm{KL}}^{-1}$. We infer these results from the principal difference between $\hat{F}_{\mathrm{KL}}^{-1}$ and $\hat{F}_{D}^{-1}$ such that the preconditioner $\hat{F}_{\mathrm{KL}}^{-1}$ takes the information of the nonmortar side only while $\hat{F}_{D}^{-1}$ takes the information of both sides of nonmortar and mortar. In Figure 2, it is shown that the preconditioners $\hat{F}_{\mathrm{KL}}^{-1}$ and $\hat{F}_{D}^{-1}$ yield numerically scalable FETI-DP methods. Figure 3 shows the estimated constant C in (1.1) for various H/h. In this figure, we see the $\log^2$ growth of the condition numbers of the operators $\hat{F}_{\mathrm{KL}}^{-1}F_{\mathrm{DP}}$ and $\hat{F}_{D}^{-1}F_{\mathrm{DP}}$, which is optimal in the standard substructuring methods.

Now, the numerical results using B_r, which implements the mortar matching condition, are provided in Table 2. Even though the grids are matching, we may use the mortar matching condition instead of the exact matching condition. We do this experiment to know how the mortar matching condition deteriorates the performance of the preconditioners for the FETI-DP method with matching grids. We compare $\hat{F}_{\mathrm{KL}}^{-1}$, $\hat{F}_{D}^{-1}$, and $\hat{F}_{\mathrm{DW}}^{-1}$ and do not test $\hat{F}_{\mathrm{KW}}^{-1}$ because $\hat{F}_{\mathrm{DW}}^{-1}$

Table 2. Results on matching grids with the mortar matching condition.

		No Preconditioner		$\hat{F}_{\text{KL}}^{-1}$		$\hat{F}_{D}^{-1}$		$\hat{F}_{\text{DW}}^{-1}$	
N	$\frac{H}{h}$	Iter.	Error (Factor)	Iter.	Cond.	Iter.	Cond.	Iter.	Cond.
4×4	4	12	$3.23\text{e}-3$	15	4.63	10	$> 1.65\text{e}+1$	6	2.10
	8	28	$8.05\text{e}-4$ (0.249)	17	7.56	20	$> 2.93\text{e}+1$	7	3.05
	16	49	$2.01\text{e}-4$ (0.250)	18	11.35	40	$> 3.74\text{e}+1$	8	3.82
	32	67	$5.03\text{e}-5$ (0.250)	20	15.46	59	$> 3.05\text{e}+2$	9	5.51
	64	93	$1.26\text{e}-5$ (0.250)	22	21.80	63	$> 4.52\text{e}+2$	10	6.48
	128	123	$3.14\text{e}-6$ (0.249)	25	27.13	68	$> 5.15\text{e}+2$	11	12.69
8×8	4	24	$8.05\text{e}-4$	17	5.13	20	$> 5.67\text{e}+1$	10	2.72
	8	45	$2.01\text{e}-4$ (0.250)	21	8.49	30	$> 7.55\text{e}+1$	12	4.31
	16	82	$5.03\text{e}-5$ (0.250)	24	12.36	56	$> 1.31\text{e}+2$	13	5.75
	32	112	$1.26\text{e}-5$ (0.250)	27	18.07	82	$> 3.55\text{e}+2$	15	6.94
	64	157	$3.14\text{e}-6$ (0.249)	29	24.69	88	$7.77\text{e}+2$	17	8.82
16×16	4	28	$2.01\text{e}-4$	18	5.21	20	$> 9.50\text{e}+1$	10	2.87
	8	54	$5.03\text{e}-5$ (0.250)	22	8.54	35	$> 6.44\text{e}+1$	12	4.29
	16	96	$1.26\text{e}-5$ (0.250)	25	12.52	55	$> 1.97\text{e}+2$	14	6.30
	32	138	$3.14\text{e}-6$ (0.249)	29	17.42	84	$> 3.30\text{e}+2$	16	6.96

and $\hat{F}_{\text{KW}}^{-1}$ are identical up to a constant factor in the case that the elliptic problem does not have jumps of coefficients. In this table, we also observe the optimal order of convergence $O(h^2)$. We see that the CG iteration numbers with the preconditioners $\hat{F}_{\text{KL}}^{-1}$ and $\hat{F}_{\text{DW}}^{-1}$ are much smaller than that without a preconditioner, and we get the best performance results of $\hat{F}_{\text{DW}}^{-1}$ among the preconditioners. On the other hand, the CG iteration number with $\hat{F}_{D}^{-1}$ is much larger than those with $\hat{F}_{\text{KL}}^{-1}$ and $\hat{F}_{\text{DW}}^{-1}$ even though it is smaller than that without a preconditioner. Especially, comparing with Table 1, we observe that the numbers of the CG iterations without a preconditioner and with $\hat{F}_{D}^{-1}$ are much larger than those in Table 1 while $\hat{F}_{\text{KL}}^{-1}$ gives the same numerical results and $\hat{F}_{\text{DW}}^{-1}$ also gives the same results as $\hat{F}_{D}^{-1}$ in Table 1. We estimate the condition numbers from the CG coefficients [12]. It causes numerical instability when the CG coefficients are small. Therefore, sometimes, we are able to estimate the growth of the condition number instead of the exact condition number.

4.2. Performances on Nonmatching Grids

In this section, we provide the numerical results for the FETI-DP operators on nonmatching grids. The experiments are performed for both comparably and noncomparably nonmatching grids.

Table 3 provides the numerical results of the FETI-DP methods for comparably nonmatching grids with the mortar matching condition. We consider problem (4.1) with $\alpha(x,y) = 1$ and the exact solution $u_{\text{exact}}(x,y) = \sin(\pi x)y(1-y)$. To get comparably nonmatching grids, we take n_i random nodes with the restriction

$$h_i \leq 1.5\frac{H_i}{n_i+1}$$

on each edge of the subdomain Ω_i, and generate meshes on each subdomain. Here, H_i is the size of the subdomain Ω_i, n_i is the number of nodes on each edge excluding end points, and h_i is the maximum size of the meshes on each edge of the subdomain Ω_i. Then, this restriction satisfies the assumption of quasi-uniform triangulation.

By the same reason as in the case of Table 2, we only compare the three preconditioners $\hat{F}_{\text{KL}}^{-1}$, $\hat{F}_{D}^{-1}$, and $\hat{F}_{\text{DW}}^{-1}$. We still observe that the CG iteration number for the FETI-DP operator with $\hat{F}_{\text{DW}}^{-1}$ and the condition number of the operator $\hat{F}_{\text{DW}}^{-1}F_{\text{DP}}$ are the smallest. The experiment

Table 3. Results on comparably nonmatching grids.

		No Preconditioner		$\hat{F}_{\mathrm{KL}}^{-1}$		$\hat{F}_D^{-1}$		$\hat{F}_{\mathrm{DW}}^{-1}$	
N	n_i+1	Iter.	Error (Factor)	Iter.	Cond.	Iter.	Cond.	Iter.	Cond.
4×4	4	39	3.90e − 3	16	5.48	53	> 1.82e + 2	10	2.49
	8	94	9.46e − 4 (0.243)	19	9.80	202	> 4.41e + 3	11	4.01
	16	181	2.35e − e − 4 (0.248)	20	15.59	410	> 2.19e + 4	13	5.92
	32	248	5.86e − 5 (0.249)	22	20.74	518	6.91e + 4	15	7.59
	64	355	1.46e − 5 (0.249)	24	26.97	638	> 5.94e + 4	17	10.91
	128	434	3.65e − 6 (0.250)	27	36.50	718	> 1.13e + 5	18	13.27
8×8	4	56	9.68e − 4	19	6.04	88	3.12e + 2	12	2.73
	8	144	2.40e − 4 (0.248)	23	10.31	353	7.05e + 3	14	4.14
	16	264	5.96e − 5 (0.248)	26	15.22	655	> 2.64e + 4	17	5.88
	32	381	1.49e − 5 (0.250)	29	20.78	821	> 3.65e + 4	20	8.06
	64	506	3.71e − 6 (0.249)	32	27.50	938	> 6.03e + 4	22	10.68
16×16	4	66	2.46e − 4	20	6.81	111	4.36e + 2	12	2.86
	8	172	5.95e − 5 (0.246)	24	10.77	411	6.07e + 3	15	4.36
	16	324	1.48e − 5 (0.249)	27	15.19	711	2.37e + 4	17	6.30
	32	477	3.68e − 6 (0.249)	30	20.45	943	5.86e + 4	20	8.66

for $\hat{F}_{\mathrm{KL}}^{-1}$ also shows that the CG iteration number is much smaller than that without a preconditioner. In the case of $\hat{F}_D^{-1}$, we observe that the CG iteration number is much larger than that without a preconditioner. From the numerical results for the FETI method by Stefanica and Klawonn [11], we remark that when the Dirichlet preconditioner employing mortar matching condition is applied to the FETI operator, it also does not work effectively on both matching grids and nonmatching grids as our numerical results show for the FETI-DP method. Figure 4 shows the numerical scalabilities of the FETI-DP methods with $\hat{F}_{\mathrm{KL}}^{-1}$ and $\hat{F}_{\mathrm{DW}}^{-1}$. In Figure 5 we observe that the estimated constant C for $\hat{F}_{\mathrm{KL}}^{-1}$ and $\hat{F}_{\mathrm{DW}}^{-1}$ is getting stable around 0.8 and 0.3, respectively. It demonstrates the optimal condition number estimates of the FETI-DP methods with $\hat{F}_{\mathrm{KL}}^{-1}$ and $\hat{F}_{\mathrm{DW}}^{-1}$.

Until now, we have considered the numerical experiments performed with the elliptic problem (4.1) of which coefficients do not have jumps across the subdomain boundaries. Now, we

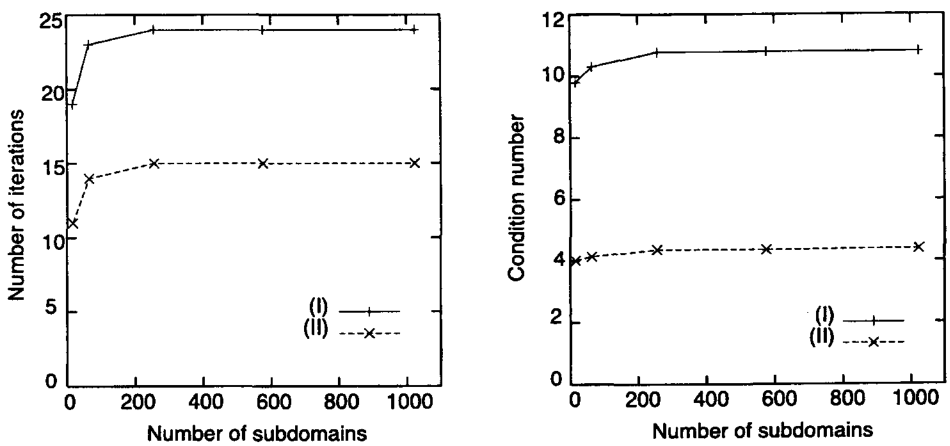

Figure 4. Numerical scalability on comparably nonmatching grids: $H/h = 8$, (I) $= \hat{F}_{\mathrm{KL}}^{-1}$, (II) $= \hat{F}_{\mathrm{DW}}^{-1}$.

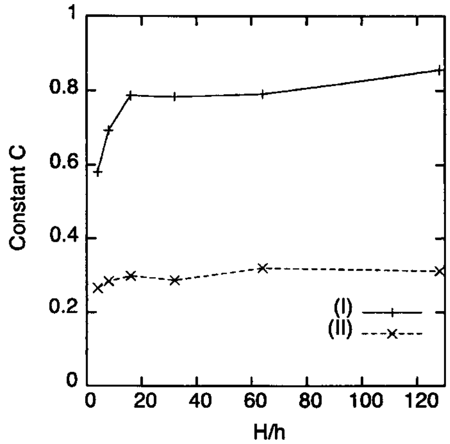

Figure 5. Estimation of $C \approx \kappa / \max(1 + \log(H_i/h_i))^2$ on comparably nonmatching grids: $N = 4 \times 4$, $\mathsf{I} = \hat{F}_{\mathrm{KL}}^{-1}$, $\mathsf{II} = \hat{F}_{\mathrm{DW}}^{-1}$.

perform the numerical experiments with problem (4.1) that have the jumps of coefficients across subdomain boundaries. We consider the cases of 2×2, 4×4, and 8×8 subdomains. In addition, for each case, we choose the test problem of which solution belongs to $H^1(\Omega)$ that is the function space required by the theory of finite elements. From now on, for convenience, we distinguish each subdomain by Ω_{ij} instead of Ω_i. The order of indices of subdomains is explained graphically in Figure 6. Then, the coefficients are determined by the following:

$$\alpha(x,y)|_{\Omega_{ij}} = \rho_{ij} = \begin{cases} 1, & \text{if both } i \text{ and } j \text{ are even,} \\ 250, & \text{if } i \text{ is odd and } j \text{ even,} \\ 5000, & \text{if } i \text{ is even and } j \text{ is odd,} \\ 10, & \text{if both } i \text{ and } j \text{ are odd,} \end{cases}$$

and we take the exact solution

$$u_{\text{exact}}(x,y) = \begin{cases} \left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right) \dfrac{\sin(\pi x)\sin(\pi y)}{\alpha(x,y)}, & \text{for } N = 2 \times 2, \\ \left(x - \frac{1}{4}\right)\left(x - \frac{3}{4}\right)\left(y - \frac{1}{4}\right)\left(y - \frac{3}{4}\right) \times \dfrac{\sin(2\pi x)\sin(2\pi y)}{\alpha(x,y)}, & \text{for } N = 4 \times 4, \\ \dfrac{\sin(8\pi x)\sin(8\pi y)}{\alpha(x,y)}, & \text{for } N = 8 \times 8. \end{cases}$$

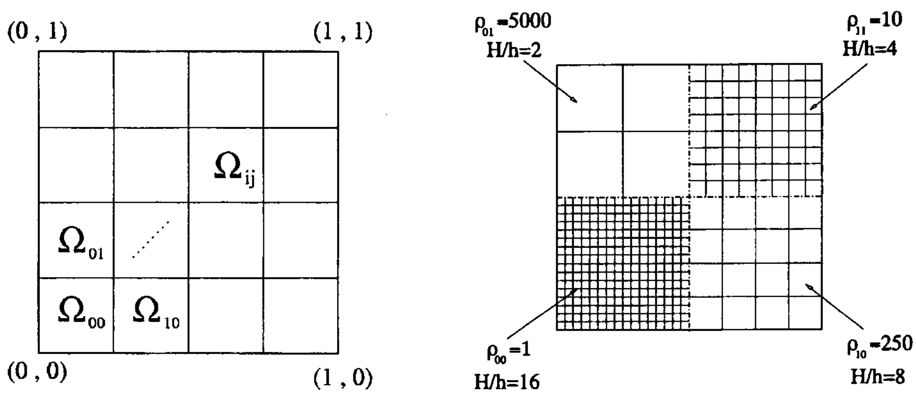

Figure 6. Subdomain index and noncomparably nonmatching grids with piecewise constant ρ_{ij}.

To get noncomparably nonmatching grids, we take uniform grids on all subdomains with the condition that the ratio

$$\frac{h_{ij}}{h_{kl}} \approx \sqrt[4]{\frac{\rho_{ij}}{\rho_{kl}}}$$

for $\partial\Omega_{ij} \cap \partial\Omega_{kl} \neq \emptyset$, where h_{ij} and h_{kl} denote the mesh sizes corresponding to Ω_{ij} and Ω_{kl}, respectively (see [13]). For an example, when $N = 2 \times 2$ we obtain a triangulation as in Figure 6. In [13], it is shown numerically that the choice of nonmortar sides are quite crucial for the problem with jumps of coefficients and a good approximation solution is obtained when the Lagrange multiplier space has higher dimension. That is, the subdomain boundary which has finer grids than the adjacent subdomain boundary should be chosen as a nonmortar side. Hence, for $|\partial\Omega_{ij} \cap \partial\Omega_{kl}| \neq 0$, we choose $\Omega_{ij}|_{\partial\Omega_{ij}\cap\partial\Omega_{kl}}$ as a nonmortar side if the number of nodes on $\Omega_{ij}|_{\partial\Omega_{ij}\cap\partial\Omega_{kl}}$ is larger than those on $\Omega_{kl}|_{\partial\Omega_{ij}\cap\partial\Omega_{kl}}$.

Tables 4 and 5 provide the numerical results for $\hat{F}_{\mathrm{KL}}^{-1}$, $\hat{F}_{\mathrm{DW}}^{-1}$, and $\hat{F}_{\mathrm{KW}}^{-1}$ on noncomparably nonmatching grids with mortar matching condition. We do not test $\hat{F}_{D}^{-1}$ here because it does not work efficiently even on comparably nonmatching grids. For the numerical tests of $\hat{F}_{\mathrm{KW}}^{-1}$, we

Table 4. Results on noncomparably nonmatching grids with the mortar matching condition.

		No Preconditioner		$\hat{F}_{\mathrm{KL}}^{-1}$	$\hat{F}_{\mathrm{KW}}^{-1}$	$\hat{F}_{\mathrm{DW}}^{-1}$
N	$\max\left(\frac{H_i}{h_i}\right)$	Iter.	Error (Factor)	Iter.	Iter.	Iter.
2×2	16	47	$1.76e-4$	4	4	24
	32	55	$4.39e-5$ (0.249)	4	4	36
	64	74	$1.10e-5$ (0.251)	4	4	68
	128	96	$2.74e-6$ (0.249)	4	4	92
	256	116	$6.86e-7$ (0.250)	4	4	108
4×4	16	96	$1.79e-3$	5	5	132
	32	104	$4.46e-4$ (0.249)	5	5	124
	64	127	$1.11e-4$ (0.249)	5	5	177
	128	164	$2.78e-5$ (0.250)	6	6	200
8×8	16	92	$1.73e-e-3$	5	5	167
	32	127	$4.33e-4$ (0.250)	5	5	223
	64	168	$1.08e-4$ (0.249)	5	5	290

Table 5. Comparison of $\hat{F}_{\mathrm{KW}}^{-1}$ for different γ: noncomparably nonmatching grids with the mortar matching condition.

		$\gamma = \frac{1}{2}$	$\gamma = 1$	$\gamma = 2$	$\gamma = 10$	$\hat{F}_{\mathrm{KL}}^{-1}$
N	$\max\left(\frac{H_i}{h_i}\right)$	Iter.	Cond. (Iter.)	Cond. (Iter.)	Cond. (Iter.)	Cond. (Iter.)
2×2	16	16	1.09 (4)	1.04 (4)	1.05 (4)	1.05 (4)
	32	24	1.15 (5)	1.06 (4)	1.06 (4)	1.06 (4)
	64	32	1.22 (5)	1.07 (4)	1.07 (4)	1.07 (4)
	128	39	1.30 (6)	1.08 (4)	1.09 (4)	1.09 (4)
	256	47	1.38 (6)	1.09 (4)	1.10 (4)	1.10 (4)
4×4	16	54	1.35 (7)	1.09 (5)	1.10 (5)	1.10 (5)
	32	51	1.49 (7)	1.13 (5)	1.14 (5)	1.14 (5)
	64	64	1.70 (8)	1.17 (5)	1.18 (5)	1.18 (5)
	128	74	1.92 (8)	1.23 (6)	1.23 (6)	1.23 (6)
8×8	16	63	1.33 (7)	1.10 (5)	1.10 (5)	1.10 (5)
	32	81	1.54 (8)	1.13 (5)	1.14 (5)	1.14 (5)
	64	97	1.79 (10)	1.18 (5)	1.18 (5)	1.18 (5)

choose $\gamma = 1/2$, 1, 2, and 10 for the diagonal scaling matrix D_r in (3.2). Especially, the results of the case $\gamma = 10$ are provided in Table 4 with the results of $\hat{F}_{\mathrm{KL}}^{-1}$ and $\hat{F}_{\mathrm{DW}}^{-1}$. It is observed that the CG iteration number with $\hat{F}_{\mathrm{DW}}^{-1}$ is much larger than those with $\hat{F}_{\mathrm{KL}}^{-1}$ and $\hat{F}_{\mathrm{KW}}^{-1}$. It is even larger than that without a preconditioner for $N = 4 \times 4$ and $N = 8 \times 8$. This fact proves numerically that the meshes on each edge of the adjacent subdomains should be comparable in order that the preconditioner $\hat{F}_{\mathrm{DW}}^{-1}$ yields the FETI-DP method with optimal condition number bound, as it was proved in [2]. However, in the cases of $\hat{F}_{\mathrm{KL}}^{-1}$ and $\hat{F}_{\mathrm{KW}}^{-1}$ with $\gamma = 10$, the CG algorithm finds the approximation solution of which error attains discretization level within a few iterations. In Table 5, the iteration numbers are equal as well as the estimated condition numbers. We also observe that the CG iteration number of $\hat{F}_{\mathrm{KW}}^{-1}$ is getting smaller as γ is getting larger. When $\gamma = 1/2$, the CG iteration number is somewhat large even though it is smaller than the case without a preconditioner. However, the iteration number decreases drastically when $\gamma = 1$, and it becomes equal to the results of $\hat{F}_{\mathrm{KL}}^{-1}$ when $\gamma = 10$.

4.3. Heuristic Comparison of $\hat{F}_{\mathrm{KL}}^{-1}$ and $\hat{F}_{\mathrm{KW}}^{-1}$ with $\gamma = \infty$

Recall the diagonal scaling matrix D_r in (3.2) and $\hat{F}_{\mathrm{KW}}^{-1}$ with B_r instead of $\tilde{B}_r$ in (3.3). For a more specific explaination, we assume that $\Omega_i|_{\Gamma_{ij}}$ is a nonmortar side and $\Omega_j|_{\Gamma_{ij}}$ is a mortar side for $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j \neq \emptyset$, which implies that $\rho_i < \rho_j$. Then, for the nodal points on $\Omega_i|_{\Gamma_{ij}}$, the elements of $D_{i,r}$ and $D_{j,r}$ are determined as

$$\zeta_i := \frac{\rho_i^\gamma}{\rho_i^\gamma + \rho_j^\gamma}, \qquad \zeta_j := \frac{\rho_j^\gamma}{\rho_i^\gamma + \rho_j^\gamma},$$

respectively. Hence,

$$\zeta_i^{-1} \to \infty, \quad \zeta_j^{-1} \to 1, \qquad \text{as } \gamma \to \infty. \tag{4.2}$$

Therefore the effect of the matrix $D_{j,r}^{-1}$ can be ignored when γ is large. We may write

$$B_r D_r^{-1} B_r^t = \begin{pmatrix} B_{r,s} & B_{r,m} \end{pmatrix} \begin{pmatrix} D_{r,s}^{-1} & 0 \\ 0 & D_{r,m}^{-1} \end{pmatrix} \begin{pmatrix} B_{r,s}^t \\ B_{r,m}^t \end{pmatrix},$$

where the subscripts s and m denote submatrices on nonmortar sides and mortar sides, respectively. From (4.2), it holds that

$$B_r D_r^{-1} B_r^t \to B_{r,s} D_{r,s}^{-1} B_{r,s}^t, \qquad \text{as } \gamma \to \infty. \tag{4.3}$$

Similarly, we obtain

$$B_r D_r^{-1} = \begin{pmatrix} B_{r,s} & B_{r,m} \end{pmatrix} \begin{pmatrix} D_{r,s}^{-1} & 0 \\ 0 & D_{r,m}^{-1} \end{pmatrix} \to \begin{pmatrix} B_{r,s} D_{r,s}^{-1} & 0 \end{pmatrix}, \qquad \text{as } \gamma \to \infty. \tag{4.4}$$

Therefore, from (4.3) and (4.4), it follows that

$$\hat{F}_{\mathrm{KW}}^{-1} \to \begin{pmatrix} B_{r,s}^{-t} & 0 \end{pmatrix} S_{rr} \begin{pmatrix} B_{r,s}^{-1} \\ 0 \end{pmatrix} = \hat{F}_{\mathrm{KL}}^{-1}, \qquad \text{as } \gamma \to \infty.$$

This is a distinguished feature of the preconditioner $\hat{F}_{\mathrm{KL}}^{-1}$.

In the aspect of the cost of calculation, $\hat{F}_{\mathrm{KL}}^{-1}$ is somewhat better than $\hat{F}_{\mathrm{KW}}^{-1}$ because $\hat{F}_{\mathrm{KL}}^{-1}$ does not require the information of mortar sides at all, and also it does not use the matrix D_r.

5. CONCLUSION

In this paper, we have compared four preconditioners. Among them, two were originally developed for matching grids and the other two were developed for FETI-DP methods on nonmatching grids. However, we have not applied all preconditioners for all numerical experiments because some preconditioners become identical for some cases. In particular, the preconditioner introduced by Kim and Lee has a distinguished feature. It just uses the flux information of the nonmortar side. We have observed that the preconditioner $\hat{F}_{\mathrm{DW}}^{-1}$ introduced by Dryja and Widlund works effectively for all experiments except noncomparably nonmatching grids and good performance results of the preconditioner $\hat{F}_{\mathrm{KL}}^{-1}$ which was introduced by Kim and Lee. Furthermore, we have observed that its efficiency catches up with that of $\hat{F}_{\mathrm{KW}}^{-1}$ when we consider the elliptic problem of which coefficients have jumps across the subdomain boundaries on noncomparably nonmatching grids. In this case, the performances of $\hat{F}_{\mathrm{KL}}^{-1}$ and $\hat{F}_{\mathrm{KW}}^{-1}$ with $\gamma = 10$ are almost the same. In fact, $\hat{F}_{\mathrm{KL}}^{-1}$ is the limit form of $\hat{F}_{\mathrm{KW}}^{-1}$ as γ approaches ∞. Considering the cost of the calculation to implement the preconditioners $\hat{F}_{\mathrm{KW}}^{-1}$ and $\hat{F}_{\mathrm{KL}}^{-1}$, we have a somewhat better result for $\hat{F}_{\mathrm{KL}}^{-1}$ because it does not require implementing the information of mortar sides at all, and does not require the diagonal scaling matrix.

In practice, it is useful to allow noncomparably nonmatching grids across subdomain boundaries because many elliptic problems appearing in the real world have nonconstant coefficients and we have to deal with this problem numerically by putting the coefficients constant on each subdomain independently. Considering this fact, it seems that the preconditioner $\hat{F}_{\mathrm{KL}}^{-1}$ is the most useful preconditioner in a practical sense.

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