

A FETI-DP formulation for the three dimensional
elasticity problem with mortar methods

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The Neumann-Dirichlet preconditioner

- FETI-DP methods for 2D elliptic problems with mortar matching (Kim and Lee (2002))

$$\kappa(M_{ND}^{-1}F_{DP}) \leq C \max_{i=1,\dots,N} \left\{ (1 + \log(H_i/h_i))^2 \right\}$$

$$M_{ND}^{-1} = \sum_{i=1}^N B^{(i)} D^{(i)} S^{(i)} D^{(i)} B^{(i)t},$$

n : dofs on nonmortar, m : dofs on remaining parts

$$D^{(i)} = \begin{pmatrix} D_{mm}^{(i)} & 0 \\ 0 & D_{nn}^{(i)} \end{pmatrix},$$

$$D_{mm}^{(i)} = 0, \quad D_{nn}^{(i)} = \left(B_n^{(i)t} B_n^{(i)} \right)^{-1}.$$

- Efficient one for the problems with discontinuous coefficients
- Extended to 3D elliptic, and 2D Stokes problem
- Well connected to the BDDC method

(BDDC method) M_{BDDC}^{-1} \cdots M_{ND}^{-1} (FETI-DP method)

(E_D, P_D : average and jump operators)

$$E_D + P_D = I,$$

$$E_D^2 = E_D, \quad P_D^2 = P_D,$$

$$E_D P_D = P_D E_D = 0$$

Elasticity problem

Partition of domain $\Omega \in \mathbf{R}^3$

$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_N \quad (D: \text{Dirichlet B.C. } N: \text{natural B.C. })$$

$$\bar{\Omega} = \cup_{i=1}^N \bar{\Omega}_i \quad (\text{geometrically conforming})$$

Find $\mathbf{u} \in [H_D^1(\Omega)]^3$ such that

$$\sum_{i=1}^N \left(\int_{\Omega_i} G_i \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, d\mathbf{x} + \int_{\Omega_i} G_i \beta_i \nabla \cdot \mathbf{u} \nabla \cdot \mathbf{v} \, d\mathbf{x} \right) = \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in [H_D^1(\Omega)]^3, \quad (1)$$

$$\langle \mathbf{F}, \mathbf{v} \rangle = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} + \int_{\partial\Omega_N} \mathbf{g} \cdot \mathbf{v} \, ds$$

$$\varepsilon(\mathbf{u})_{lk} := \frac{1}{2} \left(\frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l} \right) \quad (\text{strain tensor})$$

$$G_i = E_i / (1 + \nu_i), \quad \beta_i = \nu_i / (1 - 2\nu_i), \quad E_i > 0, \quad \nu_i \in (0, 1/2].$$

We consider the case of **Compressible Elasticity** ($\nu_i \leq \gamma < 1/2$).

- **Finite Elements**

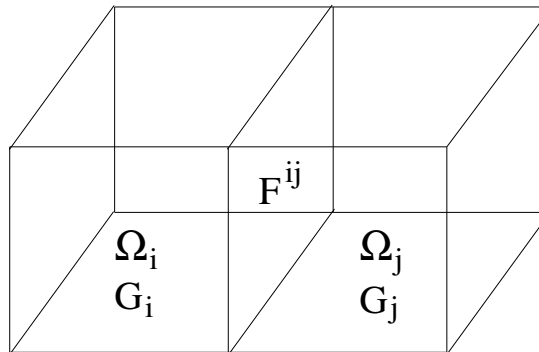
T_i : a **quasi-uniform** triangulation in Ω_i ,

X_i : P^1 -conforming finite elements based on T_i ,

$\{T_i\}_{i=1}^N$: **nonmatching** across interfaces,

$X = \{\mathbf{v} : \mathbf{v}|_{\Omega_i} \in X_i, \quad i = 1, \dots, N\}$ **Nonconforming**

- **Mortar Matching Condition**



Ω_i (nonmortar) , Ω_j (mortar) if $G_i \leq G_j$,

W_{ij} : finite elements on F^{ij} equipped with interior nodes of the nonmortar

M_{ij} : dual/standard Lagrange multiplier space

Mortar matching condition is

$$\int_{F^{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \boldsymbol{\lambda} ds = 0, \quad \forall \boldsymbol{\lambda} \in M_{ij}, \quad \forall F^{ij}.$$

Primal variables

- **Redundant constraints**

1. **Point-wise matching condition (matching grids)**

3D Elliptic : Face or Edge average constraints

(Klawonn, Widlund, Dryja (2002))

$$\int_{F^{ij}} (v_i - v_j) ds = 0, \quad \int_{E^{lk}} (v_i - v_j) ds = 0.$$

3D Elasticity : Edge average or momentum constraints

(Klawonn, Widlund (2004))

$$\int_{E^{lk}} (\mathbf{v}_i - \mathbf{v}_j) ds = 0, \quad \int_{E^{lk}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \mathbf{r} ds = 0. \quad (\mathbf{r} : \text{rotation})$$

2. **Mortar matching condition (nonmatching grids)**

3D Elliptic : Face average constraints (Kim (2004))

Note : Edge constraints **are not redundant** to mortar matching

- Why we need more redundant constraints than three face average constraints for the elasticity problem?

1. Ker(ε) has six rigid body motions $\{\mathbf{r}_i\}_{i=1}^6$ as its basis:

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{r}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{r}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{r}_4 = \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}, \mathbf{r}_5 = \begin{pmatrix} -x_3 \\ 0 \\ x_1 \end{pmatrix}, \mathbf{r}_6 = \begin{pmatrix} 0 \\ x_3 \\ -x_2 \end{pmatrix},$$

2. Vertex constraints are not strong enough to get a good condition number bound

- **Projected momentum constraints**

1. On the interface $F^{ij} = \partial\Omega_i \cap \partial\Omega_j$, we consider

$$\mathbf{r}_4 = \frac{1}{H} \begin{pmatrix} x_2 - \hat{x}_2 \\ -x_1 + \hat{x}_1 \\ 0 \end{pmatrix}, \quad \mathbf{r}_5 = \frac{1}{H} \begin{pmatrix} -x_3 + \hat{x}_3 \\ 0 \\ x_1 - \hat{x}_1 \end{pmatrix}, \quad \mathbf{r}_6 = \frac{1}{H} \begin{pmatrix} 0 \\ x_3 - \hat{x}_3 \\ -x_2 + \hat{x}_2 \end{pmatrix},$$

where $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \in F^{ij}$ and $H = \text{diam}(F^{ij})$.

2. Projection $Q : [L^2(F^{ij})]^3 \rightarrow M_{ij}$

$$\int_{F^{ij}} (Q(\mathbf{w}) - \mathbf{w}) \cdot \phi \, ds = 0 \quad \forall \phi \in W_{ij}.$$

3. We get **three projected momentum constraints**;

$$\int_{F^{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot Q(\mathbf{r}_l) \, ds = 0, \quad l = 4, \dots, 6.$$

- Primal variables

Adding more redundant constraints (primal variables) to the FETI-DP formulation causes **bigger coarse problem** that is becoming **the bottle neck** of the computation.

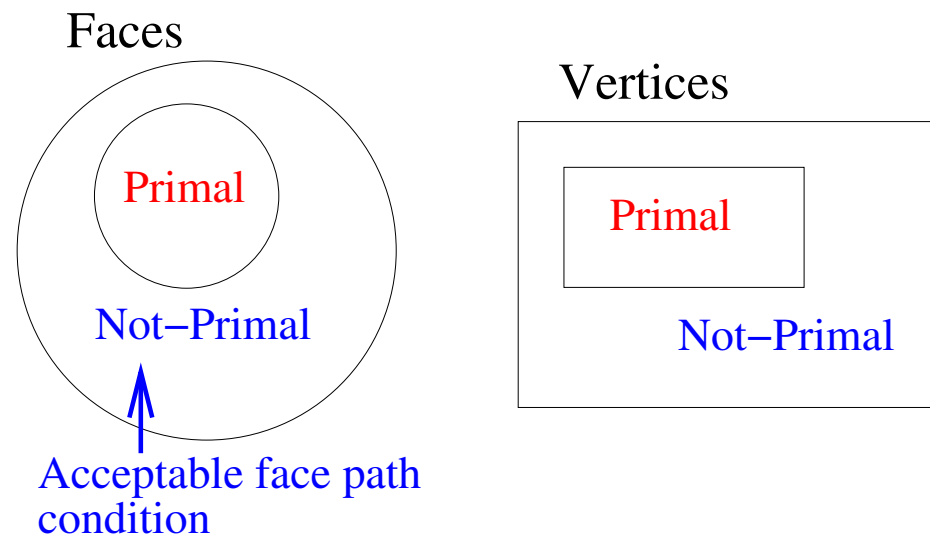


Figure 1: Set of primal variables

Primal faces : face average and projected momentum

Primal vertices : pointwise matching

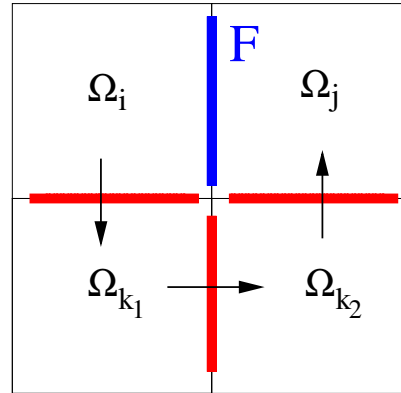


Figure 2: Acceptable face path between Ω_i and Ω_j (red: primal, blue: not-primal)

Definition 1 (Acceptable face path)

For a pair of subdomains (Ω_i, Ω_j) with a common face F and $G_i \leq G_j$,

An acceptable face path $\{\Omega_i, \Omega_{k_1}, \dots, \Omega_{k_n}, \Omega_j\}$ is a path from Ω_i to Ω_j

through **primal faces** with the **coefficients** G_{k_l} satisfying

$$TOL * \{(1 + \log(H_i/h_i))^{-1}(1 + \log(H_{k_l}/h_{k_l}))^2\} * G_{k_l} \geq G_i.$$

FETI-DP formulation

- **Finite elements with primal constraints**

Finite elements over the **interfaces** (W) and the **nonmortar interfaces** (W_n)

$$W = \prod_{i=1}^N W_i, \quad (W_i = X_i|_{\partial\Omega_i}), \quad W_n = \prod_{ij \text{ nonmortar}} W_{ij}$$

.

$\widetilde{W} := \{ \mathbf{w} \in W : \mathbf{w} \text{ satisfies } \mathbf{vertex constraints} \text{ at the } \mathbf{primal} \text{ vertices} \\ \text{and } \mathbf{face constraints} \text{ across the } \mathbf{primal} \text{ faces} \},$

$\widetilde{W}_n := \{ \mathbf{w}_n \in W_n : \mathbf{w}_n \text{ has the } \mathbf{zero averages and momentums} \text{ on primal faces} \}.$

$$\int_{F^{ij}} \mathbf{w}_n ds = 0,$$

$$\int_{F^{ij}} \mathbf{w}_n \cdot Q(\mathbf{r}_l) ds = 0, \quad l = 4, \dots, 6.$$

Note : \mathbf{w} (**zero extension** of \mathbf{w}_n into W) $\in \widetilde{W}$.

- Notation (c : primal vertices, r : the remaining part)

$$\mathbf{w}^{(i)} (\in W_i) = \begin{pmatrix} \mathbf{w}_r^{(i)} \\ \mathbf{w}_c^{(i)} \end{pmatrix}, \quad S^{(i)} = \begin{pmatrix} S_{rr}^{(i)} & S_{rc}^{(i)} \\ S_{cr}^{(i)} & S_{cc}^{(i)} \end{pmatrix}, \quad B^{(i)} = \begin{pmatrix} B_r^{(i)} & B_c^{(i)} \end{pmatrix}$$

- Restriction

$$R_c^{(i)} : V_c \text{ (Set of primal vertices)} \rightarrow V_c|_{\Omega_i}$$

- Assemble unknowns

$$\mathbf{w}_r = \begin{pmatrix} \mathbf{w}_r^{(1)} \\ \vdots \\ \mathbf{w}_r^{(N)} \end{pmatrix}, \quad \mathbf{w}_c \text{ with } R_c^{(i)} \mathbf{w}_c = \mathbf{w}_c^{(i)}, \quad \forall i.$$

- Assemble mortar matching matrix

$$B_r = \begin{pmatrix} B_r^{(1)} & \dots & B_r^{(N)} \end{pmatrix}, \quad B_c = \sum_{i=1}^N B_c^{(i)} R_c^{(i)}.$$

- Assemble stiffness matrix

$$S_{rr} = \begin{pmatrix} S_{rr}^{(1)} & & \\ & \ddots & \\ & & S_{rr}^{(N)} \end{pmatrix}, S_{rc} = \begin{pmatrix} S_{rc}^{(1)} R_c^{(1)} \\ \vdots \\ S_{rc}^{(1)} R_c^{(N)} \end{pmatrix}, S_{cc} = \sum_{i=1}^N R_c^{(i)t} S_{cc}^{(i)} R_c^{(i)}.$$

c : unknowns at primal vertices

r : unknowns at the remaining part

- mortar matching condition

$$B_r \mathbf{w}_r + B_c \mathbf{w}_c = 0. \quad (2)$$

- redundant constraints

$$R^t (B_r \mathbf{w}_r + B_c \mathbf{w}_c) = 0. \quad (3)$$

- mixed problem

$$\begin{pmatrix} S_{rr} & S_{rc} & B_r^t R & B_r \\ S_{cr} & S_{cc} & B_c^t R & B_c \\ R^t B_r & R^t B_c & 0 & 0 \\ B_r & B_c & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{w}_r \\ \mathbf{w}_c \\ \mu \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{g}_r \\ \mathbf{g}_c \\ 0 \\ 0 \end{pmatrix}.$$

- FETI-DP equation

$$F_{DP} \lambda = d.$$

- **Preconditioner** M_{ND}^{-1}

$$\langle M_{ND}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w}_n \in \widetilde{W}_n} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2}{\langle S\mathbf{w}, \mathbf{w} \rangle}, \quad (4)$$

$$S = \text{diag}_{i=1, \dots, N} \left(S^{(i)} \right), \quad B = \begin{pmatrix} B^{(1)} & \dots & B^{(N)} \end{pmatrix}$$

\mathbf{w} : zero extension of \mathbf{w}_n . (Note that $\mathbf{w} \in \widetilde{W}$)

- **Lower bound**

$$\langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \max_{\mathbf{w} \in \widetilde{W}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2}{\langle S\mathbf{w}, \mathbf{w} \rangle}. \quad (5)$$

Therefore we have the lower bound

$$\langle M_{ND}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \leq \max_{\mathbf{w} \in \widetilde{W}} \frac{\langle B\mathbf{w}, \boldsymbol{\lambda} \rangle^2}{\langle S\mathbf{w}, \mathbf{w} \rangle} = \langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle. \quad (6)$$

We will show the upper bound

$$\langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \leq C \langle M_{ND}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle.$$

Condition number analysis

Assumption on mesh size $G_i \leq G_j$

$$\frac{h_j}{h_i} \leq \left(\frac{G_j}{G_i} \right)^\gamma, \quad 0 \leq \gamma \leq 1.$$

Mortar projection $\pi_{ij} : [L^2(F^{ij})]^3 \rightarrow W^{ij}$

$$\int_{F^{ij}} (\pi_{ij}(\mathbf{w}) - \mathbf{w}) \cdot \boldsymbol{\psi} \, ds = 0, \quad \forall \boldsymbol{\psi} \in M^{ij}.$$

- **Primal face** F

Lemma 1 For $F (= \partial\Omega_i \cap \partial\Omega_j)$, a primal face with $G_i \leq G_j$, and $\mathbf{w} \in \widetilde{W}$, we have

$$G_i \|\pi_{ij}(\mathbf{w}_i - \mathbf{w}_j)\|_{H_{00}^{1/2}(F)}^2 \leq C \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 |\mathbf{w}_i|_{S_i}^2 + \left(1 + \log \frac{H_j}{h_j}\right)^2 |\mathbf{w}_j|_{S_j}^2 \right\},$$

where $|\mathbf{w}_i|_{S_i}^2 = \langle S^{(i)} \mathbf{w}_i, \mathbf{w}_i \rangle$.

- **Not Primal face F**

Assumptions on mesh sizes and the acceptable face path

$$\frac{h_j}{h_i} \leq \left(\frac{G_j}{G_i} \right)^\gamma, \quad 0 \leq \gamma \leq 1$$

$$TOL * \left\{ (1 + \log(H_i/h_i))^{-1} (1 + \log(H_{k_l}/h_{k_l})) \right\} * G_{k_l} \geq G_i$$

Lemma 2 For $F(= \partial\Omega_i \cap \Omega_j)$, not primal face with $G_i \leq G_j$, we assume that there is an acceptable face path $\{\Omega_i, \Omega_{k_1}, \dots, \Omega_{k_n}, \Omega_j\}$. Then we have

$$G_i \|\pi_{ij}(\mathbf{w}_i - \mathbf{w}_j)\|_{H_{00}^{1/2}(F)}^2 \leq C(TOL) \sum_{l \in N_{ij}} \left(1 + \log \frac{H_l}{h_l} \right)^2 |\mathbf{w}_l|_{S_l}^2,$$

where $N_{ij} = \{i, k_1, \dots, k_n, j\}$.

- **Upper bound**

Lemma 3 *We have*

$$\langle F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle \leq C \max_{i=1, \dots, N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\} \langle M_{ND}\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle,$$

where $C = C(TOL, L)$ with L (maximum face path length).

- **Condition number bound**

Theorem 2 *For the elasticity problem with discontinuous material coefficients, we obtain*

$$\kappa(M_{ND}^{-1}F_{DP}) \leq C(TOL, L) \max_{i=1, \dots, N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\}.$$

Algorithm for selecting primal faces

Definition 3 (Essentially primal face) A face $F = \partial\Omega_i \cap \partial\Omega_j$ is essentially primal, if there is no acceptable face path for (Ω_i, Ω_j) with all faces except the face F chosen as primal faces.

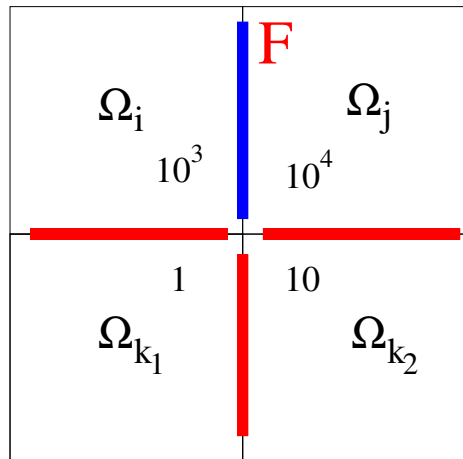


Figure 3: Essentially primal face $F = \partial\Omega_i \cap \partial\Omega_j$ when $TOL = 10$, $\frac{H_l}{h_l} = 4$

$$TOL * \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^{-1} \left(1 + \log \frac{H_{k_l}}{h_{k_l}} \right)^2 \right\} * G_{k_l} \geq G_i$$

Algorithm ($TOL, L, \{G_l, H_l, h_l\}$) given

- **Step 1.** Determine **essentially primal faces** and add them into the primal face set P .
- **Step 2.** Determine **not-primal faces** based on the set P .
- **Step 3. Order** the undetermined faces **decreasingly** according to the **ratio of coefficients** across the faces $F (= \partial\Omega_i \cap \partial\Omega_j)$ according to the **number of neighbors** what the subdomains Ω_i and Ω_j have.
- **Step 4. Do until every undetermined face F determined**
 - Add **undetermined face F** into **primal face set P**
 - Then determine **not primal faces** based on the **updated** primal face set P

End

Test example

$\Omega = [0, 1]^3$ partitioned into N^3 cubical subdomains

$L = 6, \quad TOL = 10$

In case **constant coefficient** $G_i = 1.0$ for all i

In case **random coefficient** $1, 10, 10^2, 10^3$ are randomly distributed

N^3	Total	Opimal ($N^3 - 1$)	Primal (const)	Primal (random)
2^3	12	7	7	8
4^3	144	63	68	89
6^3	540	215	246	322
8^3	1344	511	646	804
10^3	2700	999	1300	1598

Table 1: Number of primal faces for the constant coefficient and randomly distributed coefficients

Conclusion

- For the **compressible elasticity** problem with **discontinuous** material coefficients by introducing **average and momentum constraints**

$$\kappa(M_{ND}^{-1}F_{DP}) \leq C(TOL, L) \max_{i=1, \dots, N} \left\{ (1 + \log \frac{H_i}{h_i})^2 \right\}$$

- Reduced the number of primal variables with the concept of **primal and not-primal faces**
- **Geometrically nonconforming subdomain partition** by introducing **Change of Basis formulation**
- **Three-level BDDC method** in order to solve the **huge coarse problem** efficiently

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