

RESEARCH STATEMENT

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SUMMARY

In the last two decades domain decomposition has emerged as a powerful methodology for solution of partial differential equations using massively parallel computers. In such algorithms the global solution is constructed by repeatedly solving smaller local subdomain problems. Concerted research effort has led to the recent introduction of powerful FETI-DP [15] and BDDC [7] types of domain decomposition algorithms. Fast convergence and scalability have been proved for these algorithms, and they have been successfully applied to the solution of large structural engineering problems, the Maxwell's equations and problems in fluid dynamics. In practical applications non-matching discretization in different subdomains is often necessary. Standard domain decomposition techniques do not apply to such systems and mortar methods have been developed to address this problem, however their applications leads to particularly challenging linear systems [1, 4, 2].

I have developed FETI-DP type algorithms for such systems with proven optimal bounds for convergence and scalability. I have further extended these algorithms to three-dimensional elliptic problems, the Stokes equations and compressible elasticity problems with heterogeneous material parameters. An important component in my work has been the construction of auxiliary coarse-level problems that are crucial to the scalability and fast convergence of domain decomposition algorithms. My future research will focus on the development of BDDC type algorithms for incompressible elasticity and the three-dimensional Navier-Stokes equations.

The following sections explain these projects in detail and outline additional directions of mathematical and computational interest in my research.

DOMAIN DECOMPOSITION METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

Perhaps the earliest example of a domain decomposition method is the Schwarz alternating method for the solution of the classical boundary value problem for harmonic functions [28]. The problem domain is decomposed into two or more overlapping subdomains and the solution is obtained as the limit of an iterative process, where in each iteration local subdomain problems are solved and information about the local solutions is exchanged between neighboring subdomains.

Research in domain decomposition has flourished in the past twenty five years, motivated primarily by the need to solve complex scientific and engineering problems using large parallel computer systems. The linear system arising from the discretization of the given partial differential equation is solved by a preconditioned iterative Krylov space method. The basic idea is to construct a preconditioner with a guaranteed fast convergence rate, using subproblem solutions as building blocks. Schwarz-type methods for both overlapping and non-overlapping subdomain partition have been developed and analyzed [9, 8, 10]. The non-overlapping methods have evolved with the introduction of more scalable and more flexible methods such as

the Neumann-Neumann, FETI, FETI-DP and BDDC algorithms [30, 17, 15, 7]. In particular, FETI-DP and BDDC utilize continuity constraints across the subdomain partition to incorporate a coarse-level problem description in the preconditioner.

FETI-DP AND BDDC METHODS

The finite element tearing and interconnecting (FETI) method was first introduced by Farhat and Roux [17] for second order elliptic problems in two dimensions. The domain is partitioned into a set of non-overlapping subdomains, which share common boundaries (interfaces). In FETI methods a separate set of interface unknowns is assigned to each subdomain. Continuity of the solution across interfaces is imposed weakly using Lagrange multipliers, leading to a saddle point problem. The local unknowns are then eliminated and the resulting linear system for the Lagrange multipliers is solved iteratively. FETI type algorithms have been successfully applied to the solution of large industrial problems using highly parallel computers. FETI methods have been extensively studied both mathematically and experimentally [16, 5, 24, 13], eventually leading to the introduction of dual-primal FETI methods (FETI-DP) [15], where strong continuity constraints are imposed upon selected primal unknowns on the interfaces and weak continuity constraints are imposed on the remaining interface unknowns.

The introduction of continuity constraints on primal variables in FETI-DP leads to an algebraic construction of a coarse-level problem. In this respect FETI-DP is different from a typical domain decomposition method, where the coarse problem is constructed using an auxiliary coarse-level triangulation of the domain. In FETI-DP the coarse problem is obtained in the space of Lagrange multipliers after the elimination of all other unknowns. The construction of the coarse problem is essential to the scalability of the algorithm. The condition number κ of the FETI-DP preconditioned system satisfies

$$(1) \quad \kappa \leq C \left(1 + \log \frac{H}{h} \right)^2,$$

where H is the diameter of subdomains, h is the typical mesh size of a subdomain, and C is a constant that does not depend on H and h . Combining this with the well known bound on the convergence rate of Krylov space methods [18] implies that the rate of convergence for the preconditioned system does not depend on the number of subdomains. Experimental studies have confirmed the superior scalability and improved convergence rate of FETI-DP type algorithms [14, 25, 31].

Recently the balancing domain decomposition with constraints (BDDC) method was introduced by Dohrmann [7]. In BDDC a special coarse-level basis is constructed for the space of energy-minimizing functions that satisfy a set of given interface continuity constraints. The preconditioner is constructed by adding the solution of the problem in the coarse space to the weighted average of the solutions of the remaining local problems. The preconditioned linear system of BDDC is dual to that of FETI-DP when the same set of primal constraints is used. It was proved that these two linear systems have the same spectra [27, 26]. In contrast to FETI-DP, a preconditioned linear system for primal unknowns is solved in BDDC, and the preconditioner contains a coarse level component. Thus, the coarse problem itself can be solved using a Krylov space method. Such approach leads to the recently developed three-level BDDC method [32] which turns out to be as scalable as the original BDDC algorithm.

LINEAR SYSTEMS ARISING FROM MORTAR DISCRETIZATION

Mortar methods were first introduced by Bernardi, Maday and Patera [3] for solving elliptic problems with nonconforming discretizations. Nonconforming approximations occur when subdomain finite element triangulations do not match across subdomain interfaces, or when finite elements and spectral elements are used in neighboring subdomains, or when the neighboring subdomains use spectral elements of different orders. As a result, mortar matching conditions are imposed on the solution at subdomain interfaces. These are weak continuity conditions with respect to a carefully chosen Lagrange multiplier space. Mortar discretization typically leads to two types of linear systems: those resulting from finite elements that satisfy mortar matching conditions and saddle point systems obtained when mortar matching conditions are used as constraints.

The first type of linear systems is solved by iterative methods with a Schwarz-type preconditioner constructed from the solutions of subdomain problems and a coarse-level problem. In this case the main difficulty is the construction of the coarse-level problem, which is essential to the scalability of the algorithm. This difficulty arises because the partition of a complex substructure into subdomains may be geometrically non-conforming. I have constructed a Schwarz type preconditioner with a coarse space that is independent of the subdomain partition and ensures the scalability of the preconditioned system [23]. The coarse problem is constructed using an auxiliary domain such as a hexagon, or a tetrahedron, containing the original complex structure. This domain is decomposed into smaller hexagons or tetrahedra which serve as coarse finite elements. The coarse space is obtained by projecting the nodal basis functions of the coarse finite elements onto the subspace of the original structure finite elements that satisfy mortar matching conditions. I have proved that this projection is continuous with respect to the Sobolev norms and have established the condition number bound (1) for the preconditioned system. This work was carried out by a collaboration with Professor Olof Widlund.

This approach outlined above does not apply to saddle point problems obtained from mortar discretization. In particular, such systems arise in fluid dynamics and three-dimensional elasticity with heterogeneous material parameters. The type of systems is similar to the saddle point system occurring in FETI-DP-type algorithms [17, 15]. The application of FETI-DP to such systems makes it possible to construct a coarse-level component for the preconditioner since the coarse problem in FETI-DP is constructed algebraically by eliminating primal unknowns. I have developed FETI-DP type algorithms for the Stokes problem and the three-dimensional elasticity problems with heterogeneous material parameters. This work is described in the following section.

FETI-DP ALGORITHMS FOR SYSTEMS ARISING FROM MORTAR DISCRETIZATION

Recently FETI algorithms have been developed for two and three-dimensional elliptic problems with mortar discretizations [29], and numerical evidence suggests that they satisfy the condition number bound (1) also. FETI-DP algorithms for mortar discretization of two-dimensional elliptic problems have been introduced and analyzed [11, 12]. The condition number bound (1) was established under the assumption that mesh sizes of neighboring subdomains are comparable. Unfortunately this assumption typically does not hold in most practical applications [33]. I have developed a FETI-DP algorithm for mortar discretization of two-dimensional elliptic problems with coefficients which are discontinuous across subdomain interfaces [21]. The bound (1) is shown to hold without any restriction on the mesh

sizes of neighboring subdomains and is independent of the problem coefficients. This has been further confirmed in numerical experiments [6].

I have further extended this algorithm to three-dimensional elliptic problems, the two-dimensional Stokes problem and three-dimensional compressible elasticity problems with discontinuous material parameters [19, 22, 20]. The main difficulty in the application of FETI-DP to these problems is the choice of primal constraints for nonconforming approximations. These constraints should be compatible with the mortar matching conditions. For three-dimensional elliptic problem I have introduced additional face-average matching constraints. For the Stokes equations I have shown that edge-average matching constraints ensure that the solution on the interface is consistent the divergence-free condition. In the case of compressible elasticity, in addition to the face average matching constraints I have used momentum matching constraints to ensure the non-singularity of the local subdomain problems. For each of these problems I have established the condition number bound (1). In particular, for the compressible elasticity, the constant in (1) is independent of the material parameters.

FUTURE WORK

While much progress has been made in application of domain decomposition techniques to the development of highly parallel algorithms, many challenging problem remain. These include some problems of almost incompressible elasticity, the Maxwell's equations and non-linear problems such as the solution of the Navier-Stokes equations.

Nonconforming discretization enables us to use a special type of finite elements that approximate the solution of complex problems, such as almost incompressible elasticity and the Navier-Stokes, with a reliable accuracy. I plan to extend my algorithm developed for the compressible elasticity to incompressible elasticity problems with nonconforming discretization.

FETI-DP algorithms for three-dimensional elliptic problems and three-dimensional compressible elasticity problems employ a large number of primal constraints, leading to a coarse problem whose solution is expensive. I have recently been studying a BDDC algorithm for three-dimensional elliptic problems which employs the same set of primal constraints as in my FETI-DP algorithm [19]. The purpose of this work is to construct a three-level BDDC algorithm which would facilitate the solution of the coarse problem. Efficient choice of constraints in BDDC algorithms is of great interest, in particular for the incompressible elasticity and for the Stokes equations in three dimensions.

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