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모르타르 방법으로 이산화된 FETI-DP 형식의 preconditioner에 관한 연구

Preconditioners for FETI-DP formulations with mortar methods

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Abstract

In this dissertation, we consider FETI methods which are known as the most efficient domain decomposition method especially for solving large scale problems. In FETI methods, Lagrange multipliers are introduced to enforce the continuity of solutions across subdomain interfaces. This gives a mixed problem with the continuity condition as constraints. After eliminating unknowns other than the Lagrange multipliers, the resulting linear system is solved using the preconditioned conjugate gradient method. There are three variants of FETI methods, FETI, two-level FETI and dual-primal FETI(FETI-DP) method. Until now, FETI methods have been developed for the problems discretized with conforming finite elements. Among them, we extend FETI-DP methods to the problems with nonconforming discretizations, that arise from nonmatching triangulations across subdomain interfaces. The nonmatching triangulations are important for problems with corner singularities, contact problems as well as multi-physics problems. Moreover, the generation of meshes can be done independently in each subdomain. To resolve the nonconformity of the approximation, we consider mortar methods, which gives the same order of accuracy as conforming finite elements. In the mortar methods, the Lagrange multiplier space is introduced to enforce the continuity of solutions across the subdomain interfaces. The saddle point formulation of mortar methods gives a similar linear system to the mixed formulation of the FETI methods. The linear system is ill-conditioned. Moreover, it is difficult to find a good preconditioner for this system. We apply the FETI-DP method to solving this linear system efficiently and to finding a good preconditioner easily.

This dissertation concerns elliptic problems both in $2D$ and $3D$, and Stokes problem in $2D$. Especially, redundant continuity constraints are introduced for $3D$ elliptic problems and Stokes problem. The Lagrange multipliers to the redundant constraints are treated as the primal variables in the FETI-DP formulation. This redundant constraints accelerate the convergence of FETI-DP methods. We propose Neumann-Dirichlet preconditioners for the FETI-DP formulations of those problems considered in this dissertation. The Neumann-Dirichlet preconditioner follows from a dual norm on the Lagrange multiplier space. To define the dual norm, we consider a duality pairing between the Lagrange multipliers and finite elements on nonmortar sides. A norm for the finite elements on nonmortar sides are defined by using the discrete harmonic extension or the Stokes extension. We show that the preconditioner gives the condition number bound $C \max_{i=1,\cdots,N} \left\{(1+\log(H_i/h_i))^2\right\}$, where C is a constant independent of meshes and the number of subdomains. Here, H_i and h_i are the subdomain size and mesh size associated with Ω_i, and N is the number of subdomains. For the elliptic problems with discontinuous coefficients, we can also show that the constant C is not depending on the coefficients. In addition, numerical results are provided.

Contents

List of Tables

List of Figures

1. Introduction

Domain decomposition methods for solving partial differential equations have been developed from the idea of Schwarz alternating method [45] which is an iterative method for the solution of classical boundary value problems for harmonic functions. In the method, moving alternately from one subdomain to the other subdomain, similar problems are solved successively and the solutions formed by the iterative process converge to the solution of the original single domain problem.

In fact, this iterative process can be regarded as a preconditioner for the boundary value problem in the original domain. The preconditioner is essentially composed of operators which solve local problems in each subdomain. Hence, the goal of domain decomposition methods is to develop a good preconditioner using the local solvers from each subdomain. There are Schwarz methods for overlapping decompositions, substructuring methods and FETI (Finite Element Tearing and Interconnecting) methods for nonoverlapping decompositions. Among them, FETI methods are most efficient and scalable especially when we solve large scale problems using parallel machine. The scalability means that a method is robust to the increase of subdomains with the fixed number of unknowns in each subdomain. Usually, we need a coarse finite element space to obtain scalability of the method. In FETI methods, coarse problem is naturally induced from the FETI formulation without forming a special coarse finite element space.

The FETI method was first introduced by Farhat and Roux[29] for solving the elastostatic problems. It is another variant of substructuring iterative methods. The main idea is using Lagrange multipliers to match the solutions continuously across subdomain boundaries. This gives a mixed problem. After eliminating unknowns other than Lagrange multipliers, they obtained a linear system for the Lagrange multipliers. In fact, this linear system, so called the FETI operator, is symmetric and positive definite(s.p.d.) and solved using the preconditioned conjugate gradient method(PCGM). We call these whole processes as the FETI method. Further, they introduced a Dirichlet preconditioner and presented the numerical scalability of the FETI method for second order elliptic problems.

Mandel and Tezaur [37] analyzed that the condition number of the FETI operator with the Dirichlet preconditioner is bounded by $C\,(1+\log{(H/h)})^m$ with $m \leq 3$ for second order elliptic problems in $2D$ and $3D$ both, where H and h denote the sizes of subdomains and meshes, respectively and C is a constant independent of mesh size and subdomain size. For the same problem, Klawonn and Widlund[31] proposed a new preconditioner using diagonal scaling matrix and showed that the bound of condition number is $C\,(1+\log{(H/h)})^2$. Moreover, they allowed jumps of coefficients of elliptic problems across subdomain boundaries. However, for fourth order problems, it was observed that the condition number grows faster than $O(1+\log{(H/h)})^3$. Farhat *et al.* [25, 27] developed the two level FETI method and they showed that this method is numerically scalable for fourth order elliptic problems like as second order elliptic problems.

The dual-primal FETI(FETI-DP) method was introduced in [26] with the similar idea to the two level FETI method. The idea is to use primal variables at corner points and Lagrange multipliers on edges to match solutions continuously across subdomain boundaries. In the FETI-DP method, unknowns other than the primal variables at corners and the Lagrange multipliers are eliminated first. Then, the linear system for primal variables at corners and Lagrange multipliers follows. After eliminating the primal variables at corners, we obtain the resulting linear system of Lagrange multipliers, which is called a FETI-DP operator. This operator is also s.p.d. and solved using PCGM as in the FETI method. However, we have nonsingular local problems and a global corner problem in the FETI-DP operator. These make the implementation of the FETI-DP method easier than the FETI method. Moreover, the global corner problem fulfills the role of a coarse solver, which globally transmits information between subdomains. They also showed numerically that the FETI-DP method is scalable with respect to the mesh size, the subdomain size and the number of elements per subdomain for second and fourth order elliptic problems both. Mandel and Tezaur[38] analyzed that the condition number of the FETI-DP method is bounded by $C\,(1+\log{(H/h)})^2$ for both second and fourth order elliptic problems in $2D$. For $3D$ elliptic problems with heterogeneous coefficients, Klawonn *et el.* [32] obtained the same bound of the condition number. In addition, the FETI-DP method was applied to solving Stokes problem and Navier-Stokes problem by Li [34, 35].

Recently, FETI(-DP) methods are applied to the problems discretized with nonconforming finite elements [21, 22, 42, 48, 49]. Especially, the nonconforming finite elements arising from nonmatching triangulations across subdomain interfaces are considered. Nonmatching discretizations are important for multiphysics simulations, contact-impact problems, the generation of meshes and partitions aligned with jumps in diffusion coefficients, *hp*-adaptive methods, and special discretizations in the neighborhood of singularities (corners or joints). Of many methods for nonmatching methods, including [20] and [44], we consider mortar methods to resolve the nonconformity of approximations. In mortar methods, orthogonality relations between the jumps in the traces across subdomain interfaces are satisfied using a discrete Lagrange multiplier space. Then, the mortar methods give the same accuracy of approximations as conforming finite elements with the same polynomial order. The sparse linear systems that arise in mortar methods are similar to the systems solved by FETI methods on conforming discretizations [23, 29]. Hence, FETI(-DP) methods can be applied to solving this linear system efficiently.

In [42, 48, 49], numerical study shows that FETI methods with mortar discretizations are efficient and the preconditioned FETI operator seems to have condition number bound $C(1+\log(H/h))^2$. After then, Dryja and Widlund [21] showed that the Dirichlet preconditioner gives a condition number bound $(1+\log(H/h))^2$ with the Neumann-Dirichlet ordering of substructures, where H and h denote the maximum diameter of subdomains and minimum size of meshes of all subdomains, respectively. In general cases, that is, without considering ordered substructures, they obtained $(1+\log(H/h))^4$ for the condition number bound. Moreover, in [22], they proposed a different preconditioner which is similar to the one in [31], and proved the condition number bound $(1+\log(H/h))^2$. However, in their analysis, they imposed a restriction that the sizes of meshes between neighboring subdomains are comparable. This restriction is impractical when the coefficients of elliptic problems are highly discontinuous between subdomains (see Wohlmuth[54]).

In this dissertation, we extend FETI-DP methods to the problems with mortar discretizations. For the elliptic problems both in $2D$ and $3D$, we obtain the FETI-DP formulation differently with that of Dryja and Widlund [21, 22] and propose a Neumann-Dirichlet preconditioner which gives the condition number bound $C(1 + \log(H/h))^2$ without the restriction on the mesh size between neighboring subdomains. Moreover, for the elliptic problems with heterogeneous coefficients the condition number bound is shown to be independent of the coefficients. The Neumann-Dirichlet preconditioner follows from a dual norm on the Lagrange multiplier space. The dual norm is defined by using a duality pairing between the Lagrange multiplier space and finite elements on nonmortar sides, and a norm for the finite elements on nonmortar sides. The norm for the finite element function on nonmortar sides is given by the discrete harmonic extension of that function. In [32], it was shown that only considering corners as primal variables is inefficient for $3D$ problems. Hence, redundant mortar matching constraints are essentially needed for $3D$ problems to obtain the same condition number bound as $2D$ problems. Moreover, the corresponding Lagrange multipliers are treated as primal variables in the FETI-DP formulation.

For Stokes problem, we derive the FETI-DP operator with mortar matching constraints and show that the Neumann-Dirichlet preconditioner gives the condition number bound $C(1+\log(H/h))^2$. In the FETI-DP formulation, we add redundant continuity constraints to the coarse problems following the idea of Li [34]. These constraints are introduced to solve the Stokes problem correctly and efficiently.

This dissertation is organized as follows: In Chapter 2, we introduce Sobolev spaces and finite elements, and in Chapter 3, we overview FETI(-DP) methods. Mortar methods are explained in Chapter 4. Chapter 5 and Chapter 6 are devoted to FETI-DP formulations of the elliptic problems in $2D$ and $3D$, and the analysis of the condition number bound for the Neumann-Dirichlet preconditioner. In Chapter 7, we extend the method to the Stokes problem. Numerical results are presented in Chapter 8.

In the following, we make no distinction between a finite element function and the corresponding vector of nodal values, i.e., we use the same symbol v both for the finite element function and the vector of nodal values. Similarly, we use the same notation for a finite element function space and a space of vectors of nodal values. Moreover, the constant C is a generic constant which varies from place to place and does not depend on the mesh size h and the subdomain size H.

2. Sobolev spaces and finite elements

2.1 Sobolev spaces

Let $\Omega \subset \mathbb{R}^n (n = 2, 3)$ be a bounded polygonal($n = 2$) or polyhedral($n = 3$) domain and $L^2(\Omega)$ be the space of square integrable functions defined in Ω equipped with the norm $\| \cdot \|_{0,\Omega}$:

$$\|v\|_{0,\Omega}^2 := \int_\Omega v^2 \, dx.$$

The space $L_0^2(\Omega)$ is a set of functions in $L^2(\Omega)$ with zero average. The space $H^1(\Omega)$ is a set of functions in $L^2(\Omega)$, which are square integrable up to the first weak derivatives, and the norm is given by

$$\|v\|_{1,\Omega} := \left(\int_\Omega \nabla v \cdot \nabla v \, dx + \frac{1}{d_\Omega^2} \int_\Omega v^2 \, dx \right)^{1/2},$$

where d_Ω denotes the diameter of Ω. For any set A, we denote d_A as the diameter of the set A.

Now, we introduce Sobolev spaces defined on the boundary $\partial\Omega$ of the domain Ω. Let $\Sigma \subset \partial\Omega$. For $w \in L^2(\Sigma)$, we define

$$|w|_{1/2,\Sigma}^2 := \int_\Sigma \int_\Sigma \frac{|w(x) - w(y)|^2}{|x - y|^n} \, ds(x) \, ds(y).$$

Then $H^{1/2}(\partial\Omega)$ is the trace space of $H^1(\Omega)$ normed by

$$\|w\|_{1/2,\partial\Omega}^2 := |w|_{1/2,\partial\Omega}^2 + \frac{1}{d_{\partial\Omega}} \|w\|_{0,\partial\Omega}^2.$$

For any $F \subset \partial\Omega$, $H_{00}^{1/2}(F)$ is the set of functions in $L^2(F)$ such that the zero extensions of the functions into $\partial\Omega$ are contained in $H^{1/2}(\partial\Omega)$. The norm for $v \in H_{00}^{1/2}(F)$ is given by

$$\|v\|_{H_{00}^{1/2}(F)} := \left(|v|_{H_{00}^{1/2}(F)}^2 + \frac{1}{d_F} \|v\|_{0,F}^2 \right)^{1/2},$$

where

$$|v|^2_{H^{1/2}_{00}(F)} := |v|^2_{H^{1/2}(F)} + \int_F \frac{v(x)^2}{\mathrm{dist}(x, \partial F)}\, ds.$$

The space $H^{1/2}_{00}(F)$ can be obtained by Hilbert scaling between the spaces $L^2(F)$ and $H^1_0(F)$ or by the real method of interpolation between those spaces (see Lions and Magenes [36]). From Section 4.1 in [56], we have the following relation:

$$C_1 \|\widetilde{v}\|_{1/2,\partial\Omega} \leq \|v\|_{H^{1/2}_{00}(F)} \leq C_2 \|\widetilde{v}\|_{1/2,\partial\Omega} \quad \forall v \in H^{1/2}_{00}(F), \tag{2.1}$$

where the constants C_1 and C_2 are independent of F and $\widetilde{v}$ is the zero extension of v into $\partial\Omega$. For the product spaces $[H^{1/2}(\partial\Omega)]^2$ and $[H^{1/2}_{00}(F)]^2$, norms are defined using the product norms and the inequalities (2.1) also hold.

In general, we use $W^m_p(\Omega)$ to denote the Sobolev space with m-th weak derivatives in L^p-norm. The norm is defined by

$$\|v\|_{W^m_p(\Omega)} := \left(\sum_{0 \leq k \leq m} |v|^p_{W^k_p(\Omega)} \right)^{1/p},$$

and the semi-norm $|\cdot|_{W^k_p(\Omega)}$ is defined by

$$|v|_{W^k_p(\Omega)} := \left(\sum_{|\alpha|=k} \int_\Omega |D^\alpha v(x)|^p\, dx \right)^{1/p}.$$

Here, $\alpha = (\alpha_1, \cdots, \alpha_n)$ denotes a multi-index , $|\alpha| = \sum_{i=1}^n \alpha_i$ and $D^\alpha v(x)$ is the weak derivative of $v(x)$ corresponding to the multi-index α. Note that we write the Sobolev space with scaled H^1-norm as $H^1(\Omega)$ and the usual Sobolev space as $W^1_2(\Omega)$.

2.2 Approximation by interpolation and inverse inequalities

In this section, we introduce several interpolation operators and review the approximation properties and the inverse inequalities for the finite element functions. These results are used to analyze the approximation order of finite element methods.

Definition 2.1 *A domain Ω is said to be star-shaped with respect to B if, for every $x \in \Omega$, the closed convex hull of $\{x\} \bigcup B$ is contained in Ω.*

For a star-shaped domain Ω, let

$$\rho_{\max} = \sup\{\rho : \Omega \text{ is star-shaped with respect to a ball of radius } \rho\}.$$

Then, we state the following well-known result by Bramble and Hilbert [14, 15]:

Lemma 2.2 (Bramble-Hilbert) *Let B be a ball in Ω such that Ω is star-shaped with respect to B and such that its radius $\rho > (1/2)\rho_{\max}$. Then, for $u \in W_p^{m+1}(\Omega)$ with $p \geq 1$, there exists $Q^m u$ of polynomial of degree m such that*

$$|u - Q^m u|_{W_p^k(\Omega)} \leq C d_\Omega^{m+1-k} |u|_{W_p^{m+1}(\Omega)}, \quad 0 \leq k \leq m+1.$$

The polynomial $Q^m u$ is obtained from the Taylor polynomial of degree m of u averaged over B, that is,

$$Q^m u(x) = \int_B T_y^m u(x) \phi(y)\, dy,$$

where

$$T_y^m u(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha u(y)(x-y)^\alpha,$$

and $\phi(x) \in C_0^\infty(\mathbb{R}^n)$ is a *cut-off function* with $\text{supp}(\phi) = \overline{B}$ and $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$.

Now, we introduce finite elements in Ω. Let Ω^h be a triangulation of Ω with maximum diameter h. We assume that Ω^h is regular, that is, there exists a constant σ independent of h such that

$$h_\kappa \leq \sigma \rho_\kappa \quad \forall \kappa \in \Omega^h,$$

where h_κ is the diameter of κ and ρ_κ is the diameter of the circle inscribed in κ. For each triangle κ, we consider Σ_κ as the principal lattice of order m in κ:

$$\Sigma_\kappa = \left\{ x = \sum_{j=1}^{n+1} \lambda_j a_j \; : \; \sum_{j=1}^{n+1} \lambda_j = 1 \text{ with } \lambda_j \in \{0, 1/m, \cdots, (m-1)/m, 1\} \right\},$$

where $a_j \in \mathbb{R}^n$ denotes a vector corresponding to the j-th vertex of κ. Then, we define

$$X^h(\Omega) := \left\{ v \in C^0(\Omega) \,:\, v|_\kappa \in P_m(\kappa) \quad \forall \kappa \in \Omega^h \right\},$$

where $P_m(\kappa)$ is the set of polynomials of degree up to m associated with Σ_κ.

Then, for $v \in C^0(\Omega)$, we define the nodal value interpolation $I^h v(x) \in X^h(\Omega)$ by

$$I^h v(x_l) = v(x_l) \quad \forall x_l \in \Sigma_\kappa,\ \forall \kappa \in \Omega^h.$$

From the Sobolev imbedding theorem [3], we have

$$W_p^k(\Omega) \hookrightarrow C^j(\Omega) \text{ with } 0 \le j < k - n/p.$$

Applying the above inclusion with $k = 1, 2$ and $j = 0$, we obtain

$$W_p^2(\Omega) \hookrightarrow C^0(\Omega) \text{ with } p > n/2,$$
$$W_p^1(\Omega) \hookrightarrow C^0(\Omega) \text{ with } p > n.$$

Hence, we have the following approximation properties for the nodal value interpolation $I^h v(x)$:

Lemma 2.3 *For all $v \in W_p^{s+1}(\Omega)$ with $p > n/2$ and $1 \le s \le m$, we have*

$$|v - I^h v|_{W_p^k(\Omega)} \le C h^{s+1-k} |v|_{W_p^{s+1}(\Omega)}, \quad 0 \le k \le s+1.$$

Moreover, for all $v \in W_p^1(\Omega)$ with $p > n$, we have

$$|v - I^h v|_{W_p^k(\Omega)} \le C h^{1-k} |v|_{W_p^1(\Omega)}, \quad k = 0, 1.$$

For a non-smooth function $v(x)$, interpolation operators with the same approximation order as the nodal value interpolation were developed by Clément [19] and Scott and Zhang [46]. Both of them use the average values of $v(x)$ near a nodal point to obtain the interpolation. The interpolation by Clément does not regenerate the functions in X^h, where as the interpolation by Scott and Zhang does regenerate the functions in X^h. Both fit the zero boundary condition of $v(x)$, where as the interpolation $Q^m v(x)$ by Bramble and Hilbert dose not fit the zero boundary condition. The interpolation by Scott and Zhang can also fit more general boundary conditions.

Further, the idea of Scott and Zhang is generalized to construct Lagrange multipliers with dual basis (see [53]).

We do not give the exact forms of those interpolations and only state the approximation properties of those interpolations. Let $Qv(x)$ and $\widetilde{I}v(x)$ be the Clément, and Scott and Zhang interpolations, respectively.

Lemma 2.4 *For $v \in W_p^{s+1}(\Omega)$ with $0 \leq s \leq m$, there exists $Qv(x) \in X^h(\Omega)$ such that*

$$\|v - Qv\|_{W_p^k(\Omega)} \leq Ch^{s+1-k}\|u\|_{W_p^{s+1}(\Omega)}, \quad 0 \leq k \leq s+1.$$

Lemma 2.5 *For $v \in W_p^{s+1}(\Omega)$ with $0 \leq s \leq m$, there exists $\widetilde{I}v(x) \in X^h(\Omega)$ such that*

$$|v - \widetilde{I}v|_{W_p^k(\Omega)} \leq Ch^{s+1-k}|v|_{W_p^{s+1}(\Omega)}, \quad 0 \leq k \leq s+1.$$

Now, we discuss relations among various norms on a finite element space X^h. For a regular triangulation Ω^h, we add an additional assumption that there exists a constant γ independent of h such that

$$\gamma h \leq h_\kappa \quad \forall \kappa \in \Omega^h.$$

We call a regular triangulation Ω^h with the above property as a quasi-uniform triangulation. We have the following inverse inequalities for a finite element space X^h associated with a quasi-uniform triangulation Ω^h.

Lemma 2.6 *For $v \in X^h$, let $v|_\kappa \in W_p^l(\kappa) \bigcap W_q^k(\kappa)$ with $1 \leq p, q \leq \infty$ and $0 \leq k \leq l$. Then there exists a constant C independent of h and κ such that*

$$\|v\|_{W_p^l(\kappa)} \leq Ch^{k-l+n/p-n/q}\|v\|_{W_q^k(\kappa)}.$$

Using the above result when $m = 1$, that is, X^h is a piecewise linear finite element space, and $p = q = 2$, $l = 1$, we obtain

$$\|v\|_{1,\Omega} \leq Ch^{-1}\|v\|_{0,\Omega}. \tag{2.2}$$

Moreover, we have

$$\begin{aligned} \|v\|_{1/2,\partial\Omega} &\le Ch^{-1/2}\|v\|_{0,\partial\Omega}, \\ \|v\|_{1,\partial\Omega} &\le Ch^{-1/2}\|v\|_{1/2,\partial\Omega}. \end{aligned} \tag{2.3}$$

The above results are shown by Bramble *et al.* [17] and Xu [55].

2.3 Vertex-edge-face lemmas for finite element functions

In this section, we introduce several inequalities related to the interpolation of functions on a part of $\partial\Omega$, that is, a face, an edge, or a vertex. Those inequalities are essentially used to analyze the condition number bound of substructuring methods, Neumann-Neumann methods or FETI(-DP) methods.

Let Ω^h be a quasi-uniform triangulation of Ω with maximum diameter h and $X^h(\Omega)$ be a piecewise linear finite element space associated with Ω^h. Then, we consider subsets of $\partial\Omega$, faces, edges and vertices. The faces and edges are open subset of $\partial\Omega$, that is, those sets do not include their boundaries. For $\Omega \subset \mathbb{R}^2$, edges are considered as faces. We use F, E and V to denote a face, an edge, and a vertex of $\partial\Omega$, respectively.

Recall the nodal value interpolation $I^h v(x)$ in Section 2.2. Let N^h be the set of nodes in Ω^h and $X^h(\partial\Omega)$ be the space of functions in $X^h(\Omega)$ restricted on $\partial\Omega$. For a set $A \in \partial\Omega$, we define a nodal value interpolation $I_A^h w(x) \in X^h(\partial\Omega)$ as follows:

$$I_A^h w(x) = \begin{cases} w(x) & \text{if } x \in N^h \bigcap A, \\ 0 & \text{otherwise.} \end{cases}$$

We have the following famous lemmas for the above interpolation [56]. In the following, H denotes the diameter of Ω and the constant C is a generic constant independent of H and h.

Lemma 2.7 (Vertex lemma) *Let V be a vertex of $\partial\Omega$. Then for any $w \in X^h(\partial\Omega)$, we have*

$$\|I_V^h w\|_{1/2,\partial\Omega} \le Ch^{(n-1)/2}|w(V)| \le C\left(1+\log\frac{H}{h}\right)^{1/2}\|w\|_{1/2,\partial\Omega}.$$

Lemma 2.8 (Edge lemma) *Assume that $n = 3$ and E is an edge of $\partial\Omega$. Then for any $w \in X^h(\partial\Omega)$,*

$$\|I_E^h w\|_{1/2,\partial\Omega} \leq C\|w\|_{0,E} \leq C\left(1 + \log\frac{H}{h}\right)^{1/2} \|w\|_{1/2,\partial\Omega}.$$

Lemma 2.9 (Face lemma) *Let F be a face $(n = 3)$ or an edge $(n = 2)$ of $\partial\Omega$. Then for any $w \in X^h(\partial\Omega)$,*

$$\|I_F^h w\|_{1/2,\partial\Omega} \leq C\left(1 + \log\frac{H}{h}\right) \|w\|_{1/2,\partial\Omega}.$$

Lemma 2.10 *Let F be a face $(n = 3)$ or an edge $(n = 2)$ of $\partial\Omega$. Then we have*

$$\|I_F^h 1\|_{1/2,\partial\Omega} \leq CH^{(n-2)/2}\left(1 + \log\frac{H}{h}\right)^{1/2}.$$

Lemma 2.11 *Let $n = 2$. For any edge $E \subset \partial\Omega$ and any $w \in X^h(\partial\Omega)$,*

$$\|w - I_{\partial\Omega} w\|_{H_{00}^{1/2}(E)} \leq C\left(1 + \log\frac{H}{h}\right) |w|_{1/2,\partial\Omega},$$

where $I_{\partial\Omega} w = w$ on the corners of $\partial\Omega$ and is linear on each edge of $\partial\Omega$.

3. Overview of FETI methods

FETI methods are iterative substructuring methods with Lagrange multipliers, which are known as the most efficient parallel methods for large scale problems. There are three variants of FETI methods, that is, FETI, two-level FETI and FETI-DP(dual-primal FETI) method. FETI method has been developed into two-level FETI method and FETI-DP method to solve more general problems efficiently and easily.

3.1 A model problem

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$. We consider the following elliptic problem:
For $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} -\Delta u = f \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{on} \quad \partial\Omega. \end{aligned} \tag{3.1}$$

Let Ω^h be a regular triangulation of Ω. With the triangulation Ω^h, we consider the following P_1-conforming finite elements

$$X := \left\{ v \in H_0^1(\Omega) \cap C^0(\Omega) \ : \ v|_\tau \in P_1(\tau) \quad \forall \tau \in \Omega^h \right\}. \tag{3.2}$$

Then, the Galerkin approximation of (3.1) becomes:
Find $u \in X$ such that

$$a(u, v) = f(v) \quad \forall v \in X, \tag{3.3}$$

where

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) := \int_\Omega f v \, dx.$$

We decompose Ω into nonoverlapping subdomains $\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega}_i$ and assume that the boundaries of each subdomain do not divide the triangles in Ω^h. Hence, we obtain

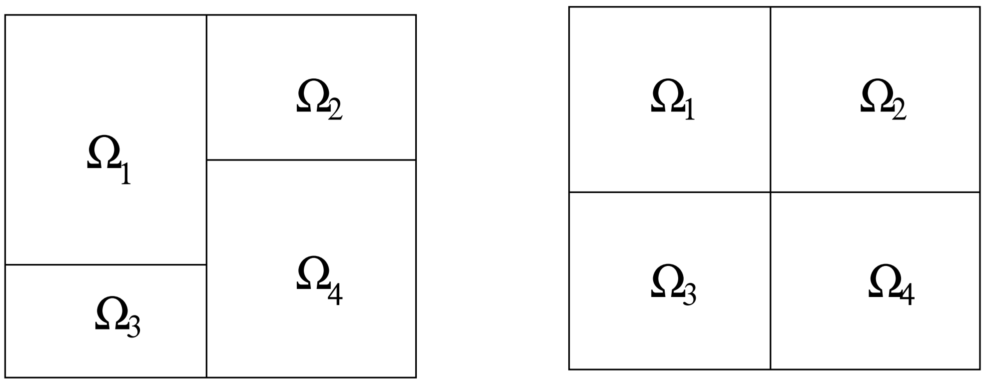

Figure 3.1: Geometrically nonconforming(left) and conforming(right) partitions

the triangulation Ω_i^h from Ω^h, that is, $\Omega_i^h = \Omega^h \cap \Omega_i$. Moreover, we assume that the partition is geometrically conforming, which means that each subdomain intersects with neighboring subdomains on the whole edge or at a vertex (see Figure 3.1). From the triangulation Ω_i^h, we define the following finite element space in each subdomain Ω_i:

$$X_i := \left\{ v \in H_D^1(\Omega_i) \cap C^0(\Omega_i) \ : \ v|_\tau \in P_1(\tau) \quad \forall \tau \in \Omega_i^h \right\},$$

where $H_D^1(\Omega_i)$ is the set of Sobolev H^1-functions in Ω_i with zero trace value on $\partial\Omega_i \cap \partial\Omega$, that is, $v|_{\partial\Omega_i \cap \partial\Omega} = 0$.

3.2 FETI method

Let

$$\widetilde{X} := \prod_{i=1}^{N} X_i. \tag{3.4}$$

Then, for $v \in \widetilde{X}$, the function values are not continuous across the subdomain interfaces. FETI method was developed from the idea that the solution u of (3.3) is obtained by solving a constraint minimization problem in $\widetilde{X}$.

Let $\partial\Omega_i^h$ denote a set of nodes in $\partial\Omega_i$ from the triangulation Ω_i^h. Note that $X \subset \widetilde{X}$ and $\widetilde{X} \not\subseteq H_0^1(\Omega)$. Hence, we consider the following matching conditions for

$v = (v_1, \cdots, v_N) \in \widetilde{X}$

$$v_i(x) = v_j(x) \quad \forall x \in \partial\Omega_i^h \cap \partial\Omega_j^h, \quad \forall i, j = 1, \cdots, N, \; i \neq j. \tag{3.5}$$

Since Ω_i^h is inherited from Ω^h, we have

$$X = \left\{ v \in \widetilde{X} \;:\; v \text{ satisfies (3.5)} \right\}. \tag{3.6}$$

The bilinear form $a(\cdot, \cdot)$ is s.p.d. on X. Hence, the problem (3.3) can be written into the following minimization problem in X

$$J(u) = \min_{v \in X} J(v), \tag{3.7}$$

where $J(v) = \frac{1}{2}a(v, v) - f(v)$. For $v \in X$, we can rewrite $a(v, v)$ and $f(v)$ into

$$a(v, v) = \sum_{i=1}^{N} a_i(v, v), \quad f(v) = \sum_{i=1}^{N} f_i(v), \tag{3.8}$$

with

$$a_i(u, v) = \int_{\Omega_i} \nabla u \cdot \nabla v \, dx, \quad f_i(v) = \int_{\Omega_i} f v \, dx.$$

Then, from (3.6), we write (3.7) into the following constraint minimization problem in $\widetilde{X}$:

$$J(u) = \min_{\substack{v \in \widetilde{X}, \\ v \text{ satisfies (3.5)}}} J(v). \tag{3.9}$$

In the FETI method, we solve the problem (3.9) with a saddle point formulation.

Now, we define notations of matrices. Let K_i be the stiffness matrix from the bilinear form $a_i(\cdot, \cdot)$ and f_i be the load vector from $f_i(\cdot)$. Then we rewrite the functional $J(v)$ into the following matrix vector product form:

$$J(v) = \frac{1}{2} \sum_{i=1}^{N} v_i^t K_i v_i - \sum_{i=1}^{N} v_i^t f_i. \tag{3.10}$$

We also rewrite the matching condition (3.5) into the following matrix vector product form:

$$\widetilde{B} u = 0, \tag{3.11}$$

where $\widetilde{B} = \begin{pmatrix} \widetilde{B}_1 & \cdots & \widetilde{B}_N \end{pmatrix}$ and each $\widetilde{B}_i$ is a matrix which has -1,0 and 1 as entries, with the number of columns equal to the number of nodes in the triangulation Ω_i^h and the number of rows equal to the number of constraints in the matching condition (3.5). Let M be the Lagrange multiplier space defined by

$$M = \text{Range}(\widetilde{B}).$$

Then, the constraint minimization problem (3.9) can be written as the following saddle point problem:

Find $(u, \lambda) \in \widetilde{X} \times M$ such that

$$L(u, \lambda) = \max_{\mu \in M} \min_{v \in \widetilde{X}} L(v, \mu), \tag{3.12}$$

where

$$L(v, \mu) = \sum_{i=1}^{N} \left(\frac{1}{2} v_i^t K_i v_i - v_i^t f_i \right) + (\widetilde{B} v)^t \mu.$$

Taking Euler-Lagrangian in the above saddle point problem, we get

$$K_i u_i + \widetilde{B}_i^t \lambda = f_i \quad \forall i = 1, \cdots, N, \tag{3.13}$$

$$\sum_{i=1}^{N} \widetilde{B}_i u_i = 0. \tag{3.14}$$

For the floating subdomain Ω_i, that is, $\partial\Omega_i \cap \partial\Omega = \emptyset$, (3.13) becomes a full Neumann boundary value problem and the matrix K_i has a null space. Hence, to solve (3.13) for u_i, we need the following admissible condition for λ:

$$f_i - \widetilde{B}_i^t \lambda \in \text{Range}(K_i) \quad \forall i = 1, \cdots, N.$$

From the fact that K_i's are symmetric, the above condition is equivalent to

$$f_i - \widetilde{B}_i^t \lambda \perp \text{Ker}(K_i) \quad \forall i = 1, \cdots, N. \tag{3.15}$$

Now, we define the admissible set

$$\mathcal{A} := \left\{ \mu \in M \ : \ f_i - \widetilde{B}_i^t \mu \perp \text{Ker}(K_i) \quad \forall i = 1, \cdots, N \right\}$$

and

$$\mathcal{V} := \left\{ \mu \in M \ : \ \widetilde{B}_i^t \mu \perp \mathrm{Ker}(K_i) \quad \forall i = 1, \cdots, N \right\}.$$

Then, taking some $\lambda_0 \in \mathcal{A}$, we have

$$\mathcal{A} = \{\lambda_0 + \nu \ : \ \nu \in \mathcal{V}\}. \tag{3.16}$$

The solution λ of the saddle point problem (3.12) should be in the admissible set $\mathcal{A}$, so that we consider the saddle point problem (3.12) in $\widetilde{X} \times \mathcal{A}$. Let P be a projection operator from M onto $\mathcal{V}$. Then, from the relation (3.16) and taking Euler-Lagrangian in the saddle point problem (3.12) on the set $\widetilde{X} \times \mathcal{A}$, the solution $(u, \lambda) \in \widetilde{X} \times \mathcal{A}$ of the problem (3.12) satisfies

$$K_i u_i + \widetilde{B}_i^t \lambda = f_i \quad \forall i = 1, \cdots, N, \tag{3.17}$$

$$P^t \sum_{i=1}^{N} \widetilde{B}_i u_i = 0. \tag{3.18}$$

Define K_i^+ as a pseudo inverse of K_i and R_i as the matrix whose columns are basis of $\mathrm{Ker}(K_i)$. Then the solution u_i of (3.17) has the following form:

$$u_i = K_i^+ (f_i - \widetilde{B}_i^t \lambda) + R_i \alpha_i, \tag{3.19}$$

where α_i is a vector, which will be chosen later and determine solution u_i uniquely. Substituting u_i into (3.18) with $P^t(\sum_{i=1}^N \widetilde{B}_i R_i) = 0$ and letting $\lambda = \lambda_0 + \nu$ with $\nu \in \mathcal{A}$, we obtain the following equation for ν:

$$F\nu = d, \tag{3.20}$$

where

$$F = P^t \sum_{i=1}^{N} \widetilde{B}_i K_i^+ \widetilde{B}_i^t P, \quad d = P^t \sum_{i=1}^{N} \widetilde{B}_i (K_i^+ f_i - K_i^+ \widetilde{B}_i^t \lambda_0).$$

We call F the FETI operator. In the FETI method, after solving for ν in (3.20) and then substituting ν into (3.19), we obtain the solution u_i's.

Remark 3.1 *The projection operator P and $\lambda_0 \in \mathcal{A}$ can be chosen in the following way. Let $R = diag_{i=1,\cdots,N}(R_i)$ and $G = \widetilde{B}R$. Then it can be shown that $G^t G$ is invertible. Since $\lambda_0 \in \mathcal{A}$, we have*

$$R_i^t \widetilde{B}_i^t \lambda_0 = R_i^t f_i \quad \forall i = 1, \cdots, N.$$

From the above relation, λ_0 satisfies

$$G^t \lambda_0 = R^t f, \tag{3.21}$$

with $f = \begin{pmatrix} f_1^t & \cdots & f_N^t \end{pmatrix}^t$*. Let* $\lambda_0 = G\beta$ *for some* β *and then substituting* λ_0 *in* (3.21)*, we obtain*

$$G^t G\beta = R^t f.$$

Hence, we can find $\lambda_0 \in \mathcal{A}$ *such that*

$$\lambda_0 = G(G^t G)^{-1} R^t f.$$

Moreover, we can compute the projection operator $P = I - G(G^t G)^{-1} G^t$*. Then* P *is the* l^2*-orthogonal projection from* M *onto* $\mathcal{V}$*.*

Remark 3.2 *After solving* ν *in* (3.20)*,* u_i*'s are computed from* (3.19)*. In* (3.19)*, each* α_i *is obtained from the condition that* $u = (u_1^t, \cdots, u_N^t)^t$ *satisfies* $\widetilde{B} u = 0$*. Hence, we get*

$$\alpha = -(G^t G)^{-1} G^t \widetilde{B} K^+ (f - \widetilde{B}^t \lambda),$$

where $\alpha = (\alpha_1^t, \cdots, \alpha_N)^t$*.*

Since $\widetilde{B}^t \nu \perp \mathrm{Ker}(K_i)$ for $\nu \in \mathcal{V}$, it can be shown that F is a s.p.d. operator on $\mathcal{V}$. Hence we use the conjugate gradient method(CGM) to solve (3.20). In CGM, the condition number of the operator F determines the reduction of relative errors at each iteration. More precisely,

$$\langle F(\nu - \nu_n), \nu - \nu_n \rangle^{\frac{1}{2}} \leq C \left(\frac{\sqrt{\kappa(F)} - 1}{\sqrt{\kappa(F)} + 1} \right)^n \langle F(\nu - \nu_0), \nu - \nu_0 \rangle^{\frac{1}{2}},$$

where $\langle \cdot, \cdot \rangle$ denotes the l^2-inner product, ν_0 is the initial iterate, ν_n is the n-th iterate of CGM and $\kappa(F)$ is the condition number of the operator F. The smaller $\kappa(F)$ is, the faster CGM converges. Therefore, we consider a preconditioner $\widehat{F}^{-1}$ for F to reduce the condition number of the system

$$\widehat{F}^{-1/2} F \widehat{F}^{-1/2} \zeta = \widehat{F}^{-1/2} d, \quad \nu = \widehat{F}^{-1/2} \zeta.$$

The following is the preconditioned conjugate gradient method (PCGM) to solve (3.20).

$k = 0$

ν_0 is given

$r_0 = d - F\nu_0.$

while $(r_0 \neq 0)$

Solve $\widehat{F} z_k = r_k$ $(z_k = \widehat{F}^{-1} r_k)$

$k = k + 1$

if $k == 1$

$p_1 = z_0$

else

$\beta_k = r_{k-1}^t z_{k-1} / r_{k-2}^t z_{k-2}$

$p_k = z_{k-1} + \beta_k p_{k-1}$

end

$\alpha_k = r_{k-1}^t z_{k-1} / p_k^t F p_k$

$\nu_k = \nu_{k-1} + \alpha_k p_k$

$r_k = r_{k-1} - \alpha_k F p_k$

Preconditioned Conjugate Gradient Method(PCGM)

The FETI method was first introduced by Farhat and Roux [29] for second order elasticity problems in $2D$. They observed that this method is numerically scalable without considering a coarse space that is essentially needed for other domain decomposition methods to achieve the scalability. It was realized that the projection operator P plays the role of coarse solver in the FETI method.

After then, Farhat, Mandel and Roux [28] showed that the following condition number bound for the FETI operator F for second order elasticity problems:

$$\kappa(F) \leq C \frac{H}{h},$$

where H and h denote the size of subdomains and meshes, respectively and C is a constant independent of H and h. From this bound, we can see that the condition number of the FETI operator does not grow when the number of subdomains increases maintaining the ratio of H and h, that is, the sizes of subdomain problems are fixed. Hence, we can solve the problem (3.1) more accurately adding more subdomains with fixed bound of condition number for the operator F. This property is called scalability. Furthermore, they introduced the Dirichlet preconditioner $\widehat{F}_D^{-1}$ such that

$$\widehat{F}_D^{-1} = PD^{-1}P^t \quad \text{with } D^{-1} = \sum_{i=1}^{N} \widetilde{B}_i \begin{pmatrix} 0 & 0 \\ 0 & S_{bb}^i \end{pmatrix} \widetilde{B}_i^t.$$

Here, S_{bb}^i is a Schur complement matrix which is obtained from K_i after eliminating interior unknowns.

Mandel and Tezaur [37] analyzed that the condition number of FETI operator with Dirichlet preconditioner is bounded by $C\,(1 + \log(H/h))^m$ with $m = 2$ or 3, for second order elliptic problems in $2D$. With a different preconditioner, Klawonn and Widlund [31] showed that the bound of the condition number is $C\,(1 + \log(H/h))^2$ and generalized the result for the elliptic problems with heterogeneous coefficients.

FETI method was extended to time dependent problems [24], advection-diffusion problems [51] and plate-bending problems [39]. For plate-bending problems, the condition number of the FETI operator with Dirichlet preconditioner grows faster than $O((1 + \log(H/h))^3)$. Since the plate-bending problems are of fourth order, tearing the approximate solution at the cross point causes the drawback compared with second order problems.

Farhat *et al.* [25, 27] introduced the two-level FETI method, a modification of FETI method, for the fourth order problems. Adding additional Lagrange multipliers, which makes the solution continuously at cross points(corners) in each CGM iteration, to the original FETI formulation, they obtained the numerical scalability. Tezaur [50] analyzed that the condition number bound of the two level FETI method with Dirichlet preconditioner is $C(1 + \log(H/h))^m$ with $m = 2$ or 3.

3.3 Dual-Primal FETI(FETI-DP) method

The dual-primal FETI(FETI-DP) method was first introduced by Farhat *et al.* [26] with the similar idea to the two-level FETI method. However, the implementation

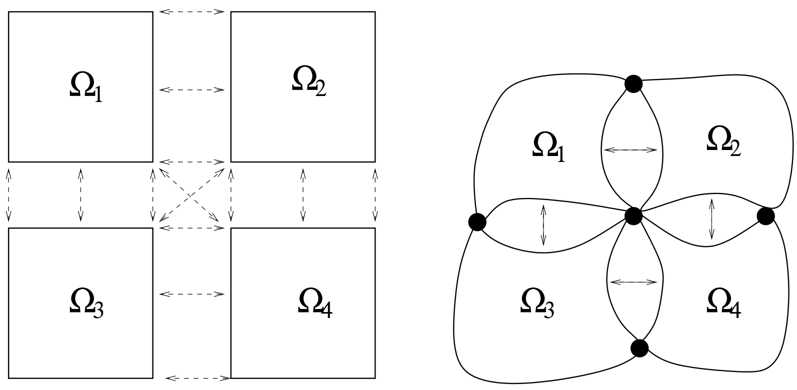

Figure 3.2: FETI vs. FETI-DP

is easier and the performance is better than existing FETI methods. The idea is using primal variables at subdomain corners to match solutions directly across the subdomain interfaces and using Lagrange multipliers to match solutions indirectly across the remaining parts of the subdomain interfaces. Hence, the continuity of the solutions at the subdomain corners holds for overall FETI-DP iterations.

In the FETI-DP method, we consider the following discrete space

$$\widetilde{X}_c = \Big\{ v \in \widetilde{X} \ : \ v \text{ is continuous at subdomain corners} \Big\},$$

where $\widetilde{X}$ is the space defined in (3.4). For $v = (v_1^t, \cdots, v_N^t)^t \in \widetilde{X}$, we may write

$$v_i = \begin{pmatrix} v_r^i \\ v_c^i \end{pmatrix} \qquad \text{for } i = 1, \cdots, N,$$

where v_r^i and v_c^i are vectors corresponding to the d.o.f. on the interior or edges, and at the corners of the subdomain Ω_i, respectively. For $v \in \widetilde{X}_c$, since v is continuous at subdomain corners and $\{\Omega_i\}_{i=1}^N$ is geometrically conforming, there exists a vector v_c such that $L_c^i v_c = v_c^i$ for $i = 1, \cdots, N$ where L_c^i is a matrix with entries 0 and 1, which restricts v_c on the corners of subdomain Ω_i. The vector v_c has the d.o.f. corresponding to the number of subdomain corners. To match v continuously on the remaining parts of the subdomain interfaces, we need the following conditions:

$$v_i(x) = v_j(x) \quad \forall x \in \partial\Omega_i^h \cap \partial\Omega_j^h \cap \Gamma_{ij}^0, \ \ \forall i, j = 1, \cdots, N, \tag{3.22}$$

where Γ_{ij}^0 is the interior part of Γ_{ij}. Then we write (3.22) as

$$\widetilde{B}_r v_r = 0, \tag{3.23}$$

where $\widetilde{B}_r = \begin{pmatrix} \widetilde{B}_r^1 & \cdots & \widetilde{B}_r^N \end{pmatrix}$ and $v_r = \begin{pmatrix} v_r^1 \\ \vdots \\ v_r^N \end{pmatrix}$. The matrix $\widetilde{B}_r^i$ has 0, 1 and -1 as components, with the number of columns equal to the number of nodes on $\partial\Omega_i^h$ excluding corners and the number of rows equal to the number of constraints in (3.22). Then we have

$$X = \left\{ v \in \widetilde{X}_c \ : \ \widetilde{B}_r v_r = 0 \right\}, \tag{3.24}$$

where X is the finite element function space defined in (3.2). From (3.24), the solution u of the problem (3.7) satisfies

$$J(u) = \min_{\substack{v \in \widetilde{X}_c \\ \widetilde{B}_r v_r = 0}} J(v). \tag{3.25}$$

We introduce

$$M = \text{Range}(\widetilde{B}_r).$$

Then, M is equal to a space of vectors which have a d.o.f equal to the number of constraints in (3.22).

In a saddle point formulation, (3.25) becomes: Find $(u, \lambda) \in \widetilde{X}_c \times M$ such that

$$L(u, \lambda) = \max_{\mu \in M} \min_{v \in \widetilde{X}_c} L(v, \mu), \tag{3.26}$$

where

$$L(v, \mu) = \sum_{i=1}^{N} \left\{ \frac{1}{2} a_i(v, v) - f_i(v) \right\} + < \widetilde{B}_r v_r, \mu > .$$

Let K_i be the stiffness matrix from $a_i(\cdot, \cdot)$ and f_i be the load vector from $f_i(\cdot)$. We may assume that K_i and f_i are ordered with

$$K_i = \begin{pmatrix} K_{rr}^i & K_{rc}^i \\ K_{cr}^i & K_{cc}^i \end{pmatrix}, \quad f_i = \begin{pmatrix} f_r^i \\ f_c^i \end{pmatrix},$$

where the subscripts r and c denote the d.o.f. on interior or edges, and at corners, respectively. Let

$$\begin{aligned}
K_{rr} &= \begin{pmatrix} K_{rr}^{i} & & \\ & \ddots & \\ & & K_{rr}^{N} \end{pmatrix}, \\
K_{rc} &= \begin{pmatrix} K_{rc}^{1} L_c^{1} \\ \vdots \\ K_{rc}^{N} L_c^{N} \end{pmatrix}, \\
K_{cr} &= K_{rc}^{t}, \\
K_{cc} &= \sum_{i=1}^{N} (L_c^{i})^{t} K_{cc}^{i} L_c^{i}, \\
u_r &= \begin{pmatrix} u_r^{1} \\ \vdots \\ u_r^{N} \end{pmatrix}, \quad f_r = \begin{pmatrix} f_r^{1} \\ \vdots \\ f_r^{N} \end{pmatrix}, \quad f_c = \sum_{i=1}^{N} (L_c^{i})^{t} f_c^{i}.
\end{aligned} \tag{3.27}$$

Using (3.27), $L(v,\mu)$ is written into

$$L(v,\mu) = \frac{1}{2} \begin{pmatrix} v_r \\ v_c \end{pmatrix}^{t} \begin{pmatrix} K_{rr} & K_{rc} \\ K_{cr} & K_{cc} \end{pmatrix} \begin{pmatrix} v_r \\ v_c \end{pmatrix} - \begin{pmatrix} v_r \\ v_c \end{pmatrix}^{t} \begin{pmatrix} f_r \\ f_c \end{pmatrix} + (\widetilde{B}_r v_r)^{t} \mu,$$

where v_c is a vector that satisfies

$$L_c^{i} v_c = v_c^{i} \quad \forall i = 1, \cdots, N$$

and $v_r = ((v_r^{1})^{t}, \cdots, (v_r^{N})^{t})^{t}$.

Taking Euler-Lagrangian in (3.26), (u, λ) satisfies

$$K_{rr} u_r + K_{rc} u_c + \widetilde{B}_r^{t} \lambda = f_r, \tag{3.28}$$

$$K_{cr} u_r + K_{cc} u_c = f_c, \tag{3.29}$$

$$\widetilde{B}_r u_r = 0. \tag{3.30}$$

Since K_{rr} is invertible, solving (3.28) for u_r we have

$$u_r = K_{rr}^{-1} (f_r - K_{rc} u_c - \widetilde{B}_r^{t} \lambda). \tag{3.31}$$

Substituting u_r into (3.30) and (3.29), we obtain

$$F_{rr}\lambda + F_{rc}u_c = d_r, \tag{3.32}$$

$$F_{rc}^t\lambda - F_{cc}u_c = -d_c, \tag{3.33}$$

where

$$\begin{aligned}
F_{rr} &= \widetilde{B}_r K_{rr}^{-1} \widetilde{B}_r^t,\\
F_{rc} &= \widetilde{B}_r K_{rr}^{-1} K_{rc},\\
F_{cr} &= F_{rc}^t,\\
F_{cc} &= K_{cc} - K_{cr}K_{rr}^{-1}K_{rc},\\
d_r &= \widetilde{B}_r K_{rr}^{-1} f_r,\\
d_c &= f_c - K_{cr}K_{rr}^{-1} f_r.
\end{aligned}$$

It can be shown that F_{cc} is invertible. Solving (3.33) for u_c, we obtain

$$u_c = F_{cc}^{-1}(F_{cr}\lambda + d_c). \tag{3.34}$$

Then, substituting u_c into (3.32), we obtain the following equation for λ:

$$(F_{rr} + F_{rc}F_{cc}^{-1}F_{cr})\lambda = d_r - F_{rc}F_{cc}^{-1}d_c. \tag{3.35}$$

We call

$$F_{DP} = F_{rr} + F_{rc}F_{cc}^{-1}F_{cr}$$

a FETI-DP operator. It is shown that F_{DP} is a s.p.d. operator. Hence, with a suitable preconditioner, (3.35) is solved for λ using the PCGM. After solving for λ, u_c and u_r are obtained from (3.34) and (3.31). As a preconditioner for the operator F_{DP}, we consider the following Dirichlet preconditioner:

$$\widehat{F}_{DP}^{-1} = \sum_{i=1}^{N} \widetilde{B}_r^i \begin{pmatrix} 0 & 0 \\ 0 & S_{rr}^i \end{pmatrix} (\widetilde{B}_r^i)^t,$$

where S_{rr}^i is a Schur complement operator obtained from K_{rr}^i after eliminating interior unknowns.

For second and fourth order elastostatic problems, FETI-DP method with Dirichlet preconditioner is more robust and computationally efficient than existing FETI methods, particularly when the number of subdomains is very large. In the FETI-DP method, gluing the solution at corners, we do not have floating subdomain problems as in the FETI methods. Hence we do not need a projection operator to eliminate the null space of the floating subdomain problems. It was observed that F_{cc}^{-1} plays a role of coarse solver in the FETI-DP method. That is, F_{cc}^{-1} globally transmits information among the subdomains at each FETI-DP iteration. The bound of condition number for the FETI-DP operator with Dirichlet preconditioner is

$$\kappa(\widehat{F}_{DP}^{-1} F_{DP}) \leq C \left(1 + \log \frac{H}{h}\right)^2 ,$$

which was analyzed by Mandel and Tezaur [38] for both second and fourth order elliptic problems.

For $3D$ problems, the FETI-DP method with the Dirichlet preconditioner needs modifications to get the optimal condition number bound as in $2D$ problems. Farhat *et al.* [26] extended the FETI-DP method to $3D$ problems by adding redundant constraints to the coarse problem and obtained the numerical scalability as in $2D$ problems. The Lagrange multipliers corresponding to the redundant constraints are treated as the primal variables in the FETI-DP formulation. Hence, the coarse problem is enlarged compared with the original FETI-DP method. They called the FETI-DP method with redundant constraints, which are added to the coarse problem, as the augmented FETI-DP method.

Klawonn, Widlund and Dryja [32] showed that with a different preconditioner the condition number of the FETI-DP method is bounded by $C(1 + \log(H/h))^2$ for heterogenous coefficient elliptic problems in $3D$. From the connection with the existing substructuring iterative method for $3D$ problems, they showed that when using only the d.o.f. at corners as primal variables, FETI-DP method is not effective. They also, as Farhat *et al.* did in [26], added the redundant constraints to the coarse problem and showed that FETI-DP method for $3D$ elliptic problems has the same condition number bound as $2D$ problems. Moreover, they proposed an algorithm choosing optimal primal variables with respect to the jumps of coefficients of elliptic problems. The condition number bound was also shown to be $C(1 + \log(H/h))^2$ for the case with the optimal primal variables.

Extensions of the FETI-DP method to the (Navier-)Stokes problem were done by Li[34, 35] both in $2D$ and $3D$ cases. In the FETI-DP formulation, to solve the Stokes problem more correctly and effectively at each FETI-DP iteration, the redundant constraints are added to the coarse problem. Moreover, it is shown that with a Dirichlet preconditioner, the bound of condition number of the FETI-DP operator is $C(1+\log(H/h))^2$. The Dirichlet preconditioner consists of local Stokes problems on each subdomain.

3.4 Augmented FETI-DP method

In this section, we briefly review the augmented FETI-DP method, which was developed for solving $3D$ problems more efficiently. Further, we will use the augmented FETI-DP formulation for solving the Stokes problem. In the FETI-DP formulation, the continuity of the solution across the subdomain interfaces is enforced by the Lagrange multipliers:

$$\widetilde{B}_r u_r = 0. \tag{3.36}$$

Hence, the continuity of the solution holds when the FETI-DP iteration has converged.

To accelerate the convergence of the FETI-DP method, we consider redundant constraints

$$Q^t \widetilde{B}_r u_r = 0, \tag{3.37}$$

where Q is some chosen matrix with a full column rank. Let $N = \text{Range}(Q^t \widetilde{B}_r)$ and $M = \text{Range}(\widetilde{B}_r)$. Define

$$M_Q = \left\{ \lambda \in M \ : \ Q^t \lambda = 0 \right\}.$$

Then, introducing the Lagrange multipliers $\mu \in U$ and $\lambda \in M$ for the constraints (3.37) and (3.36), respectively, and then taking the Euler-Lagrangian to the saddle point formulation

$$\max_{\mu \in U, \lambda \in M} \min_{v \in \widetilde{X}_c} \left\{ J(v) + < Q^t \widetilde{B}_r v_r, \mu > + < \widetilde{B}_r v_r, \lambda > \right\},$$

we obtain the followings:

Find $(u, \lambda, \mu) \in \widetilde{X}_c \times M \times U$ such that

$$\begin{aligned}
&K_{rr}u_r + K_{rc}u_c + \widetilde{B}_r^t\lambda + \widetilde{B}_r^t Q\mu = f_r,\\
&K_{cr}u_r + K_{cc}u_c = f_c,\\
&Q^t\widetilde{B}_r u_r = 0,\\
&\widetilde{B}_r u_r = 0.
\end{aligned} \tag{3.38}$$

Let

$$\begin{aligned}
&\widetilde{u}_c = \begin{pmatrix} u_c \\ \mu \end{pmatrix}, \quad \widetilde{f}_c = \begin{pmatrix} f_c \\ 0 \end{pmatrix},\\
&\widetilde{K}_{cc} = \begin{pmatrix} K_{cc} & 0 \\ 0 & 0 \end{pmatrix},\\
&\widetilde{K}_{rc} = \begin{pmatrix} K_{rc} & \widetilde{B}_r^t Q \end{pmatrix}, \quad \widetilde{K}_{cr} = \widetilde{K}_{rc}^t.
\end{aligned}$$

We consider $\widetilde{u}_c$ as a primal variable like in the original FETI-DP formulation and rewrite (3.38) into

$$\begin{aligned}
&K_{rr}u_r + \widetilde{K}_{rc}\widetilde{u}_c = f_r,\\
&\widetilde{K}_{cr}u_r + \widetilde{K}_{cc}\widetilde{u}_c = \widetilde{f}_c,\\
&\widetilde{B}_r u_r = 0.
\end{aligned}$$

Then, eliminating unknowns u_r and then $\widetilde{u}_c$, the FETI-DP operator follows:

$$(F_{rr} + \widetilde{F}_{rc}\widetilde{F}_{cc}^{-1}\widetilde{F}_{rc}^t)\lambda = d_r - \widetilde{F}_{rc}\widetilde{F}_{cc}^{-1}\widetilde{d}_c, \tag{3.39}$$

where

$$\begin{aligned}
&\widetilde{F}_{rc} = \widetilde{B}_r K_{rr}^{-1}\widetilde{K}_{rc},\\
&\widetilde{F}_{cc} = \widetilde{K}_{cc} - \widetilde{K}_{cr}\widetilde{K}_{rr}^{-1}\widetilde{K}_{rc},\\
&\widetilde{d}_c = \widetilde{f}_c - \widetilde{K}_{cr}K_{rr}^{-1}f_r,
\end{aligned}$$

and the other terms are the same as those of the original FETI-DP formulation. The invertibility of $\widetilde{F}_{cc}$ follows from the fact that Q has full column rank. Since $\widetilde{F}_{cc}$ contains F_{cc} as a diagonal block, we call this method as the augmented FETI-DP method. When $Q = 0$, the augmented FETI-DP formulation degenerates to that of the basic FETI-DP method. Let $F_{DP}^{A} = F_{rr} + \widetilde{F}_{rc}\widetilde{F}_{cc}^{-1}\widetilde{F}_{rc}^{t}$. It can be shown that F_{DP}^{A} is s.p.d. on M_Q. Therefore, the solution λ in (3.39) is uniquely determined in M_Q.

4. Mortar methods

Mortar methods were first introduced by Bernardi, Maday and Patera [11] for nonconforming discretizations of the elliptic problems in $2D$ coupling finite element and spectral methods. The methods were extended to coupling the finite elements with nonmatching triangulations across subdomain interfaces and the spectral methods with different orders between subdomains. In this dissertation, we consider the mortar method for the finite elements with nonmatching triangulations across subdomain interfaces and call it the mortar finite element method. Nonmatching discretizations are important for multiphysics simulations, contact-impact problems, the generation of meshes and partitions aligned with jumps in diffusion coefficients, hp-adaptive methods, and special discretizations in the neighborhood of singularities (corners or joints).

In the mortar methods, the orthogonality relations between jumps in the traces across subdomain interfaces and Lagrange multipliers are imposed to obtain the optimality of approximation like as conforming discretizations. Hence, the choice of Lagrange multiplier space is crucial in the mortar methods. Until now, several Lagrange multiplier spaces have been developed for the mortar finite element methods. Among them, the standard Lagrange multiplier space was naturally induced from the finite elements on nonmortar sides. However, the basis of the mortar finite elements obtained from the standard Lagrange multiplier space are not locally supported like as the finite element basis. In a mixed formulation of the mortar methods, the Lagrange multipliers approximate the normal derivative of the solution. From this observation, the normal derivative of the solution can not be well approximated by using the standard Lagrange multiplier space which consists of continuous functions. Hence, a Lagrange multiplier space with dual basis was introduced by Wohlmuth [53]. The locality of basis for mortar finite elements holds for this type of Lagrange multiplier space. Hence, the implementation and the analysis of the mortar methods are easier than those of the standard Lagrange multiplier space.

4.1 Nonconforming approximation

In this section, we give two formulations of mortar method. They are the nonconforming formulation and the saddle-point formulation. We consider a simple elliptic problem (3.1). We assume that Ω is bounded polygonal(polyhedral) domain in $\mathbb{R}^n (n = 2, 3)$ and decomposed into nonoverlapping polygonal(polyhedral) subdomains $\{\Omega_i\}_{i=1}^N$, which are geometrically conforming. Each subdomain Ω_i is associated with a quasi-uniform triangulation $\Omega_i^{h_i}$ with maximum diameter h_i. On the subdomain interfaces, these triangulations may not be aligned. Let

$$X_h := \prod_{i=1}^N X_i$$

with X_i defined in Section 3.1. Since the meshes are nonmatching across the subdomain boundaries, X_h is not contained in $H_0^1(\Omega)$. Hence, we need an appropriate condition to find a good approximation u_h in X_h for the solution $u \in H_0^1(\Omega)$ of the problem (3.1). The mortar finite element method was developed for this purpose. Before going into the mortar element method, we give a brief review where the idea comes from.

In the following, we regard $\|\cdot\|_{1,\Omega_i}$ and $\|\cdot\|_{1/2,\Omega_i}$ as usual Sobolev norms without scaling factor. Let us define

$$\widetilde{H} := \prod_{i=1}^N H_D^1(\Omega_i)$$

equipped with the norm

$$\|v\|_{\widetilde{H}} := \left(\sum_{i=1}^N \|v_i\|_{1,\Omega_i}^2 \right)^{1/2}.$$

We introduce the following Sobolev space:

$$H_0(div, \Omega) = \left\{ \mathbf{q} \in [L^2(\Omega)]^2 \;:\; \nabla \cdot \mathbf{q} \in L^2(\Omega),\; \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0 \right\}$$

normed by

$$\|\mathbf{q}\|_{H(div,\Omega)} = \left(\|\mathbf{q}\|_{0,\Omega}^2 + \|\nabla \cdot \mathbf{q}\|_{0,\Omega}^2 \right)^{1/2}.$$

Define

$$M = \left\{ \psi \in (\psi_i)_{i=1}^N \in \prod_{i=1}^N H^{-1/2}(\partial\Omega_i) \ : \ \exists \mathbf{q} \in H_0(div, \Omega) \text{ such that } \psi_i = \mathbf{q} \cdot \mathbf{n}_i, \ \forall i \right\}$$

normed by

$$\|\psi\|_M = \inf_{\substack{\mathbf{q} \,\in\, H_0(div,\,\Omega) \\ \mathbf{q} \cdot \mathbf{n}_i \,=\, \psi, \ \forall i}} \|\mathbf{q}\|_{H(div,\Omega)}.$$

Note that $H^{-1/2}(\partial\Omega_i)$ is the dual space for $H^{1/2}(\partial\Omega_i)$ and $H^{1/2}(\partial\Omega_i)$ is a function space that is composed of traces of functions in $H^1(\Omega_i)$. We consider bilinear form $\widetilde{b}(\cdot,\cdot) : \widetilde{H} \times M \to \mathbb{R}$ such that

$$\widetilde{b}(v, \psi) := \sum_{i=1}^N \int_{\partial\Omega_i} v_i \psi_i \, ds.$$

Then we can characterize $H_0^1(\Omega)$ as (see [43])

$$H_0^1(\Omega) = \left\{ v \in \widetilde{H} \ : \ \widetilde{b}(v, \psi) = 0 \quad \forall \psi \in M \right\}. \tag{4.1}$$

Using the same idea as (4.1), we consider the following condition on X with suitable M_{ij}:

$$\int_{\Gamma_{ij}} (v_i - v_j)\lambda_{ij} \, ds = 0 \quad \forall \lambda_{ij} \in M_{ij}, \ \forall i, j = 1, \cdots, N, \tag{4.2}$$

where $(v_1, \cdots, v_N) \in X$. The space M_{ij}'s will be defined later. On each interface $\Gamma_{ij} (= \Omega_i \cap \Omega_j)$, we determine one as a nonmortar side and the other as a mortar side. Then, we define

$$\begin{aligned} m_i &:= \{ j \ : \ \Omega_i \text{ is the nonmortar side of } \Gamma_{ij} \}, \\ s_i &:= \{ j \ : \ \Omega_i \text{ is the mortar side of } \Gamma_{ij} \}. \end{aligned} \tag{4.3}$$

Let

$$M_h = \prod_{i=1}^N \prod_{j \in m_i} M_{ij}$$

and a bilinear form $b(\cdot,\cdot) : X_h \times M_h \to \mathbb{R}$

$$b(v, \mu) := \sum_{i=1}^N \sum_{j \in m_i} \int_{\Gamma_{ij}} (v_i - v_j)\mu \, ds. \tag{4.4}$$

Then we define

$$V_h := \{v \in X_h \ : \ b(v,\mu) = 0 \quad \forall \mu \in M_h\}\,. \tag{4.5}$$

Recall the definitions of $a(v,v)$ and $f(v)$ in (3.8). Then the nonconforming formulation of the problem (3.1) becomes:

Find $u_h \in V_h$ satisfying

$$a(u_h, v) = f(v) \quad \forall v \in V_h. \tag{4.6}$$

This formulation was first introduced by Bernardi *et al.* [11]. After then, considering the mortar matching condition as constraints, the following saddle-point formulation was introduced in [5]:

Find $(u_h, \lambda_h) \in (X_h, M_h)$ such that

$$\begin{aligned} a(u_h, v) + b(v, \lambda_h) &= f(v) \quad \forall v \in X_h, \\ b(\mu, u_h) &= 0 \quad \forall \mu \in M_h. \end{aligned} \tag{4.7}$$

If the space M_h is suitably chosen, both of these two formulations have unique solutions and those solutions are the same. The space M_h is associated with the triangulations inherited from the nonmortar sides of interfaces. The inf-sup condition of the space $X_h \times M_h$ is essential for the unique solvability of the formulation (4.7). The solution λ_h in (4.7) approximates λ, the normal derivative of the solution u on the subdomain interfaces. Further, if the inf-sup constant is not depending on the mesh size and the space M_h has an approximation property like as the standard finite elements, then the error $\lambda - \lambda_h$ in $(H_{00}^{1/2})'$-norm has the same order of approximation as the H^1-norm of $u - u_h$. The approximation order of $u - u_h$ is also determined by the choice of M_h.

4.2 Lagrange multiplier spaces

In this section, we state the abstract multipliers conditions which will give the suitable Lagrange multiplier space ([30], [54]). In the following, C is a generic constant which does not depend on the triangulations and Γ_{ij}.

Let us define

$$W_{ij}^0 := \{w \in H_0^1(\Gamma_{ij}) \ : \ w = v|_{\Gamma_{ij}} \text{ for } v \in X_i\}. \tag{4.8}$$

Then the abstract conditions for the Lagrange multiplier space M_{ij} are

(A.1) $1 \in M_{ij}$

(A.2) W_{ij}^0 and M_{ij} have the same dimension.

(A.3) There is a constant C such that

$$\|\phi\|_{0,\Gamma_{ij}} \leq C \sup_{\psi \in M_{ij}} \frac{(\phi, \psi)_{\Gamma_{ij}}}{\|\psi\|_{0,\Gamma_{ij}}} \quad \forall \phi \in W_{ij}^0.$$

(A.4) For $\mu \in H^{k-1/2}(\Gamma_{ij})$, there exists $\mu_h \in M_{ij}$ such that

$$\|\mu - \mu_h\|_{0,\Gamma_{ij}}^2 \leq C h_i^{2k-1} |\mu|_{k-1/2,\Gamma_{ij}}^2,$$

where k is the order of finite elements in X_i.

Now, we define a mortar projection operator, which is essential in the analysis of the mortar methods.

Definition 4.1 *The mortar projection* $\pi_{ij} : L^2(\Gamma_{ij}) \to W_{ij}^0$ *is defined by*

$$\int_{\Gamma_{ij}} (w - \pi_{ij} w) \mu \, ds = 0 \quad \forall \mu \in M_{ij}.$$

The condition (A.1) gives the coercivity of the bilinear form $a(\cdot, \cdot)$ in V_h, which is independent of number of subdomains and mesh size. From (A.2) and (A.3), we can see that the mortar projection is well-defined. Furthermore, from (A.2) and (A.3), the continuity of the mortar projection in $H_{00}^{1/2}$-norm can be shown. Then, we can see that the inf-sup constant of $X_h \times M_h$ is independent of mesh size from the continuity of the projection operator. Hence, both problems (4.6) and (4.7) have unique solutions. The approximation order of the space V_h is calculated by using the Lagrange interpolation and the continuity of the mortar projection and it is the same as the conforming finite elements. For the error $u - u_h$, we can obtain the optimal order of approximation using the approximation property of the space V_h and (A.4). For the error $\lambda - \lambda_h$, the order of approximation can be shown by the inf-sup condition and (A.4).

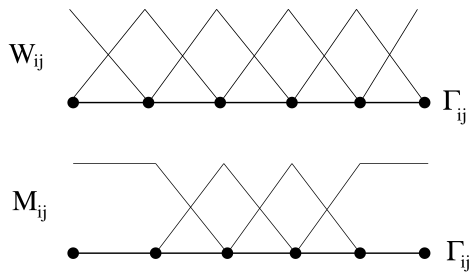

Figure 4.1: Basis functions for W_{ij} and M_{ij}(standard)

Now, we illustrate several Lagrange multiplier spaces that satisfy above conditions (A.1)-(A.4). The standard Lagrange multiplier space was first introduced in [11] for the elliptic problems in $2D$. After then, Belgacem and Maday [9] extended the result to $3D$ problems.

First, we consider a two-dimensional case. On Γ_{ij} with $j \in m_i$, we define

$$W_{ij} := \{w \ : \ w = v|_{\Gamma_{ij}} \text{ for } v \in X_i\}. \tag{4.9}$$

and let

$$\{\phi_0,\ \phi_1, \cdots, \phi_L,\ \phi_{L+1}\}$$

be the nodal basis functions for W_{ij}. Moreover, we assume that the basis functions are sequentially ordered according to the location of the nodes on Γ_{ij}. From the basis functions in W_{ij}, M_{ij} is defined as

$$M_{ij} := \text{span}\{\phi_0 + \phi_1,\ \phi_2, \cdots, \phi_{L-1}, \phi_L + \phi_{L+1}\}.$$

The basis for W_{ij} and M_{ij} are illustrated in Figure 4.1. The standard Lagrange multiplier space is similarly defined for the higher order finite elements or three-dimensional cases. For the case of higher order finite elements, let us assume that each subdomain Ω_i is associated with P_k-conforming finite elements. Let T_{ij} be the triangulation of Γ_{ij} inherited from the nonmortar side of Γ_{ij}. Then, M_{ij} is defined by

$$\begin{aligned} M_{ij} := \{\mu \ : \ \mu|_\tau \in P_l(\tau), \text{ if } \tau \cap \partial\Gamma_{ij} = \emptyset, l = k \\ \text{otherwise, } l = k-1, \forall \tau \in T_{ij}\}. \end{aligned}$$

For the three-dimensional cases, common faces are considered as the interfaces of subdomains. Let us assume that Ω_i is equipped with the P_1-conforming finite elements. The interface $\Gamma_{ij}(=\partial\Omega_i \cap \partial\Omega_j)$ consists of triangulations induced from the nonmortar side. We distinguish nodes on the interior of Γ_{ij} and the boundary of Γ_{ij}. Let I and B be the sets of nodes on the interior and the boundary of Γ_{ij}, respectively. For each $a \in B$, we assume that there exist $N_a(\geq 1)$ interior nodes which are vertices of triangles with a as a vertex and denote them by $\{a_q\}_{q=1}^{N_a}$. For each $a \in B$, we choose positive real numbers $\{c_q^a\}_{q=1}^{N_a}$ such that $\sum_{q=1}^{N_a} c_q^a = 1$. Then the Lagrange multiplier space is defined as

$$M_{ij} := \left\{ \mu \in W_{ij} \;:\; \mu = \sum_{a\in I} \mu(a)\phi_a + \sum_{a\in B} (\sum_{q=1}^{N_a} c_q^a \mu(a_q))\phi_a \right\},$$

where ϕ_a is the nodal basis function at the node a.

The standard Lagrange multiplier space M_{ij} is contained in the finite element space on the nonmortar side of Γ_{ij}. Hence, it consist of continuous functions. From the observation that λ_h approximates the normal derivative of the solution on the interfaces, the standard space M_{ij} is not correct one to approximate the normal derivative because the normal flux of u may not be continuous on the interfaces even though $u \in H^2(\Omega)$. To overcome the discrepancy, the Lagrange multiplier space with dual basis was developed by Wohlmuth [53]. The concept of dual basis was first introduced in [46]. In [53], it was also shown that the Lagrange multiplier space with the dual basis gives the same approximation property as the standard one. Further, Kim *et al.* [30] generalized the result to three-dimensional problems.

First, we consider $2D$ case. The interface Γ_{ij} is equipped with the triangulation T_{ij} from the nonmortar side. Let $\{\phi_l\}_{l=1}^n$ be the nodal basis for W_{ij}^0. These basis functions are sequentially ordered according to the location of nodes. Then, the dual basis $\{\psi_l\}_{l=1}^n$ is defined by

$$\int_{\Gamma_{ij}} \phi_l \psi_k \, ds = \delta_{lk} \int_{\Gamma_{ij}} \phi_l \, ds \quad \forall l, k = 1, \cdots, n,$$

and $1 \in \operatorname{span}\{\psi_1, \cdots, \psi_n\}$.

We follow [53] to give an example of dual basis. For $\tau \in T_{ij}$, let ϕ_l and ϕ_{l+1} be the nodal basis functions at the end points of τ. On τ whose end points do not

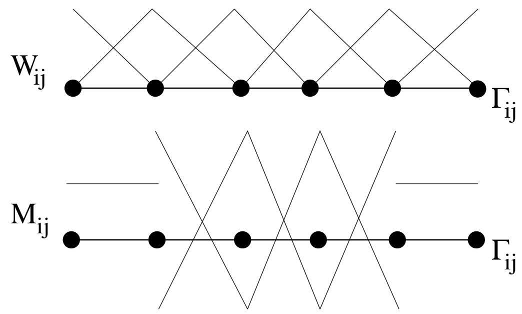

Figure 4.2: Basis functions for W_{ij} and M_{ij}(dual)

intersect with $\partial\Gamma_{ij}$, we find $\psi_{l,\tau} = a_1\phi_l|_\tau + a_2\phi_{l+1}|_\tau$ and $\psi_{l+1,\tau} = b_1\phi_l|_\tau + b_2\phi_{l+1}|_\tau$ such that

$$\int_\tau \phi_m \psi_{s,\tau}\, ds = \delta_{ms} \int_\tau \phi_m\, ds \text{ for } m, s = l, l+1. \tag{4.10}$$

Then, we obtain $(a_1, a_2) = (2, -1)$ and $(b_1, b_2) = (-1, 2)$. On τ whose one end point intersects with $\partial\Gamma_{ij}$, we let, say l, be the index of the point which does not intersect with $\partial\Gamma_{ij}$. Then, $\psi_{l,\tau}$ is given by

$$\psi_{l,\tau} = 1,$$

which satisfy the condition (4.10). For $\phi_l \in W_{ij}^0$, whose support consists of triangles τ_{l-1} and τ_l in T_{ij} with $m = 1$ or 2, ψ_l is defined as

$$\psi_l|_{\tau_i} = \psi_{l,\tau_i} \text{ for } i = l-1, l,$$

and zero on the remaining part of Γ_{ij}. From the construction, it can be seen easily that $\{\psi_i\}_{i=1}^n$ is a dual basis of $\{\phi_i\}_{i=1}^n$. The dual basis of M_{ij} and nodal basis of W_{ij} are illustrated in Figure 4.2.

The dual basis can be extended to $3D$ problems similarly. In $3D$ case, the interface $\Gamma_{ij}(= \partial\Omega_i \cap \partial\Omega_j)$ consists of two-dimensional triangulations. For $\tau \in T_{ij}$, we label the vertices of τ by $\{1, 2, 3\}$. There are four possible cases: First case is that all of three nodes are on the interior of Γ_{ij}. Second, one of them is on the boundary of Γ_{ij}. Third, two of them are on the boundary of Γ_{ij}. Fourth, all vertices are on the boundary of Γ_{ij}.

For the first case, let $\{\phi_{\tau,l}\}_{l=1}^{3}$ be a nodal basis for three vertices. We want to find $\psi_{\tau,l} = \sum_{k=1}^{3} a_{lk}\phi_{\tau,k},\ l = 1, 2, 3$ such that

$$\int_{\tau} \phi_{\tau,k}\psi_{\tau,l}\, ds = \delta_{kl} \int_{\tau} \phi_{\tau,k}\, ds, \quad k, l = 1, 2, 3.$$

Then, we obtain

$$(a_{lk}) = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

For the second case, we may assume that the node with the label 3 is on the boundary of Γ_{ij}. Let $\psi_{\tau,l} = \sum_{k=1}^{3} a_{lk}\phi_{\tau,k},\ l = 1, 2$. Then we find a_{lk}'s

$$(a_{lk}) = \begin{pmatrix} 5/2 & -3/2 & 1/2 \\ -3/2 & 5/2 & 1/2 \end{pmatrix},$$

which satisfy

$$\begin{aligned} &\int_{\tau} \phi_{\tau,k}\psi_{\tau,l}\, ds = \delta_{kl} \int_{\tau} \phi_{\tau,k}\, ds, \quad k, l = 1, 2, \\ &\psi_{\tau,1} + \psi_{\tau,2} = 1. \end{aligned}$$

For the third case, we assume that the node with the label 1 is on the interior of Γ_{ij}. Then, we let $\psi_{\tau,1} = 1$.

For the fourth case, there is no interior node. Hence, we do not have an extra $\psi_{\tau,l}$. Instead, we find an interior node x_τ, which is a vertex of a triangle $\widetilde{\tau}$ that shares a common edge with τ. Then, for $\mu \in M_{ij}$, we let $\mu|_\tau = \mu|_{\widetilde{\tau}}$.

We consider nodal basis functions $\{\phi_i\}_{i=1}^{n}$ for W_{ij}^{0} and obtain $\{\psi_i\}_{i=1}^{n}$ using the local dual basis $\{\psi_{\tau,l}\}$ similarly as is $2D$ case. If there is a triangle τ of the fourth case, we modify the dual basis function ψ_i corresponding to the interior node x_τ by extending $\psi_i = 1$ on the triangle τ. Then, M_{ij} is given by

$$M := \text{span}\{\psi_1, \cdots, \psi_n\}.$$

From the construction of the local dual basis, we can also see that $\{\psi_l\}_{l=1}^{n}$ is a dual basis for $\{\phi_l\}_{l=1}^{n}$.

There have been various extensions of the mortar element methods. To problems other than elliptic problems, such as advection-diffusion problems, Stokes problem, Maxwell equations and plate problems, the mortar element methods are also applicable[1, 6, 7, 40]. Due to its generality and optimality, the mortar methods have been widely used for problems with realistic importances[2, 4, 8, 18].

4.3 A priori error estimates

In this section, we provide proofs for the approximation properties of the mortar methods. As mentioned before, using the Lagrange multipliers satisfying (A.1)-(A.4), we can obtain the same order of approximations as conforming finite elements.

The space V_h is not a subspace of $H_0^1(\Omega)$. Hence, we are in a nonconforming setting. The uniform ellipticity of the bilinear form $a(\cdot,\cdot)$ on $V_h \times V_h$ is well known for the standard mortar space V_h; see [11]. In [10], it was shown that $a(\cdot,\cdot)$ is uniformly elliptic on $Y \times Y$, where

$$Y := \{v \in \prod_{i=1}^{N} H_D^1(\Omega_i) \ : \ \int_{\Gamma_{ij}} (v_i - v_j)\, ds = 0, \ \forall i = 1, \cdots, N, \ j \in m_i\}.$$

Further, the ellipticity constant on the space Y was shown to be independent of the number of subdomains in [47]. The approximation property of V^h to $H_0^1(\Omega)$ was also shown in [11]. In addition to the uniform ellipticity and the approximation property of V^h, we need to consider the consistency error to obtain a stable and convergent finite element discretization. In the following, C is a generic constant independent of the number of subdomains and mesh size.

Now, let us show the following stabilities of the mortar projection:

Lemma 4.2 *We have*

$$\|\pi_{ij} w\|_{0,\Gamma_{ij}} \leq C \|w\|_{0,\Gamma_{ij}} \quad \forall w \in L^2(\Gamma_{ij}),$$

and

$$\|\pi_{ij} w\|_{1,\Gamma_{ij}} \leq C \|w\|_{1,\Gamma_{ij}} \quad \forall w \in H_0^1(\Gamma_{ij}).$$

Proof. First, we show the L^2-stability. From (A.3), the definition of π_{ij} and Hölder inequality, we obtain

$$\begin{aligned}\|\pi_{ij}w\|_{0,\Gamma_{ij}} &\le C \sup_{\mu\in M_{ij}} \frac{(\pi_{ij}w,\mu)_{\Gamma_{ij}}}{\|\mu\|_{0,\Gamma_{ij}}}\\ &\le C \sup_{v\in W_{ij}^0} \frac{(w,\mu)_{\Gamma_{ij}}}{\|\mu\|_{0,\Gamma_{ij}}}\\ &\le C\|w\|_{0,\Gamma_{ij}}.\end{aligned}$$

For $w \in H_0^1(\Gamma_{ij})$, there exists $Qw \in W_{ij}^0$ such that

$$\|w - Qw\|_{0,\Gamma_{ij}} \le Ch_i\|w\|_{1,\Gamma_{ij}}, \quad \|Qw\|_{1,\Gamma_{ij}} \le C\|w\|_{1,\Gamma_{ij}} \text{ (see Lemma 2.4).}$$

Using the fact that $\pi_{ij}(Qw) = Qw$, the inverse inequality in (2.2) and the L^2-stability of the mortar projection, we have

$$\begin{aligned}\|\pi_{ij}(w - Qw)\|_{1,\Gamma_{ij}} &\le Ch_i^{-1}\|w - Qw\|_{0,\Gamma_{ij}}\\ &\le C\|w\|_{1,\Gamma_{ij}}.\end{aligned}$$

Then, from the triangle inequality, the above inequality and the approximation property of Q, we obtain

$$\|\pi_{ij}w - w\|_{1,\Gamma_{ij}} \le C\|w\|_{1,\Gamma_{ij}} \quad \forall w \in H_0^1(\Gamma_{ij}).$$

This completes the proof. ■

Remark 4.3 *Using an interpolation between $L^2(\Gamma_{ij})$ and $H_0^1(\Gamma_{ij})$, we have*

$$\|\pi_{ij}w\|_{H_{00}^{1/2}(\Gamma_{ij})} \le C\|w\|_{H_{00}^{1/2}(\Gamma_{ij})} \quad \forall w \in H_{00}^{1/2}(\Gamma_{ij}). \tag{4.11}$$

This result also holds for $3D$ case.

For $v \in \prod_{i=1}^N H_1(\Omega_i)$, let us define a broken H^1-norm as

$$\|v\|_*^2 = \sum_{i=1}^N \|v\|_{1,\Omega_i}^2.$$

From the stability of mortar projection, we have the following approximation property of the space V_h.

Lemma 4.4 *Assume that* $v|_{\Omega_i} \in H^2(\Omega_i)$ *for* $i = 1, \cdots, N$. *Then we have*

$$\inf_{v_h \in V_h} \|v - v_h\|_*^2 \leq C \sum_{i=1}^{N} h_i^2 \|v\|_{2,\Omega_i}^2.$$

Proof. Let $I^h v \in X_h$ be the Lagrange interpolation of v. Take

$$\chi = I^h v + \sum_{i=1}^{N} \sum_{j \in m_i} E_{ij} \pi_{ij} [I^h v] \in V_h,$$

where $[I^h v] = (I^h v)_i - (I^h v)_j$ and E_{ij} is an extension operator from W_{ij}^0 to X_i, which is continuous

$$\|E_{ij} w\|_{1,\Omega_i} \leq C \|w\|_{H_{00}^{1/2}(\Gamma_{ij})}$$

and $E_{ij} w = 0$ on $\partial\Omega_i \backslash \Gamma_{ij}$. The discrete harmonic extension can be such an extension operator.

Then, we have

$$\|\sum_{i=1}^{N} \sum_{j \in m_i} E_{ij} \pi_{ij} [I^h v]\|_*^2 \leq C \sum_{i=1}^{N} \sum_{j \in m_i} \|\pi_{ij} [I^h v]\|_{H_{00}^{1/2}(\Gamma_{ij})}^2.$$

We observe that $I^h v \in H_{00}^{1/2}(\Gamma_{ij})$ for $2D$ case, but not for $3D$ case. So that we analyze each case differently.

For $2D$ case, using the stability of mortar projection in $H_{00}^{1/2}$-norm and coloring argument, we obtain

$$\|\sum_{i=1}^{N} \sum_{j \in m_i} E_{ij} \pi_{ij} [I^h v]\|_*^2 \leq C \sum_{i=1}^{N} h_i^2 \|v\|_{2,\Omega_i}^2.$$

For $3D$ case, using the inverse inequality, the stability of mortar projection in L^2-norm and coloring argument, we get

$$\|\sum_{i=1}^{N} \sum_{j \in m_i} E_{ij} \pi_{ij} [I^h v]\|_*^2 \leq C \max_{i=1,\cdots,N, j \in m_i} \{(1 + h_j/h_i)\} \sum_{i=1}^{N} h_i^2 \|v\|_{2,\Omega_i}^2.$$

Then, using the above inequalities, approximation property of $I^h v$ and triangle inequality, we obtain

$$\|v - \chi\|_*^2 \leq C \sum_{i=1}^{N} h_i^2 \|v\|_{2,\Omega_i}^2.$$

This completes the proof. ■

Remark 4.5 *For 3D case, the constant C in the approximation property depends on the ratio of meshes between mortar and nonmortar sides.*

From the second Lemma of Strang [13], we have the following well-known result:

Lemma 4.6

$$\|u-u_h\|_* \leq C\left(\inf_{v_h\in V_h}\|u-v_h\|_* + \sup_{v_h\in V_h}\frac{\sum_{i=1}^N\sum_{j\in m_i}\int_{\Gamma_{ij}}\frac{\partial u}{\partial n}[v_h]\,ds}{\|v_h\|_*}\right).$$

The first term is called an approximation error and the second term is called a consistency error.

For the consistency error, we have

Lemma 4.7

$$\sup_{v_h\in V_h}\frac{\sum_{i=1}^N\sum_{j\in m_i}\int_{\Gamma_{ij}}\frac{\partial u}{\partial n}[v_h]\,ds}{\|v_h\|_*} \leq C\left(\sum_{i=1}^N h_i^2\|u\|_{2,\Omega_i}^2)\right)^{1/2}.$$

Proof. Since $v_h \in V_h$, we have

$$\int_{\Gamma_{ij}}\frac{\partial u}{\partial n}[v_h]\,ds = \int_{\Gamma_{ij}}(\frac{\partial u}{\partial n}-\mu_h)[v_h]\,ds \quad \forall \mu_h\in M_{ij}.$$

From (A.4) with $k=1$ and the definition of dual norm $(H^{1/2}(\Gamma_{ij}))'$, we get

$$\|\frac{\partial u}{\partial n}-\mu_h\|_{(H^{1/2}(\Gamma_{ij}))'} \leq Ch_i|\frac{\partial u}{\partial n}|_{1/2,\Gamma_{ij}},$$

where μ_h is chosen as the L^2-projection of $\frac{\partial u}{\partial n}$ onto M_{ij}. It follows that

$$\int_{\Gamma_{ij}}\frac{\partial u}{\partial n}[v_h]\,ds \leq Ch_i|\frac{\partial u}{\partial n}|_{1/2,\Gamma_{ij}}(|v_h|_{1,\Omega_i}+|v_h|_{1,\Omega_j}).$$

Using the above inequality, a coloring argument and a trace theorem, we obtain

$$|\sum_{i=1}^N\sum_{j\in m_i}\int_{\Gamma_{ij}}\frac{\partial u}{\partial n}[v_h]\,ds| \leq C(\sum_{i=1}^N h_i^2\|u\|_{2,\Omega_i})^{1/2}\|v_h\|_*$$

and complete the proof. ■

From Lemma 4.6, Lemma 4.4 and Lemma 4.7, we obtain the following a priori estimate for $u-u_h$:

Theorem 4.8 *Assume that* $u|_{\Omega_i} \in H^2(\Omega_i)$ *for* $i = 1, \cdots, N$. *Then, we have*

$$\|u - u_h\|_* \le C \left(\sum_{i=1}^{N} h_i^2 \|u\|_{2,\Omega_i}^2 \right)^{1/2}.$$

Now, we derive a priori estimate of $\lambda - \lambda_h$ with a suitable norm, where $\lambda = \frac{\partial u}{\partial n}$ on the interface of subdomains and λ_h is a solution of the saddle-point formulation (4.7). Let us define a norm for $\mu_h \in M$ by

$$\|\mu_h\|_{(H_{00}^{1/2}(\Gamma))'}^2 = \sum_{i=1}^{N} \sum_{j \in m_i} \|\mu_h\|_{(H_{00}^{1/2}(\Gamma_{ij}))'}^2$$

The following inf-sup condition is essential in the a priori estimate of $\lambda - \lambda_h$. From the continuity of mortar projection in $H_{00}^{1/2}$ norm, we can easily obtain the following result.

Lemma 4.9 *There exists a constant* β *independent of mesh sizes and the number of subdomains such that*

$$\inf_{0 \ne \mu_h \in M} \sup_{0 \ne v_h \in X_h} \frac{b(v_h, \mu_h)}{\|\mu_h\|_{(H_{00}^{1/2}(\Gamma))'} a(v_h, v_h)^{1/2}} \ge \beta.$$

From the above result and Lemma 4.7, we have

Theorem 4.10

$$\|\lambda - \lambda_h\|_{(H_{00}^{1/2}(\Gamma))'}^2 \le C \sum_{i=1}^{N} h_i^2 \|u\|_{2,\Omega_i}^2,$$

where C *depends on the inf-sup constant* β.

Until now, we review the a priori error estimates of mortar methods for the elliptic problem (3.1). For an elliptic problem with heterogeneous coefficients, we also obtain the similar a priori error bounds by following [53]. For that case, we obtain

$$\|u - u_h\|_*^2 \le C \sum_{i=1}^{N} C_i h_i^2 \|u\|_{2,\Omega_i}^2,$$

$$\|\lambda - \lambda_h\|_{(H_{00}^{1/2}(\Gamma))'}^2 \le C \sum_{i=1}^{N} C_i h_i^2 \|u\|_{2,\Omega_i}^2,$$

where

$$C_i = \max\left(\sup_{k\in m_i} \min\left(1+\frac{a_i}{a_k}, 1+\left(\frac{h_i}{h_k}\right)^2\right), \sup_{\substack{1\le j\le N\\ i\in m_j}} \min\left(1+\frac{a_j}{a_i}, 1+\left(\frac{h_j}{h_i}\right)^2\right)\right).$$

Here, the constant a_i is the positive coefficient of the elliptic problem in Ω_i. On Γ_{ij}, if we choose nonmortar side with smaller a_i, then we always have $C_i \le 2$ for all i. For $3D$ case, the ratio of meshes between mortar and nonmortar sides occurs in the constant C of the a priori estimates. We can also see that the term dose not give significant effect when it is multiplied by C_i's even though we choose smaller mesh size on nonmortar side. The elliptic problems with discontinuous coefficients can be approximated by the elliptic problems with heterogeneous coefficients. For the problems with continuous coefficients, the assumption of comparable sizes of meshes is reasonable in practice. Hence, the ratio of meshes can be bounded by some number in this case.

5. Elliptic problems in 2D

5.1 A model problem and finite elements

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$. We consider a FETI-DP method on nonmatching grids for the following elliptic problem:

For $f \in L^2(\Omega)$, find $u \in H^1(\Omega)$ such that

$$\begin{aligned} -\nabla \cdot (A(x)\nabla u(x)) + \beta(x)u(x) &= f(x) \ \text{ in } \Omega, \\ u(x) &= 0 \ \text{ on } \Gamma_D, \\ \boldsymbol{n} \cdot (A(x)\nabla u(x)) &= 0 \ \text{ on } \Gamma_N. \end{aligned} \tag{5.1}$$

Here, $A(x) = (\alpha_{ij}(x)) \in \mathbb{R}^{2\times 2}$ and $\boldsymbol{n}$ is the outward unit vector normal to Γ_N. We assume that $\alpha_{ij}(x), \beta(x) \in L^\infty(\Omega)$, $A(x)$ is uniformly elliptic, $\beta(x) \geq 0$ for all $x \in \Omega$ and $|\Gamma_D| \neq 0$, where $|\Gamma_D|$ denotes the measure of Γ_D.

Let Ω be partitioned into nonoverlapping polygonal subdomains $\{\Omega_i\}_{i=1}^N$. We assume that the partition is geometrically conforming, which means that the subdomains intersect with neighboring subdomains on the whole edge or at a vertex(corner). Let $\Omega_i^{h_i}$ be a quasi-uniform triangulation of the subdomain Ω_i with the maximum diameter h_i. The meshes may not be aligned across the subdomain interfaces. For each subdomain Ω_i, we introduce a P_1-conforming finite element space

$$X_i := \{v \in H_D^1(\Omega_i) \ : \ v|_\tau \in P_1(\tau), \ \tau \in \Omega_i^h\},$$

where $H_D^1(\Omega_i) := \{v \in H^1(\Omega_i) \ : \ v = 0 \text{ on } \Gamma_D \cap \partial\Omega_i\}$ and $P_1(\tau)$ is a set of polynomials of degree ≤ 1 in τ. For $(u_i, v_i) \in X_i \times X_i$, define a bilinear form

$$a_i(u_i, v_i) := \int_{\Omega_i} A(x)\nabla u_i \cdot \nabla v_i \, dx + \int_{\Omega_i} \beta(x)u_i v_i dx.$$

To get the FETI-DP formulation, we need a finite element space in Ω as follows:

$$X := \left\{ v \in \prod_{i=1}^N X_i \ : \ v \text{ is continuous at subdomain vertices} \right\}.$$

By restricting the space X_i's on the boundaries of each subdomains, we define

$$W_i := X_i|_{\partial\Omega_i} \ \forall i = 1, \cdots, N.$$

Then we let

$$W := \left\{ w \in \prod_{i=1}^{N} W_i \ : \ w \text{ is continuous at subdomain corners} \right\}. \tag{5.2}$$

Let S^i be the Schur complement matrix of the bilinear form $a_i(\cdot, \cdot)$ over the finite elements X_i. That is,

$$S^i = A^i_{BB} - A^i_{BI}(A^i_{II})^{-1}A^i_{IB},$$

where A^i is a stiffness matrix associated with the bilinear form $a(\cdot, \cdot)$ and ordered with

$$A^i = \begin{pmatrix} A^i_{II} & A^i_{IB} \\ A^i_{BI} & A^i_{BB} \end{pmatrix}.$$

Here, the subscripts I and B represent the d.o.f. on interior and boundary of Ω_i, respectively. Then, a semi-norm is defined for $w_i \in W_i$

$$|w_i|^2_{S^i} :=< S^i w_i, w_i >,$$

where $< \cdot, \cdot >$ is the l^2-inner product of vectors. For $w \in W$, since w is continuous at subdomain vertices, by summing up these semi-norms, we define a norm

$$\|w\|^2_W := \sum_{i=1}^{N} |w_i|^2_{S^i}, \quad w_i = w|_{\partial\Omega_i}. \tag{5.3}$$

Moreover, we define a subspace of W

$$W_r := \{w \in W \ : \ w \text{ vanishes at subdomain vertices}\}. \tag{5.4}$$

We note that the space X is not contained in $H^1(\Omega)$. To approximate the solution of the problem (5.1) in X, we impose the mortar matching condition (4.2) on $v \in X$ with a suitable Lagrange multiplier space satisfying the assumptions (A.1)-(A.4) in Section 4.2. On each Γ_{ij}, we determine mortar and nonmortar sides and define the index sets m_i and s_i as (4.3). We define the spaces W_{ij}, W^0_{ij} and M_{ij} as in

Section 4.2. Especially, we consider the standard Lagrange multiplier space M_{ij}. However, our theory can be extended to a general Lagrange multiplier space which satisfies the assumptions (A.1)-(A.4). Then the global Lagrange multiplier space is defined by

$$M := \prod_{i=1}^{N} \prod_{j \in m_i} M_{ij}.$$

Similarly, we let

$$W^0 := \prod_{i=1}^{N} \prod_{j \in m_i} W_{ij}^0.$$

Now, we define norms for the spaces W^0 and M. For $w_{ij} \in W_{ij}^0$, $\widetilde{w}_{ij} \in W_i$ is the zero extension of w_{ij} into $\partial\Omega_i$. Let $\widetilde{w}_i = \sum_{j \in m_i} \widetilde{w}_{ij}$ and $\widetilde{w} = (\widetilde{w}_1, \cdots, \widetilde{w}_N)$. Since $\widetilde{w}$ is continuous at subdomain vertices, $\widetilde{w} \in W$. Hence, we define a norm for $w \in W^0$ as

$$\|w\|_{W^0} := \|\widetilde{w}\|_W. \tag{5.5}$$

Let $< \cdot, \cdot >_m$ be a duality pairing between M and W^0 such that

$$< \lambda, w >_m := \sum_{i=1}^{N} \sum_{j \in m_i} \int_{\Gamma_{ij}} \lambda_{ij} w_{ij} \, ds \text{ for } (\lambda, w) \in M \times W^0. \tag{5.6}$$

Using this, we define a dual norm on M by

$$\|\lambda\|_M := \max_{w \in W^0 \setminus \{0\}} \frac{< \lambda, w >_m}{\|w\|_{W^0}}. \tag{5.7}$$

Recall the following mortar matching condition for $(v_1, \cdots, v_N) \in X$:

$$\int_{\Gamma_{ij}} (v_i - v_j) \lambda_{ij} \, ds = 0 \quad \forall \lambda_{ij} \in M_{ij}, \ \forall i = 1, \cdots, N, \ j \in m_i. \tag{5.8}$$

Now, we rewrite the mortar matching condition (5.8) into a matrix form. Let $\{\phi_k^m\}_{k=0}^{K_m+1}$ be the basis function for $W_m|_{\Gamma_{ij}}$ with $m = i, j$ and $\{\psi_l\}_{l=1}^{L}$ be the basis function for M_{ij}. Then we define matrices B_i^{ij} and B_j^{ij} with entries

$$\left(B_m^{ij}\right)_{lk} = \pm \int_{\Gamma_{ij}} \psi_l \phi_k^m \, ds, \ l = 1, \cdots, L, \ k = 0, \cdots, K_m + 1 \text{ for } m = i, j, \tag{5.9}$$

where the $+$ sign is chosen if m is the nonmortar side of Γ_{ij}, otherwise, the $-$ sign is chosen. Then we rewrite (5.8) into

$$B_i^{ij} v_i|_{\Gamma_{ij}} + B_j^{ij} v_j|_{\Gamma_{ij}} = 0, \quad \forall i = 1, \cdots, N, \ j \in m_i.$$

Let $E_{ij} : M_{ij} \to M$ be an extension operator from M_{ij} to M by zero and $R_{ij}^l : W_l \to W_l|_{\Gamma_{ij}}$ for $l = i, j$ be a restriction operator. Using these operators, we define

$$B_i = \sum_{j \in m_i} E_{ij} B_i^{ij} R_{ij}^i + \sum_{j \in s_i} E_{ji} B_i^{ji} R_{ji}^i.$$

Then the mortar matching condition (5.8) becomes

$$Bw = 0,$$

where $B = \begin{pmatrix} B_1 & \cdots & B_N \end{pmatrix}$ and $w = \begin{pmatrix} w_1^t & \cdots & w_N^t \end{pmatrix}^t$ with $w_i = v_i|_{\partial\Omega_i}$.

5.2 FETI-DP formulation

5.2.1 FETI-DP operator

In this section, we construct the FETI-DP operator for the problem (5.1) with the mortar matching condition as constraints. The derivation of FETI-DP equation for the Lagrange multipliers follows [38]. However, the FETI-DP operator with mortar matching condition is new. Dryja and Widlund[21, 22] eliminate unknowns both on interior and vertex nodal points, and impose a mortar matching condition over W_r in (5.4). Hence, the resulting solution u does not satisfy the mortar matching condition (5.8). We only eliminate interior nodal points, and impose the mortar matching condition on the function over W in (5.2).

For $w_i \in W_i$ we write

$$w_i = \begin{pmatrix} w_r^i \\ w_c^i \end{pmatrix},$$

where r and c stand for the nodal values on the edges and vertices. From now on, we use the subscript symbol r and c to represent the degrees of freedom(d.o.f.) on edges and at vertices, respectively. Define W_c as the set of vectors which have d.o.f.

corresponding to the union of subdomain vertices, that is, global corner points. For $w = (w_1, \cdots, w_N) \in W$, since w is continuous at subdomain vertices, there exists $w_c \in W_c$ such that $L_c^i w_c = w_c^i$ for $i = 1, \cdots, N$, where the matrix L_c^i consists of 0 and 1 and restricts the value of w_c on the vertices of subdomain Ω_i. Hence, for $w = (w_1, \cdots, w_N) \in W$, we write

$$w_i = \begin{pmatrix} w_r^i \\ L_c^i w_c \end{pmatrix} \ \forall i, \quad \text{for some } w_c \in W_c.$$

Recall that S^i is the Schur complement matrix obtained from the bilinear form $a_i(\cdot, \cdot)$ and let g^i be the Schur complement forcing vector obtained from $\int_{\Omega_i} f v_i \, dx$. The matrix S^i and vector g^i are ordered into

$$S^i = \begin{pmatrix} S_{rr}^i & S_{rc}^i \\ S_{cr}^i & S_{cc}^i \end{pmatrix}, \quad g^i = \begin{pmatrix} g_r^i \\ g_c^i \end{pmatrix}.$$

Let $B_{i,r}$ and $B_{i,c}$ be matrices that consist of the columns of B_i corresponding to the nodal points on edges and at vertices, respectively.

Then, the saddle-point formulation of the problem (5.1) with the mortar constraints gives:

Find $(w_r, w_c, \lambda) \in W_r \times W_c \times M$ such that

$$S_{rr} w_r + S_{rc} w_c + B_r^t \lambda = g_r, \tag{5.10}$$

$$S_{cr} w_r + S_{cc} w_c + B_c^t \lambda = g_c, \tag{5.11}$$

$$B_r w_r + B_c w_c = 0, \tag{5.12}$$

where

$$S_{rr} = \text{diag}_{i=1,\cdots,N} \left(S_{rr}^i \right),$$

$$S_{rc} = \begin{pmatrix} S_{rc}^1 L_c^1 \\ \vdots \\ S_{rc}^N L_c^N \end{pmatrix},$$

$$S_{cr} = S_{rc}^t,$$

$$
\begin{aligned}
&S_{cc} = \sum_{i=1}^{N} (L_c^i)^t S_{cc}^i L_c^i, \\
&B_r = (B_{1,r}, \cdots, B_{N,r}), B_c = \sum_{i=1}^{N} B_{i,c} L_c^i, \\
&g_r = \begin{pmatrix} g_r^1 \\ \vdots \\ g_r^N \end{pmatrix}, \; g_c = \sum_{i=1}^{N} (L_c^i)^t \; g_c^i, \; w_r = \begin{pmatrix} w_r^1 \\ \vdots \\ w_r^N \end{pmatrix}.
\end{aligned}
$$

Since S_{rr} is invertible, we solve (5.10) for w_r to get

$$
w_r = S_{rr}^{-1} \left(g_r - S_{rc} w_c - B_r^t \lambda \right).
$$

After substituting w_r into (5.12) and (5.11), we obtain

$$
\begin{aligned}
B_r S_{rr}^{-1} B_r^t \lambda + \left(B_r S_{rr}^{-1} S_{rc} - B_c \right) w_c &= B_r S_{rr}^{-1} g_r, \\
\left(S_{cr} S_{rr}^{-1} B_r^t - B_c^t \right) \lambda - \left(S_{cc} - S_{cr} S_{rr}^{-1} S_{rc} \right) w_c &= -\left(g_c - S_{cr} S_{rr}^{-1} g_r \right).
\end{aligned}
$$

Let

$$
\begin{aligned}
F_{I_{rr}} &= B_r S_{rr}^{-1} B_r^t, \\
F_{I_{rc}} &= B_r S_{rr}^{-1} S_{rc} - B_c, \\
F_{I_{cr}} &= S_{cr} S_{rr}^{-1} B_r^t - B_c^t \left(= F_{I_{rc}}^t \right), \\
F_{I_{cc}} &= S_{cc} - S_{cr} S_{rr}^{-1} S_{rc}, \\
d_r &= B_r S_{rr}^{-1} g_r, \\
d_c &= g_c - S_{cr} S_{rr}^{-1} g_r.
\end{aligned}
\tag{5.13}
$$

Then (λ, w_c) satisfies

$$
\begin{pmatrix} F_{I_{rr}} & F_{I_{rc}} \\ F_{I_{cr}} & -F_{I_{cc}} \end{pmatrix} \begin{pmatrix} \lambda \\ w_c \end{pmatrix} = \begin{pmatrix} d_r \\ -d_c \end{pmatrix}.
$$

Eliminating w_c in the above equation, we obtain

$$
\left(F_{I_{rr}} + F_{I_{rc}} F_{I_{cc}}^{-1} F_{I_{cr}} \right) \lambda = d_r - F_{I_{rc}} F_{I_{cc}}^{-1} d_c. \tag{5.14}
$$

Here, $F_{DP} = F_{I_{rr}} + F_{I_{rc}} F_{I_{cc}}^{-1} F_{I_{cr}}$ is called the FETI-DP operator for the problem (5.1). In Section 5.3, it will be shown that F_{DP} is a s.p.d. operator. Hence, the equation (5.14) will be solved by the PCGM.

Remark 5.1 *In the formulation by Dryja and Widlund [21, 22], the mortar constraints are*

$$B_r w_r = 0.$$

Hence, letting $B_c = 0$ *in* (5.13), (5.14) *gives the FETI-DP operator developed by Dryja and Widlund.*

5.2.2 Preconditioner

From now on, we find an operator $\widehat{F}_{DP}$ that gives

$$< \widehat{F}_{DP}\lambda, \lambda >= \|\lambda\|^2_{(W^0)'}. \tag{5.15}$$

Then, the operator $\widehat{F}_{DP}^{-1}$ will be proposed as a preconditioner for F_{DP}.

Let $E^i_{ij} : W^0_{ij} \rightarrow W_i$ be an extension operator by 0 and $R_{ij} : W^0 \rightarrow W^0_{ij}$ be a restriction operator. We have

$$\widetilde{w}_{ij} = E^i_{ij} w_{ij} \ \text{ for } \ w_{ij} \in W^0_{ij},$$

where $\widetilde{w}_{ij} \in W_i$ is the zero extension of w_{ij} into $\partial\Omega_i$. Then, by (5.5) and (5.3), we get

$$\|w\|^2_{W^0} = \sum_{i=1}^{N} \left\langle S^i \left(\sum_{j \in m_i} E^i_{ij} R_{ij} w \right), \sum_{j \in m_i} E^i_{ij} R_{ij} w \right\rangle.$$

Let $E^i = \sum_{j\in m_i} E^i_{ij} R_{ij}$, then the above relation is written into

$$\|w\|^2_{W^0} =< \widehat{S} w, w > \quad \text{with} \quad \widehat{S} = \sum_{i=1}^{N} (E^i)^t S^i E^i. \tag{5.16}$$

Assume that Ω_i is the nonmortar side of Γ_{ij}. We recall that

$$(B^{ij}_i)_{lk} = \int_{\Gamma_{ij}} \xi^{ij}_l \phi^{ij}_k \, ds, \ l = 1, \cdots, L, \ k = 0, 1, \cdots, K_i + 1.$$

Since Ω_i is the nonmortar side of Γ_{ij}, we have $K_i = L$. We take $(B^{ij}_{i,r})_{lk} = (B^{ij}_i)_{lk}$ for $l, k = 1, \cdots, L$ and it gives

$$\lambda^t_{ij} B^{ij}_{i,r} w_{ij} = \int_{\Gamma_{ij}} \lambda_{ij} w_{ij} \, ds \ \text{ for } w_{ij} \in W^0_{ij}.$$

Let

$$\widehat{B} = \mathrm{diag}_{i=1,\cdots,N}\left(\mathrm{diag}_{j\in m_i}\left(B^{ij}_{i,r}\right)\right).$$

Then, the following holds for $(w,\lambda) \in W^0 \times M$:

$$\lambda^t \widehat{B} w = \sum_{i=1}^{N} \sum_{j\in m_i} \int_{\Gamma_{ij}} \lambda_{ij} w_{ij}\, ds, \tag{5.17}$$

where $\lambda_{ij} = \lambda|_{\Gamma_{ij}}$ and $w_{ij} = w|_{\Gamma_{ij}}$.

From the definition of the dual norm (5.7), (5.6), (5.17) and (5.16), we have

$$\|\lambda\|_M^2 = \max_{w\in W^0\setminus\{0\}} \frac{<\lambda, \widehat{B}w>^2}{<\widehat{S}w, w>}.$$

Since $\widehat{S}$ is symmetric and positive definite on W^0, the maximum in the above equation occurs when $\widehat{B}^t\lambda = \widehat{S}w$. This gives that

$$\|\lambda\|_M^2 =< \widehat{B}\widehat{S}^{-1}\widehat{B}^t\lambda, \lambda > .$$

Therefore, we have

$$\widehat{F}_{DP} = \widehat{B}\widehat{S}^{-1}\widehat{B}^t.$$

Then we take $\widehat{F}_{DP}^{-1} = \left(\widehat{B}\widehat{S}^{-1}\widehat{B}^t\right)^{-1}$ as a preconditioner for F_{DP} and we call it a Neumann-Dirichlet preconditioner. Since $\widehat{B}$ consists of diagonal blocks $B^{ij}_{i,r}$'s, which are invertible and symmetric, we get

$$\widehat{F}_{DP}^{-1} = \sum_{i=1}^{N} \left(\sum_{j\in m_i} R_{ij}^t (B^{ij}_{i,r})^{-1} (E^i_{ij})^t \right) S^i \left(\sum_{j\in m_i} E^i_{ij} (B^{ij}_{i,r})^{-1} R_{ij} \right).$$

Hence, the work for multiplying $\widehat{F}_{DP}^{-1}$ by a vector can be done parallely in each subdomain. Let

$$\widehat{B}_i = \sum_{j\in m_i} R_{ij}^t (B^{ij}_{i,r})^{-1} (E^i_{ij})^t.$$

Moreover, from the operator $\widehat{B}_i$, we can see that the preconditioner $\widehat{F}_{DP}^{-1}$ is different from the preconditioners in [21, 22, 29, 31, 38]. Only on the slave sides of interfaces, the function values are transferred between the spaces W_i and M. Hence, the cost needed to compute $\widehat{B}_i w_i$ and $\widehat{B}_i^t\lambda$ is reduced by half compared with other FETI(-DP) preconditioners.

5.3 Condition number bound estimation

The following well-known result is given when $a_i(u,v) = \int_{\Omega_i} \nabla u \cdot \nabla v \, dx$ (see Theorem 4.1.3 in [41]). With slight modification, we can obtain the similar result for a general case.

Lemma 5.2 *For $w_i \in W_i$, we have*

$$C_1 |w_i|^2_{1/2,\partial\Omega_i} \leq < S^i w_i, w_i > \leq C_2 \|w_i\|^2_{1/2,\partial\Omega_i},$$

where C_1 and C_2 are constants depending on $A(x)$ and $\beta(x)$, but not depending on H_i and h_i.

In the following, we obtain a formula that is useful to analyze the condition number bound of the FETI-DP operator and the result is the same as Lemma 4.3 of Mandel and Tezaur [38]. However, in our formulation, the continuity constraints are imposed on $w \in W$, that is, the d.o.f. on edges and global corners; see (5.12).

Lemma 5.3 *For $\lambda \in M$, we have*

$$\max_{w \in W \setminus \{0\}} \frac{< Bw, \lambda >^2}{\|w\|^2_W} =< F_{DP}\lambda, \lambda > .$$

Proof. We rewrite the equations (5.10)-(5.12) into

$$S_b w_b + B_b^t \lambda = g_b,$$
$$B_b w_b = 0,$$

where

$$S_b = \begin{pmatrix} S_{rr} & S_{rc} \\ S_{cr} & S_{cc} \end{pmatrix}, \quad B_b = \begin{pmatrix} B_r & B_c \end{pmatrix},$$
$$w_b = \begin{pmatrix} w_r \\ w_c \end{pmatrix}, \qquad g_b = \begin{pmatrix} g_r \\ g_c \end{pmatrix}.$$

Since S_b is invertible, elimination w_b in the above equations, we obtain

$$B_b S_b^{-1} B_b^t \lambda = B_b S_b^{-1} g_b,$$

which is the same as (5.14). Therefore, we have

$$F_{DP} = B_b S_b^{-1} B_b^t. \tag{5.18}$$

For $w \in W$, using the notations in Section 5.2, we write

$$< Bw, \lambda > = < B_r w_r + B_c w_c, \lambda >,$$
$$\|w\|_W^2 = \begin{pmatrix} w_r \\ w_c \end{pmatrix}^t \begin{pmatrix} S_{rr} & S_{rc} \\ S_{cr} & S_{cc} \end{pmatrix} \begin{pmatrix} w_r \\ w_c \end{pmatrix}.$$

Then, we have

$$\max_{w \in W \backslash \{0\}} \frac{b(w,\lambda)^2}{\|w\|_W^2} = \max_{w_b \in W_r \times W_c \backslash \{0\}} \frac{< B_b w_b, \lambda >^2}{w_b^t S_b w_b}. \tag{5.19}$$

Since S_b is s.p.d. on $W_r \times W_c$, in the R.H.S. of (5.19) the maximum occurs when $S_b w_b = {B_b}^t \lambda$. Hence we have

$$\max_{w \in W \backslash \{0\}} \frac{< Bw, \lambda >^2}{\|w\|_W^2} = < B_b S_b^{-1} B_b^t w_b, \lambda > . \tag{5.20}$$

Combining (5.20) and (5.18), we complete the proof. ■

Remark 5.4 *Since S_b is s.p.d. on $W_r \times W_c$, from* (5.18)*, we can see that F_{DP} is s.p.d. on M.*

Now, we estimate the lower bound of the condition number for the operator $\widehat{F}_{DP}^{-1} F_{DP}$.

Lemma 5.5 *For any $\lambda \in M$, we have*

$$\max_{w \in W \backslash \{0\}} \frac{< Bw, \lambda >^2}{\|w\|_W^2} \geq \|\lambda\|_M.$$

Proof. For $w \in W^0$, let $\widetilde{w} = (\widetilde{w}_1, \cdots, \widetilde{w}_N)$ be the zero extension into W. Then, it follows that

$$\max_{w \in W \backslash \{0\}} \frac{< Bw, \lambda >^2}{\|w\|_W^2} \geq \max_{w \in W^0 \backslash \{0\}} \frac{< B\widetilde{w}, \lambda >^2}{\|\widetilde{w}\|_W^2}. \tag{5.21}$$

Since $\widetilde{w}_j = 0$ on Γ_{ij}, for $j \in m_i$, we have

$$< B\widetilde{w}, \lambda > = \sum_{i=1}^{N} \sum_{j \in m_i} \int_{\Gamma_{ij}} w_{ij} \lambda_{ij} \, ds = < \lambda, w >_m, \tag{5.22}$$

where $w_{ij} = w|_{\Gamma_{ij}}$. Combining (5.22), (5.5) and (5.7), we obtain

$$\max_{w \in W^0 \setminus \{0\}} \frac{< B\widetilde{w}, \lambda >^2}{\|\widetilde{w}\|_W^2} = \max_{w \in W^0 \setminus \{0\}} \frac{< \lambda, w >_m^2}{\|w\|_{W^0}^2} = \|\lambda\|_M^2. \tag{5.23}$$

From (5.21) and (5.23), we complete the proof. ■

To estimate the upper bound of $< F_{DP}\lambda, \lambda >$, we need the following estimate for $\|w_i - w_j\|^2_{H_{00}^{1/2}(\Gamma_{ij})}$.

Lemma 5.6 *For $w \in W$, let $w_i = w|_{\partial\Omega_i}$ and $w_j = w|_{\partial\Omega_j}$. Then we have*

$$\|w_i - w_j\|^2_{H_{00}^{1/2}(\Gamma_{ij})} \leq C \max_{l \in \{i,j\}} \left\{ \left(1 + \log \frac{H_l}{h_l}\right)^2 \right\} \left(|w_i|^2_{1/2,\partial\Omega_i} + |w_j|^2_{1/2,\partial\Omega_j} \right),$$

where C is a constant independent of h_i's and H_i's and may depend on $A(x)$ and $\beta(x)$.

Proof. Let $I^H w$ be a linear function on Γ_{ij} that has the same value with w at the end points of Γ_{ij}. From Lemma 2.11, we have

$$\|w_l - I^H w_l\|_{H_{00}^{1/2}(\Gamma_{ij})} \leq C \left(1 + \log \frac{H_l}{h_l}\right) |w_l|_{1/2,\partial\Omega_l} \text{ for } l = i, j.$$

Using this, we prove the lemma. ■

Recall the definition of the mortar projection π_{ij} in Section 4.2 and the stability of π_{ij}:

$$\|\pi_{ij} v\|_{H_{00}^{1/2}(\Gamma_{ij})} \leq C \|v\|_{H_{00}^{1/2}(\Gamma_{ij})} \quad \forall v \in H_{00}^{1/2}(\Gamma_{ij}), \tag{5.24}$$

where C is a constant independent of H_i's and h_i's. Now, we estimate the upper bound of the operator $\widehat{F}_{DP}^{-1} F_{DP}$.

Lemma 5.7 *For $\lambda \in M$, we have*

$$\max_{w \in W \setminus \{0\}} \frac{< Bw, \lambda >^2}{\|w\|_W^2} \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \|\lambda\|_M^2,$$

where C is a constant depending on $A(x)$ and $\beta(x)$, but independent of h_i's and H_i's.

Proof. From the definitions of the matrix B and π_{ij} in (4.1), we have

$$< Bw, \lambda >^2 = \left(\sum_{i=1}^{N} \sum_{j \in m_i} \int_{\Gamma_{ij}} \pi_{ij}(w_i - w_j)\lambda_{ij}\, ds \right)^2 .$$

We let $z \in W^0$ be such that $z|_{\Gamma_{ij}} = \pi_{ij}(w_i - w_j)$. Then the above equation is the duality pairing between λ and z. Hence, using the definition of dual norm on λ, we get

$$< Bw, \lambda >^2 \leq \|\lambda\|_M^2 \|z\|_{W^0}^2. \tag{5.25}$$

Let $\widetilde{z} = (\widetilde{z}_1, \cdots, \widetilde{z}_N) \in W$ be the zero extension of z. Then, from (5.5), (5.3), Lemma 5.2, (2.1), (5.24) and Lemma 5.6,

$$\begin{aligned}
\|z\|_{W^0}^2 &= \sum_{i=1}^{N} < S^i \widetilde{z}_i, \widetilde{z}_i > \\
&\leq C \sum_{i=1}^{N} \|\widetilde{z}_i\|_{1/2,\partial\Omega_i}^2 \\
&\leq C \sum_{i=1}^{N} \sum_{j \in m_i} \|\pi_{ij}(w_i - w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})}^2 \\
&\leq C \sum_{i=1}^{N} \sum_{j \in m_i} \|w_i - w_j\|_{H_{00}^{1/2}(\Gamma_{ij})}^2 \\
&\leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \sum_{i=1}^{N} |w_i|_{1/2,\partial\Omega_i}^2 \\
&\leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \|w\|_W^2.
\end{aligned} \tag{5.26}$$

Here, C denotes a generic constant independent of h_i's and H_i's, which may vary from occurrence and occurrence. Combining (5.25) and (5.26), we complete the proof. ■

Since the preconditioner $\widehat{F}_{DP}^{-1}$ follows from the dual norm of $\lambda \in M$ (see (5.15)), combining Lemma 5.3, Lemma 5.5 and Lemma 5.7, we obtain the following estimate.

Theorem 5.8 *For $\lambda \in M$, we have*

$$< \widehat{F}_{DP}\lambda, \lambda > \leq < F_{DP}\lambda, \lambda > \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\} < \widehat{F}_{DP}\lambda, \lambda >,$$

where C is a constant depending on $A(x)$ and $\beta(x)$, but independent of H_i's and h_i's.

Corollary 5.9 *We have the condition number estimate*

$$\kappa \left(\widehat{F}_{DP}^{-1} F_{DP} \right) \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\},$$

where C is a constant depending on $A(x)$ and $\beta(x)$, but independent of H_i's and h_i's.

Remark 5.10 *On each Γ_{ij}, the choice of master and slave side is arbitrary.*

Remark 5.11 *In Corollary 5.9, the condition number depends on $A(x)$ and $\beta(x)$. Now we consider a problem:*

$$-\nabla \cdot (\alpha(x) \nabla u(x)) = f(x) \quad in \quad \Omega,$$
$$u = 0 \quad on \quad \partial\Omega,$$

where $\alpha(x)$ is a piecewise constant and has jumps across the subdomain boundaries, i.e., $\alpha(x) = \rho_i$ for all $x \in \Omega_i$ for some constant $\rho_i > 0$. On Γ_{ij}, we choose $\Omega_i^h|_{\Gamma_{ij}}$ as the slave side if $\rho_i \leq \rho_j$. Otherwise, we choose $\Omega_i^h|_{\Gamma_{ij}}$ as the master side. Then we have

$$C_1 \rho_i |w_i|^2_{1/2,\partial\Omega_i} \leq < S^i w_i, w_i > \leq C_2 \rho_i \|w_i\|^2_{1/2,\partial\Omega_i},$$

where C_1 and C_2 are constants independent of ρ_i's, h_i's and H_i's. Following the proof *of Lemma 5.7 and using the above inequalities instead of Lemma 5.2, we obtain*

$$
\begin{aligned}
\|z\|_{W^0}^2 &\le C \sum_{i=1}^{N} \sum_{j \in m_i} \rho_i \|w_i - w_j\|^2_{H_{00}^{1/2}(\Gamma_{ij})} \\
&\le C \sum_{i=1}^{N} \sum_{j \in m_i} \left\{ \max_{l \in \{i,j\}} \left\{ \left(1 + \log \frac{H_l}{h_l}\right)^2 \right\} \right. \\
&\qquad\qquad \left. \times \left(\rho_i |w_i|^2_{1/2,\partial\Omega_i} + \rho_i |w_j|^2_{1/2,\partial\Omega_j} \right) \right\} \\
&\le C \sum_{i=1}^{N} \sum_{j \in m_i} \left\{ \max_{l \in \{i,j\}} \left\{ \left(1 + \log \frac{H_l}{h_l}\right)^2 \right\} \right. \\
&\qquad\qquad \left. \times \left(< S^i w_i, w_i > + \frac{\rho_i}{\rho_j} < S^j w_j, w_j > \right) \right\},
\end{aligned}
$$

where C is a generic constant independent of ρ_i's, H_i's and h_i's. Since $\rho_i \le \rho_j$, we can see that the constant C in Lemma 5.7 is bounded independently of the coefficients. Hence, the condition number bound is independent of ρ_i's.

6. Elliptic problems in 3D

6.1 A model problem and finite elements

Let Ω be a bounded polyhedral domain in $\mathbb{R}^3$. We consider the same elliptic problem (5.1) with $A(x) \in \mathbb{R}^{3\times 3}$. The domain Ω is partitioned into nonoverlapping polyhedral subdomains $\{\Omega_i\}_{i=1}^N$, which are geometrically conforming. This means that each subdomain intersects with neighboring subdomains on the whole face, whole edge or at a vertex. Among them, we call faces the interfaces of subdomains and use Γ_{ij} to denote the interface of subdomain Ω_i and Ω_j. Let $\Omega_i^{h_i}$ be a quasi-uniform triangulation of Ω_i with the maximum diameter h_i. These meshes may not be aligned across the subdomain interfaces.

For each subdomain Ω_i, we introduce a finite element space X_i, W_i, X and W as in $2D$ case in Chapter 5. We define a bilinear form

$$a_i(u_i, v_i) := \int_{\Omega_i} A(x)\nabla u_i \cdot \nabla v_i \, dx + \int_{\Omega_i} \beta(x) u_i v_i dx.$$

and let S^i be the Schur complement matrix obtained from the bilinear form $a_i(\cdot,\cdot)$ over the finite elements X_i. Using this operator, a semi-norm is defined for $w_i \in W_i$:

$$|w_i|_{S^i}^2 :=< S^i w_i, w_i >,$$

where $< \cdot,\cdot >$ is the l^2-inner product of vectors. For $w \in W$, since w is continuous at subdomain vertices, by summing up these semi-norms, we define a norm

$$\|w\|_W^2 := \sum_{i=1}^N |w_i|_{S^i}^2, \quad w_i = w|_{\partial\Omega_i}. \tag{6.1}$$

On each Γ_{ij}, we determine a nonmortar side and a mortar side and define the index sets m_i and s_i; see (4.3). The spaces W_{ij}, W_{ij}^0 and M_{ij} are defined as in Section 4.2. For $3D$ case, examples of M_{ij}, which satisfy the assumptions (A.1)-

(A.4), are given in Section 4.2. Then, we define

$$W^0 := \prod_{i=1}^{N} \prod_{j \in m_i} W_{ij}^0, \tag{6.2}$$

$$M := \prod_{i=1}^{N} \prod_{j \in m_i} M_{ij}^0. \tag{6.3}$$

The space W^0 is equipped with the norm $\| \cdot \|_{W^0}$

$$\|w\|_{W^0} := \|\widetilde{w}\|_W, \tag{6.4}$$

where $\widetilde{w} \in W$ is the zero extension of w.

Using the Lagrange multiplier space M, we impose the mortar matching condition (5.8) on the space X. It is already known from the numerical results in [26] that using the primal variables at corners is not enough to get the same condition number bound as $2D$ problems. Hence, we add redundant continuity constraints to the coarse problem and follow the augmented FETI-DP formulation. The redundant constraints are

$$\int_{\Gamma_{ij}} v_i \, ds = \int_{\Gamma_{ij}} v_j \, ds \quad \forall i = 1, \cdots, N, \; j \in m_i. \tag{6.5}$$

That is, the averages of functions are the same across the common face Γ_{ij}. Since $1 \in M_{ij}$, the above constraints are redundant to the mortar constraints (5.8). Then, those constraints are written into the following algebraic equations:

$$Bw = 0$$
$$R^t B w = 0,$$

where the matrix B is defined similarly as $2D$ case and R is a matrix that gives the redundant constraints. More precisely, $R^t \lambda = 0$ means that sum of $\lambda|_{\Gamma_{ij}}$ is zero for each Γ_{ij} and R has 0 or 1 as entries.

For the $3D$ elliptic problems with conforming discretizations, Klawonn *et al.* [32] developed FETI-DP methods with various redundant constraints. They showed that the method is not competitive when only using the primal variables at corners. Additional continuity constraints on edges or on faces are needed to obtain the same condition number bound as $2D$ elliptic problems. The continuity constraints on an edge is that the averages of functions across the common edge are the same. The same is applied to a face also. From their results, it seems that the continuity constraints on edges are essential. Further, in [33], they extended the results to the case with face constraints only. In mortar context, the constraints on edges are not redundant to the mortar matching condition. We will only impose the face constraints as the redundant constraints. This is a different feature of our method from that of Klawonn *et al.* [32].

6.2 FETI-DP formulation

6.2.1 FETI-DP operator

In $3D$, we have a face, an edge or a vertex as an intersection of subdomains. Hence, we use the symbol r to represent the d.o.f. on faces and edges and c to represent the d.o.f. at corners(vertices). Then, we write

$$w_i = \begin{pmatrix} w_r^i \\ w_c^i \end{pmatrix} \text{ for } w_i \in W_i$$

and define w_c and w_r for $w \in W$ as in Section 5.2.1. The spaces W_r and W_c consist of the vectors w_r and w_c, respectively. Let U be a Lagrange multiplier space corresponding to the redundant constraints (6.5). We use that same notations of matrices and vectors as in Section 5.2.1 except that the symbol r represents the d.o.f. on faces and edges. Then, we have the following saddle point formulation of the problem (5.1):

Find $(w_r, w_c, \mu, \lambda) \in W_r \times W_c \times U \times M$ satisfying

$$\begin{aligned} S_{rr} w_r + S_{rc} w_c + B_r^t R\mu + B_r^t \lambda &= g_r, \\ S_{cr} w_r + S_{cc} w_c + B_c^t R\mu + B_c^t \lambda &= g_c, \\ R^t B_r w_r + R^t B_c w_c &= 0, \\ B_r w_r + B_c w_c &= 0. \end{aligned} \tag{6.6}$$

In the above equations, we regard $\widetilde{w}_c = \begin{pmatrix} w_c \\ \mu \end{pmatrix}$ as the primal variables in the FETI-DP formulation and follow the augmented FETI-DP formulation introduced in Section 3.4. Let

$$\begin{aligned}
K_{rr} &= S_{rr}, \\
K_{rc} &= \begin{pmatrix} S_{rc} & B_r^t R \end{pmatrix}, \quad K_{cr} = K_{rc}^t, \\
K_{cc} &= \begin{pmatrix} S_{cc} & B_c^t R \\ R^t B_c & \mathbf{0} \end{pmatrix}, \\
\widetilde{B}_c &= \begin{pmatrix} B_c & \mathbf{0} \end{pmatrix}, \quad \widetilde{g}_c = \begin{pmatrix} g_c \\ \mathbf{0} \end{pmatrix}.
\end{aligned}$$

Then we have

$$\begin{pmatrix} K_{rr} & K_{rc} & B_r^t \\ K_{cr} & K_{cc} & \widetilde{B}_c^t \\ B_r & \widetilde{B}_c & \mathbf{0} \end{pmatrix} \begin{pmatrix} w_r \\ \widetilde{w}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} g_r \\ \widetilde{g}_c \\ \mathbf{0} \end{pmatrix}. \tag{6.7}$$

Since K_{rr} is invertible, after eliminating w_r in (6.7), we obtain

$$\begin{pmatrix} -F_{cc} & F_{cl} \\ F_{cl}^t & F_{ll} \end{pmatrix} \begin{pmatrix} \widetilde{w}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} -d_c \\ d_l \end{pmatrix},$$

where

$$\begin{aligned}
F_{cc} &= K_{cc} - K_{cr} K_{rr}^{-1} K_{cr}, \\
F_{lc} &= B_r K_{rr}^{-1} K_{rc} - \widetilde{B}_c^t, \quad F_{cl} = F_{lc}^t, \\
F_{ll} &= B_r K_{rr}^{-1} B_r^t \\
d_l &= B_r S_{rr}^{-1} g_r, \quad d_c = \widetilde{g}_c - K_{cr} K_{rr}^{-1} g_r.
\end{aligned}$$

From the fact that $B_c^t R$ has a full column rank, we can show that F_{cc} is invertible. Hence, eliminating $\widetilde{w}_c$ in the above equation, the FETI-DP equation of (6.6) follows

$$F_{DP} \lambda = d_l - F_{cl}^t F_{cc}^{-1} d_c, \tag{6.8}$$

with $F_{DP} = F_{ll} + F_{cl}^t F_{cc}^{-1} F_{cl}$. we call F_{DP} the FETI-DP operator. Since, we added the redundant mortar matching constraints to the FETI-DP formulation, the solution of FETI-DP equation is not uniquely determined in M. Let us define

$$M_R := \left\{ \lambda \in M \ : \ R^t \lambda = 0 \right\}. \tag{6.9}$$

In Section 6.3, we will show that F_{DP} is s.p.d. on M_R. Hence, the solution $\lambda \in M_R$ is uniquely determined.

6.2.2 Preconditioner

Since F_{DP} is s.p.d. on M_R, we will solve (6.8) by preconditioned conjugate gradient method using a suitable preconditioner. We derive a preconditioner from the similar idea with $2D$ case.

Let us define the following subspaces equipped with the norms induced from W and W^0:

$$W_R := \left\{ w \in W \ : \ R^t B w = 0 \right\}, \tag{6.10}$$

$$W_R^0 := \left\{ w \in W^0 \ : \ R^t B \widetilde{w} = 0 \right\}, \tag{6.11}$$

where $\widetilde{w}$ is the zero extension of w into the space W. Recall the definition of the space M_R in (6.9). A duality pairing between the spaces M_R and W_R^0 is defined as

$$< \lambda, w >_m = \sum_{i=1}^{N} \sum_{j \in m_i} \int_{\Gamma_{ij}} \lambda_{ij} w_{ij} \, ds. \tag{6.12}$$

Then, a dual norm on $\lambda \in M_R$ is given by

$$\|\lambda\|_{M_R} := \max_{w \in W_R^0} \frac{< \lambda, w >_m}{\|w\|_{W^0}}. \tag{6.13}$$

Similarly to the $2D$ problems, we will find an operator $\widehat{F}_{DP}$ which gives

$$< \widehat{F}_{DP} \lambda, \lambda > = \|\lambda\|_{M_R}^2 \tag{6.14}$$

and propose $\widehat{F}_{DP}^{-1}$ as a preconditioner for the operator F_{DP}.

Now, we derive a matrix form of the operator $\widehat{F}_{DP}^{-1}$. Since the dual norm is defined on the subspaces M_R and W_R^0, we need the following l_2-orthogonal projections:

$$P_{W_R^0}^{ij} : W^0|_{\Gamma_{ij}} \rightarrow W_R^0|_{\Gamma_{ij}},$$

$$P_{M_R}^{ij} : M|_{\Gamma_{ij}} \rightarrow M_R|_{\Gamma_{ij}}.$$

From the above projection operators, the l_2-orthogonal projections $P_{W_R^0} : W \to W_R$ and $P_{M_R} : M \to M_R$ are obtained

$$P_{W_R^0} = \mathrm{diag}_{i=1}^N \mathrm{diag}_{j\in m_i}(P^{ij}_{W_R^0}),$$
$$P_{M_R} = \mathrm{diag}_{i=1}^N \mathrm{diag}_{j\in m_i}(P^{ij}_{M_R}).$$

We recall the following restriction and extension

$$R_{ij} : W^0 \to W^0_{ij},$$
$$E^i_{ij} : W^0_{ij} \to W_i.$$

and the matrices B_i^{ij} and B_j^{ij} in (5.9). We obtain the matrices $B_{i,r}^{ij}$ from B_i^{ij} after deleting columns corresponding to the d.o.f. on the boundary of Γ_{ij}. Let

$$\widehat{S} = \sum_{i=1}^N (\sum_{j\in m_i} E^i_{ij} R_{ij}) S^i (\sum_{j\in m_i} E^i_{ij} R_{ij})^t,$$
$$\widehat{B} = \mathrm{diag}_{i=1}^N \mathrm{diag}_{j\in m_i}(B_i^{ij}).$$

Then we have

$$\|w\|^2_{W^0} =< \widehat{S}_p w, w > \quad \text{for } w \in W_R^0,$$
$$< \lambda, w >_m = \lambda^t \widehat{B}_p w \quad \text{for } \lambda \in M_R,\ w \in W_R^0,$$

where

$$\widehat{S}_p = P^t_{W_R^0} \widehat{S} P_{W^0},$$
$$\widehat{B}_p = P^t_{M_R} \widehat{B} P_{W_R^0}.$$

It can be shown that $\widehat{S}_p$ and $\widehat{B}_p$ are invertible on W_R^0 and $\widehat{B}_p^t$ is invertible on M_R. Hence, the maximum in (6.13) occurs when $\widehat{S}_p w = \widehat{B}_p^t \lambda$ and this gives

$$< \widehat{B}_p \widehat{S}_p^{-1} \widehat{B}_p^t \lambda, \lambda >= \|\lambda\|^2_{M_R} \quad \text{for } \lambda \in M_R.$$

As a result, we have $\widehat{F}_{DP} = \widehat{B}_p \widehat{S}_p^{-1} \widehat{B}_p^t$. From the observation that $\widehat{B}_p$ consists of invertible block matrices $\widehat{B}_p^{ij} = (P_{M_R}^{ij})^t B_{i,r}^{ij} P_{W_R^0}^{ij}$, we get

$$\widehat{F}_{DP}^{-1} = \sum_{i=1}^{N} (\sum_{j \in m_i} E_{ij}^i (B_p^{ij})^{-1} R_{ij})^t S^i (\sum_{j \in m_i} E_{ij}^i (B_p^{ij})^{-1} R_{ij}). \tag{6.15}$$

Hence, the computation of $\widehat{F}_{DP}^{-1}\lambda$ can be done parallelly in each subdomain.

6.3 Condition number bound estimation

We have the following result as in $2D$ problems.

Lemma 6.1 *For $w_i \in W_i$, we have*

$$C_1 |w_i|_{1/2,\partial\Omega_i}^2 \leq < S^i w_i, w_i > \leq C_2 \|w_i\|_{1/2,\partial\Omega_i}^2,$$

where C_1 and C_2 are constants depending on $A(x)$ and $\beta(x)$, but independent of H_i and h_i.

Lemma 6.2 *For $\lambda \in M_R$, we have*

$$\max_{w \in W_R \backslash \{0\}} \frac{< Bw, \lambda >^2}{\|w\|_W^2} = < F_{DP}\lambda, \lambda > .$$

Proof. The saddle-point problem (6.7) is equivalent to solving the following problem

$$\max_{\lambda \in B(W_R)} \min_{w \in W_R} \left(\frac{1}{2} w^t S w + w^t g + \lambda^t B w \right),$$

where g is a vector composed of the vectors g_r and g_c in (6.6). It can be shown easily that $B(W_R) = M_R$. We recall the l^2-orthogonal projections $P_{M_R} : M \to M_R$ and $P_{W_R} : W \to W_R$. Then, taking Euler-Lagrangian in the above problem, we get

$$S_p w + B_p^t \lambda = P_{W_R}^t g,$$
$$B_p w = 0,$$

where

$$S_p = P_{W_R}^t S P_{W_R},$$
$$B_p = P_{M_R}^t B P_{W_R}.$$

We can see that S_p is s.p.d. on W_R. Hence, eliminating w in the above equations, we obtain

$$B_p S_p^{-1} B_p^t \lambda = d,$$

where $d = B_p S_p^{-1} P_{W_R}^t g$. Since this equation is obtained from the same problem with (6.7), we have

$$F_{DP} = B_p S_p^{-1} B_p^t. \tag{6.16}$$

Using the identity

$$\|w\|_W^2 =< Sw, w >$$

and the projections P_{W_R} and P_{M_R}, we can see that

$$\max_{w \in W_R \backslash \{0\}} \frac{< Bw, \lambda >^2}{\|w\|_W^2} =< B_p S_p^{-1} B_p^t \lambda, \lambda > \quad \text{for } \lambda \in M_R. \tag{6.17}$$

From (6.16) and (6.17), we prove the lemma. ■

Remark 6.3 *For $\lambda \in M_R$, $B_p^t \lambda = 0$ gives $\lambda = 0$ and S_p is s.p.d. on W_R. Hence, from* (6.16)*, we can see that F_{DP} is s.p.d. on M_R.*

Now, we estimate the lower bound of the operator F_{DP}.

Lemma 6.4 *For any $\lambda \in M_R$, we have*

$$\max_{w \in W_R \backslash \{0\}} \frac{< Bw, \lambda >^2}{\|w\|_W^2} \geq \|\lambda\|_{M_R}^2.$$

Proof. Let $\widetilde{w} \in W$ be the zero extension of $w \in W_R^0$. Then, we can see that $\widetilde{w} \in W_R$. Using the definitions of $\|\lambda\|_{M_R}$, $\|w\|_{W^0}$ and $< \lambda, w >_m$, we get

$$\begin{aligned} \|\lambda\|_{M_R}^2 &= \max_{w \in W_R^0 \backslash \{0\}} \frac{< \lambda, w >_m^2}{\|w\|_{W^0}^2} \\ &= \max_{w \in W_R^0 \backslash \{0\}} \frac{< B\widetilde{w}, \lambda >^2}{\|\widetilde{w}\|_W^2} \\ &\leq \max_{w \in W_R \backslash \{0\}} \frac{< Bw, \lambda >^2}{\|w\|_W^2}. \end{aligned}$$

This completes the proof. ■

To estimate the upper bound of the operator F_{DP}, we define an interpolation $I_0^i w_i \in W_i$ by

$$(I_0^i w_i)(x) = \begin{cases} w_i(x), & x \in \partial F \cap \partial\Omega_i^h, \\ C_F, & x \in F \cap \partial\Omega_i^h, \end{cases}$$

where $\partial\Omega_i^h$ is the set of nodes on the boundary of Ω_i and C_F is an average of w_i on the face $F \subset \partial\Omega_i$, that is,

$$C_F = \frac{\int_F w_i \, ds}{\int_F \, ds}.$$

Note that faces and edges are open sets which do not include their boundaries. In the following, C is a generic constant which does not depend on the mesh size or the number of subdomains and may depend on $A(x)$ or $\beta(x)$. Recall the definition of norms $\|\cdot\|_{H_{00}^{1/2}(F)}$ and $\|\cdot\|_{1/2,\partial\Omega_i}$ in Section 2.1. Using the definition of C_F, Hölder inequality and the definition of $\|\cdot\|_{1/2,\partial\Omega_i}$, we obtain

$$|C_F| \le C H_i^{-1/2} \|w_i\|_{1/2,\partial\Omega_i}. \tag{6.18}$$

For a set $A \subset \partial\Omega_i$, $I_A^h w_i$ denotes a nodal value interpolation of w_i on the set A. The interpolation $I_0^i w_i$ has the following approximation properties.

Lemma 6.5 *For $w_i \in W_i$, we have*

$$\|w_i - I_0^i w_i\|_{H_{00}^{1/2}(F)} \le C \left(1 + \log \frac{H_i}{h_i}\right) |w_i|_{1/2,\partial\Omega_i}, \tag{6.19}$$

$$\|I_0^i w_i - C_F\|_{0,F} \le C h_i^{1/2} \left(1 + \log \frac{H_i}{h_i}\right)^{1/2} |w_i|_{1/2,\partial\Omega_i}. \tag{6.20}$$

Proof. First, we consider

$$\begin{aligned} \|w_i - I_0^i w_i\|_{H_{00}^{1/2}(F)} &= \|I_F^h w_i - I_F^h C_F\|_{H_{00}^{1/2}(F)} \\ &\le \|I_F^h w_i\|_{H_{00}^{1/2}(F)} + |C_F| \|I_F^h 1\|_{H_{00}^{1/2}(F)}. \end{aligned}$$

Then from the Lemma 2.9, Lemma 2.10 and (6.18), we get

$$\|w_i - I_0^i w_i\|_{H_{00}^{1/2}(F)} \le C \left(1 + \log \frac{H_i}{h_i}\right) \|w_i\|_{1/2,\partial\Omega_i}.$$

Since $w_i - I_0^i w_i$ is invariant to a constant addition, we can replace the norm $\|\cdot\|_{1/2,\partial\Omega_i}$ by the semi-norm $|\cdot|_{1/2,\partial\Omega_i}$.

Now, we consider the second estimate. From the definition of $I_0^i w_i$ and the quasi-uniform assumption on the triangulation, we get

$$\begin{aligned}
\|I_0^i w_i - C_F\|_{0,F} &= \|I_{\partial F}^h (w_i - C_F)\|_{0,F} \\
&\leq C h_i^{1/2} \|I_{\partial F}^h (w_i - C_F)\|_{0,\partial F} \\
&\leq C h_i^{1/2} \sum_{E \subset \partial F} \|I_E^h (w_i - C_F)\|_{0,E} \\
&\leq C h_i^{1/2} \left(\sum_{E \subset \partial F} \|w_i\|_{0,E} + \sum_{E \subset \partial F} \|C_F\|_{0,E} \right),
\end{aligned}$$

where E is a closed edge on ∂F. Using the Lemma 2.8, we have

$$\|w_i\|_{0,E} \leq C \left(1 + \log \frac{H_i}{h_i}\right)^{1/2} \|w_i\|_{1/2,\partial\Omega_i}, \tag{6.21}$$

and

$$\|C_F\|_{0,E} \leq |E|^{1/2} |C_F|.$$

From (6.18) and $|E| \leq C H_i$, it follows that

$$\|C_F\|_{0,E} \leq C \|w_i\|_{1/2,\partial\Omega_i}. \tag{6.22}$$

From (6.21), (6.22) and the invariance of $I_0^i w_i - C_F$ to the constant addition, we complete the proof of (6.20). ■

Using the above estimates, we have the following result similarly to the $2D$ case.

Lemma 6.6 *For $w \in W_R$, we have*

$$\|\pi_{ij}(w_i - w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})} \leq C \max_{i,j} \left\{ \left(1 + \log \frac{H_l}{h_l}\right) \right\} \left(|w_i|_{1/2,\partial\Omega_i} + \left(\frac{h_j}{h_i}\right)^{1/2} |w_j|_{1/2,\partial\Omega_j} \right).$$

Proof. Using the interpolations $I_0^i w_i$ and $I_0^j w_j$, the inverse inequality (2.3) and the continuity of π_{ij}, we get

$$\begin{aligned}
\|\pi_{ij}(w_i - w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})} &\leq \|\pi_{ij}(w_i - I_0^i w_i)\|_{H_{00}^{1/2}(\Gamma_{ij})} + \|\pi_{ij}(w_j - I_0^j w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})} \\
&\quad + \|\pi_{ij}(I_0^i w_i - I_0^j w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})} \\
&\leq \|w_i - I_0^i w_i\|_{H_{00}^{1/2}(\Gamma_{ij})} + \|w_j - I_0^j w_j\|_{H_{00}^{1/2}(\Gamma_{ij})} \\
&\quad + C h_i^{-1/2} \|I_0^i w_i - I_0^j w_j\|_{0,\Gamma_{ij}}.
\end{aligned}$$

Since $w \in W_R$, we have the same C_F for w_i and w_j on $F(=\Gamma_{ij})$. Then, we have

$$\|I_0^i w_i - I_0^j w_j\|_{0,\Gamma_{ij}} \le \|I_0^i w_i - C_F\|_{0,\Gamma_{ij}} + \|I_0^j w_j - C_F\|_{0,\Gamma_{ij}}.$$

From the above equation and the approximation properties of $I_0^i w_i$ in Lemma 6.5, we obtain

$$\|\pi_{ij}(w_i - w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})} \le C\left((1+\log\frac{H_i}{h_i})|w_i|_{1/2,\partial\Omega_i} + (\frac{h_j}{h_i})^{1/2}(1+\log\frac{H_j}{h_j})|w_j|_{1/2,\partial\Omega_j}\right)$$

and complete the proof. ■

Now, we estimate the upper bound of the operator F_{DP}. Let us define

$$r_i = \max_{j\in m_i}\left\{1+\frac{h_j}{h_i}\right\} \quad \text{for } i = 1,\cdots,N.$$

Lemma 6.7 *For $\lambda \in M_R$, we have*

$$\max_{w\in W_R\backslash\{0\}} \frac{< Bw,\lambda >^2}{\|w\|_W^2} \le C \max_{i=1,\cdots,N}\left\{r_i\left(1+\log\frac{H_i}{h_i}\right)^2\right\}\|\lambda\|_{M_R}^2,$$

where C is a constant depending on $A(x)$ and $\beta(x)$, but independent of h_i's and H_i's.

Proof. From the definitions of B and π_{ij}, we have

$$< Bw,\lambda >^2 = \left(\sum_{i=1}^N \sum_{j\in m_i} \int_{\Gamma_{ij}} \pi_{ij}(w_i - w_j)\lambda_{ij}\, ds\right)^2.$$

We consider $z \in W^0$ such that $z|_{\Gamma_{ij}} = \pi_{ij}(w_i - w_j)$. Since $w \in W_R$, we can see that $z \in W_R^0$. Then the above equation is the duality pairing between $\lambda \in M_R$ and $z \in W_R^0$. Hence, using the definition of dual norm on λ, we get

$$< Bw,\lambda >^2 \le \|\lambda\|_{M_R}^2 \|z\|_{W^0}^2. \tag{6.23}$$

Let $\widetilde{z} = (\widetilde{z}_1, \cdots, \widetilde{z}_N) \in W$ be the zero extension of z. Then from (6.4), (6.1), Lemma 6.1, (2.1), (4.11) and Lemma 6.6,

$$\begin{aligned}
\|z\|_{W^0}^2 &= \sum_{i=1}^N < S^i \widetilde{z}_i, \widetilde{z}_i > \\
&\le C \sum_{i=1}^N \|\widetilde{z}_i\|_{1/2,\partial\Omega_i}^2 \\
&\le C \sum_{i=1}^N \sum_{j\in m_i} \|\pi_{ij}(w_i - w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})}^2 \\
&\le C \sum_{i=1}^N \sum_{j\in m_i} \left((1+\log\frac{H_i}{h_i})^2 |w_i|_{1/2,\partial\Omega_i}^2 + \frac{h_j}{h_i}(1+\log\frac{H_j}{h_j})^2 |w_j|_{1/2,\partial\Omega_j}^2 \right) \\
&\le C \max_{i=1,\cdots,N} \left\{ \left(1+\log\frac{H_i}{h_i}\right)^2 r_i \right\} \|w\|_W^2.
\end{aligned} \tag{6.24}$$

Here, C denotes a generic constant independent of h_i's and H_i's, which may vary from occurrence and occurrence. Combining (6.23) and (6.24), we complete the proof. ■

Remark 6.8 *When the coefficients $A(x)$ and $\beta(x)$ do not change rapidly across subdomain interfaces, it is appropriate to use triangulations which have similar mesh sizes between neighboring subdomains. Hence, in this case, we may assume that r_i is bounded independent of the mesh sizes.*

Now, we consider the following elliptic problem with discontinuous constant coefficients:

$$\begin{aligned}
-\nabla \cdot (\alpha(x)\nabla u) &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{6.25}$$

with $\alpha(x)|_{\Omega_i} = \rho_i (> 0)$ for all $i = 1, \cdots, N$. Then, we have the similar estimates to Lemma 6.1

$$C_1 \rho_i |w_i|_{1/2,\partial\Omega_i} \le < S^i w_i, w_i > \le C_2 \rho_i \|w_i\|_{1/2,\partial\Omega_i},$$

where C_1 and C_2 are constants not depending on ρ_i, h_i and H_i. Using the above bound, we follow the proofs of Lemma 6.7 and obtain

$$\|z\|_{W^0}^2 \leq C \sum_{i=1}^{N} \sum_{j \in m_i} \left(\left(1 + \log \frac{H_i}{h_i}\right)^2 |w_i|_{1/2,\partial\Omega_i}^2 + \frac{h_j}{h_i} \frac{\rho_i}{\rho_j} \left(1 + \log \frac{H_j}{h_j}\right)^2 |w_j|_{1/2,\partial\Omega_j}^2 \right), \tag{6.26}$$

where C is a constant independent of ρ_i's, h_i's and H_i's. For the same elliptic problem in $2D$, Wohlmuth [52] observed that the ratio $\frac{h_i}{h_j}$ tends to become $\left(\frac{\rho_i}{\rho_j}\right)^{1/4}$ as an adaptivity strategy is applied successively. In this stage, we make a reasonable assumption on the ratio of meshes for $3D$ problems.

Assumption on meshes: For each Γ_{ij}, we assume that

$$\frac{h_j}{h_i} \leq C \left(\frac{\rho_j}{\rho_i}\right)^{\gamma}, \text{ with } 0 \leq \gamma \leq 1, \tag{6.27}$$

where C is a constant independent of h_i's, ρ_i's and H_i's.

On Γ_{ij}, if we choose Ω_i with smaller ρ_i as a slave side, then from the above assumption and (6.26) we get

$$\begin{aligned} \|z\|_{W^0}^2 \leq & C \sum_{i=1}^{N} \sum_{j \in m_i} \left(\left(1 + \log \frac{H_i}{h_i}\right)^2 |w_i|_{1/2,\partial\Omega_i}^2 \right. \\ & \left. + \left(\frac{\rho_i}{\rho_j}\right)^{1-\gamma} \left(1 + \log \frac{H_j}{h_j}\right)^2 |w_j|_{1/2,\partial\Omega_j}^2 \right), \end{aligned}$$

where C is a constant independent of ρ_i's, h_i's and H_i's. Since the slave side has smaller ρ_i's, in the above equation $\left(\frac{\rho_i}{\rho_j}\right)^{1-\gamma} \leq 1$. Therefore, we obtain the following result.

Lemma 6.9 *With the assumption* (6.27) *on meshes, for the elliptic problem* (6.25) *we have*

$$\max_{w \in W_R \setminus \{0\}} \frac{< Bw, \lambda >^2}{\|w\|_W^2} \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \|\lambda\|_{M_R}^2,$$

where C is a constant independent of ρ_i's, h_i's and H_i's.

Remark 6.10 *The result is the same as* $2D$ *case. However, we need an additional assumption on the ratio of meshes for* $3D$ *problems.*

Now, we restrict ourselves to the elliptic problems with coefficients $A(x)$ and $\beta(x)$ that do not change rapidly across subdomain interfaces or with discontinuous coefficients ρ_i's. From Remark 6.8 and Lemma 6.9, we can see that the term r_i disappears on the condition number bound for those cases. From Lemma 6.2, Lemma 6.4, Lemma 6.7 and Lemma 6.9, we have the following result.

Theorem 6.11 *Assume that the elliptic problem has coefficients* $A(x)$ *and* $\beta(x)$ *which do not change rapidly across subdomain interfaces or the elliptic problem has discontinuous coefficients* ρ_i*'s. Then, for* $\lambda \in M_R$*,*

$$< \widehat{F}_{DP}\lambda, \lambda > \leq < F_{DP}\lambda, \lambda > \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\} < \widehat{F}_{DP}\lambda, \lambda >,$$

where C *is a constant depending on* $A(x)$ *and* $\beta(x)$*, but independent of* H_i*'s and* h_i*'s. For the elliptic problems with discontinuous coefficients* ρ_i*'s, the constant* C *is independent of the coefficients.*

From (6.14) and the above theorem, we obtain the condition number bound:

Corollary 6.12 *Under the assumption of Theorem 6.11, we have*

$$\kappa(\widehat{F}_{DP}^{-1} F_{DP}) \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\},$$

where the constant C is the same as one in the above theorem.

7. Stokes problem in 2D

In this chapter, we consider a FETI-DP formulation of the Stokes problem with mortar methods. Under the conforming discretizations, Li [34, 35] extended the FETI-DP methods to the Stokes problem and linearized Navier-Stokes problem both in $2D$ and $3D$. The analysis of the mortar methods for the Stokes problem was done by Belgacem [6].

7.1 A model problem and finite elements

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$. In the following, we consider the Stokes problem: For $\boldsymbol{f} \in [L^2(\Omega)]^2$, find $(\boldsymbol{u}, p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ satisfying

$$\begin{aligned} -\triangle \mathbf{u} + \nabla p &= \mathbf{f} \;\; \text{in } \Omega, \\ -\nabla \cdot \mathbf{u} &= 0 \;\; \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \;\; \text{on } \partial\Omega. \end{aligned} \tag{7.1}$$

We assume that Ω is partitioned into nonoverlapping bounded polygonal subdomains $\{\Omega_i\}_{i=1}^N$ and the partition is geometrically conforming. For each subdomain, we introduce the following Sobolev spaces:

$$\begin{aligned} H_D^1(\Omega_i) &:= \left\{ v \in H^1(\Omega_i) \;:\; v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega \right\}, \\ L_0^2(\Omega_i) &:= \left\{ q \in L^2(\Omega_i) \;:\; \int_{\Omega_i} q \, dx = 0 \right\}, \\ \Pi^0 &:= \left\{ q^0 \in L_0^2(\Omega) \;:\; q^0|_{\Omega_i} \text{ is a constant for each } i \right\}. \end{aligned}$$

Then, the variational form of the Stokes problem (7.1) is:

Find $\left(\boldsymbol{u}, p_I, p^0\right) \in \prod_{i=1}^N \left[H_D^1(\Omega_i)\right]^2 \times \prod_{i=1}^N L_0^2(\Omega_i) \times \Pi^0$ such that

$$
\begin{aligned}
\sum_{i=1}^N (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\Omega_i} - \sum_{i=1}^N (p_I + p^0, \nabla \cdot \boldsymbol{v})_{\Omega_i} = \sum_{i=1}^N (\boldsymbol{f}, \boldsymbol{v})_{\Omega_i} \quad \forall\, \boldsymbol{v} \in \prod_{i=1}^N \left[H_D^1(\Omega_i)\right]^2, \\
-\sum_{i=1}^N (\nabla \cdot \boldsymbol{u}, q_I)_{\Omega_i} = 0 \quad \forall\, q_I \in \prod_{i=1}^N L_0^2(\Omega_i), \\
-\sum_{i=1}^N (\nabla \cdot \boldsymbol{u}, q^0)_{\Omega_i} = 0 \quad \forall\, q^0 \in \Pi^0,
\end{aligned}
\tag{7.2}
$$

and the velocity $\boldsymbol{u}$ is continuous across the subdomain interfaces $\Gamma = \bigcup_{i,j=1}^N (\partial\Omega_i \cap \partial\Omega_j)$. Here, $(\cdot,\cdot)_{\Omega_i}$ denotes the inner product in $[L^2(\Omega_i)]^n$ for $n = 1$ or 2.

For each subdomain Ω_i, we consider a quasi-uniform triangulation $\Omega_i^{2h_i}$ with the maximum diameter $2h_i$. After bisecting each edge of triangles in $\Omega_i^{2h_i}$, we obtain a finer triangulation $\Omega_i^{h_i}$ from $\Omega_i^{2h_i}$. Note that these triangulations need not match across the subdomain interfaces. From these triangulations, we consider the inf-sup stable $P_1(h_i) - P_0(2h_i)$ finite elements in each subdomain Ω_i and let

$$
\begin{aligned}
X_i &:= \left\{ \boldsymbol{v}_i \in \left[H_D^1(\Omega_i) \cap C^0(\Omega_i)\right]^2 \;:\; \boldsymbol{v}_i|_\tau \in [P_1(\tau)]^2 \;\; \forall\, \tau \in \Omega_i^{h_i} \right\}, \\
Q_i &:= \left\{ q_i \in L^2(\Omega_i) \;:\; q_i|_\tau \in P_0(\tau) \;\; \forall\, \tau \in \Omega_i^{2h_i} \right\}, \\
Q_i^0 &:= Q_i \cap L_0^2(\Omega_i),
\end{aligned}
$$

where $P_l(\tau)$ is a set of polynomials of degree $\leq l$ in τ.

To get a FETI-DP formulation, we define the following spaces:

$$
\begin{aligned}
&X := \left\{ \boldsymbol{v} \in \prod_{i=1}^N X_i \;:\; \boldsymbol{v} \text{ is continuous at subdomain corners} \right\}, \\
&Q := \prod_{i=1}^N Q_i^0, \\
&W_i := X_i|_{\partial\Omega_i} \quad \text{for } i = 1, \cdots, N, \\
&W = \left\{ \boldsymbol{w} \in \prod_{i=1}^N W_i \;:\; \boldsymbol{w} \text{ is continuous at subdomain corners} \right\}.
\end{aligned}
$$

For $\boldsymbol{v} = (\boldsymbol{v}_1^t, \cdots, \boldsymbol{v}_N^t)^t \in X$, we write

$$\boldsymbol{v}_i = \begin{pmatrix} \boldsymbol{v}_I^i \\ \boldsymbol{v}_r^i \\ \boldsymbol{v}_c^i \end{pmatrix},$$

where the symbol I, r and c represent the d.o.f. on interior, on edges and at corners(vertices), respectively. Since $\boldsymbol{v}$ is continuous at subdomain corners, there exists a vector $\boldsymbol{v}_c$ satisfying $\boldsymbol{v}_c^i = L_c^i \boldsymbol{v}_c$ for all $i = 1, \cdots, N$, with a restriction map L_c^i. The vector $\boldsymbol{v}_c$ has the d.o.f. corresponding to the union of subdomain corners. Let

$$\boldsymbol{v}_I^t = \begin{pmatrix} (\boldsymbol{v}_I^1)^t & \cdots & (\boldsymbol{v}_I^N)^t \end{pmatrix}, \boldsymbol{v}_r^t = \begin{pmatrix} (\boldsymbol{v}_r^1)^t & \cdots & (\boldsymbol{v}_r^N)^t \end{pmatrix}.$$

We define the spaces X_I, W_r and W_c which consist of vectors $\boldsymbol{v}_I$, $\boldsymbol{v}_r$ and $\boldsymbol{v}_c$, respectively. Similarly, for $\boldsymbol{w} \in W$, we define $\boldsymbol{w}_r \in W_r$ and $\boldsymbol{w}_c \in W_c$.

Note that the space X is not contained in $[H_0^1(\Omega)]^2$. To approximate the solution of the problem (7.1) in the space X, we impose the mortar matching condition on the velocity functions. Let $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. Since the triangulations are different across Γ_{ij}, we distinguish them by choosing one as a mortar side and the other as a nonmortar side. Then the index sets m_i and s_i are defined as (4.3). We may write

$$\overline{\partial\Omega_i \setminus \partial\Omega} = (\bigcup_{j\in m_i} \Gamma_{ij}) \bigcup (\bigcup_{j \in s_i} \Gamma_{ij}).$$

Now, we define the following spaces from the finite elements on the nonmortar sides of interfaces:

$$\begin{aligned} W_{ij} &:= W_i|_{\Gamma_{ij}} \quad \text{for } j \in m_i,\ i = 1, \cdots, N, \\ W_{ij}^0 &:= \{\boldsymbol{w}_{ij} \in W_{ij} \ : \ \boldsymbol{w}_{ij} \text{ vanishes at the end points of } \Gamma_{ij}\}, \\ W^0 &:= \prod_{i=1}^N \prod_{j\in m_i} W_{ij}^0 \end{aligned}$$

and consider the Lagrange multiplier space M_{ij} introduced in Section 4.2. More precisely, the standard Lagrange multiplier space M_{ij} is defined as

$$M_{ij} := \{\boldsymbol{\psi} \in \left[C^0(\Gamma_{ij})\right]^2 \ : \ \boldsymbol{\psi}|_\tau \in [P_l(\tau)]^2, \text{ if } \tau \cap \partial\Gamma_{ij} = \emptyset,\ l = 1,$$
$$\text{otherwise } l = 0, \forall \tau \in T_{ij}\},$$

where T_{ij} is a triangulation on Γ_{ij} inherited from the nonmortar side of Γ_{ij}. Then we take the Lagrange multiplier space

$$M := \prod_{i=1}^{N} \prod_{j \in m_i} M_{ij}$$

and impose the following mortar matching condition on the velocity functions:

For $\boldsymbol{v} = (\boldsymbol{v}_1, \cdots, \boldsymbol{v}_N) \in X$, $\boldsymbol{v}$ satisfies that

$$\int_{\Gamma_{ij}} (\boldsymbol{v}_i - \boldsymbol{v}_j) \cdot \boldsymbol{\lambda}_{ij} \, ds = 0 \quad \forall \boldsymbol{\lambda}_{ij} \in M_{ij}, \forall\, i = 1, \cdots, N, \forall j \in m_i. \tag{7.3}$$

Let us define the spaces

$$V := \{\boldsymbol{v} \in X \;:\; \boldsymbol{v} \text{ satisfies } (7.3)\},$$
$$P := \left\{q \in L_0^2(\Omega) \;:\; q|_{\Omega_i} \in Q_i \;\; \forall\, i = 1, \cdots, N\right\}$$

for the velocity and pressure, respectively. The space P is written into a direct sum of the L^2-orthogonal spaces Q and Π^0, that is,

$$P = Q \oplus \Pi^0.$$

When Hood-Taylor finite elements $P_2(h) - P_1(h)$ are used for each subdomain, the spaces M, V and P are defined similarly to the $P_1(h) - P_0(2h)$ finite elements. It was shown in [6] that the best approximation property holds for the approximation space $V \times P$ with Hood-Taylor finite elements. The inf-sup constant of the space $V \times P$ is crucial in the analysis of the approximation order. If the inf-sup constant is independent of mesh size and subdomain size then the best approximation property holds. In [6], it was shown that the inf-sup constant is independent of the mesh size. However, it was not proved for the subdomain size. Following the similar idea to Belgacem [6], we can see that the inf-sup constant of the space $V \times P$ with $P_1(h) - P_0(2h)$ finite elements is independent of the mesh size . For the subdomain size H, we compute the inf-sup constant numerically and observe that the constant seems to be independent of H (see Section 8.2).

Now, we rewrite (7.3) into a matrix form. Let B_i^{ij} be a matrix with entries

$$(B_i^{ij})_{lk} = \pm \int_{\Gamma_{ij}} \boldsymbol{\psi}_l \cdot \boldsymbol{\phi}_k \, ds \quad \forall l = 1, \cdots, L, \forall k = 1, \cdots, K, \tag{7.4}$$

where $\{\boldsymbol{\psi}_l\}_{l=1}^L$ is a basis for M_{ij} and $\{\boldsymbol{\phi}_k\}_{k=1}^K$ is a nodal basis for $W_i|_{\Gamma_{ij}}$. Here, $W_i|_{\Gamma_{ij}}$ means the restriction of functions in W_i on Γ_{ij}. In (7.4), the +sign is chosen if $\Omega_i|_{\Gamma_{ij}}$ is a nonmortar side, otherwise the −sign is chosen. Then we rewrite (7.3) as

$$B_i^{ij}\boldsymbol{v}_i|_{\Gamma_{ij}} + B_j^{ij}\boldsymbol{v}_j|_{\Gamma ij} = \mathbf{0} \quad \forall i = 1, \cdots, N, \forall j \in m_i. \tag{7.5}$$

Define $E_{ij} : M_{ij} \to M$ to be an extension operator by zero and $R_{ij}^l : W_l \to W_l|_{\Gamma_{ij}}$ for $l = i, j$ to be a restriction operator and let $B_i = \sum_{j \in m_i \cup s_i} E_{ij} B_i^{ij} R_{ij}^i$. Then (7.5) becomes

$$B\boldsymbol{w} = \mathbf{0}, \tag{7.6}$$

where

$$B = \begin{pmatrix} B_1 & \cdots & B_N \end{pmatrix},$$
$$\boldsymbol{w} = \begin{pmatrix} \boldsymbol{w}_1^t & \cdots & \boldsymbol{w}_N^t \end{pmatrix}^t \text{ with } \boldsymbol{w}_i = \boldsymbol{v}_i|_{\partial\Omega_i},\ \forall i = 1, \cdots, N.$$

Let $B_{i,r}$ and $B_{i,c}$ be matrices that consist of the columns of B_i corresponding to the d.o.f. on edges and corners, respectively. Then, using the notations introduced in Section 7.1, (7.6) is written into

$$B_r\boldsymbol{w}_r + B_c\boldsymbol{w}_c = \mathbf{0}, \tag{7.7}$$

where $B_r = \begin{pmatrix} B_{1,r} & \cdots & B_{N,r} \end{pmatrix}$ and $B_c = \sum_{i=1}^N B_{i,c} L_c^i$.

7.2 FETI-DP formulation

7.2.1 FETI-DP operator

In this section, we formulate a FETI-DP operator with the continuity constraints (7.7) which are obtained from the mortar matching condition (7.3). To solve the Stokes problem efficiently and correctly, we will add the redundant continuity constraints to the coarse problem:

$$\int_{\Gamma_{ij}} (\boldsymbol{v}_i - \boldsymbol{v}_j)\, ds = \mathbf{0} \quad \forall i = 1, \cdots, N, \forall j \in m_i. \tag{7.8}$$

In the FETI-DP method, the mortar matching condition holds when the solution has converged. Hence, the convergence of the FETI-DP method is enhanced by adding the redundant constraints to the coarse problem. When preconditioning the FETI-DP operator, we solve a Dirichlet problem, i.e. a local Stokes problem, in each subdomain. Furthermore, the compatibility condition of the local Stokes problem follows from the redundant constraints.

We rewrite (7.8) as

$$R^t(B_r\boldsymbol{w}_r + B_c\boldsymbol{w}_c) = \mathbf{0}, \tag{7.9}$$

where the matrix R has the number of columns corresponding to two times of the number of Γ_{ij}'s(interfaces) and rows corresponding to the d.o.f. on the space M and has entries 1 and 0. For $\boldsymbol{\lambda} \in M$, at each interior nodal point of Γ_{ij}, $\boldsymbol{\lambda}|_{\Gamma_{ij}}$ has two components corresponding to horizonal and vertical parts of velocity function. For $\boldsymbol{\lambda} \in M$, $R^t\boldsymbol{\lambda} = \mathbf{0}$ means that for all Γ_{ij}, the sums of $\boldsymbol{\lambda}|_{\Gamma_{ij}}$ corresponding to each horizonal and vertical parts of velocity function are zero.

Let U be the Lagrange multiplier space corresponding to the constraints (7.9) and for $\boldsymbol{\mu} \in U$, $\boldsymbol{\mu}|_{\Gamma_{ij}}$ has two components that correspond to the constraints for horizontal velocity and vertical velocity. Introducing Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ to enforce the constraints (7.7) and (7.9), the followings are induced from the Galerkin approximation to (7.2):

Find $(\boldsymbol{u}_I, p_I, \boldsymbol{u}_r, \boldsymbol{u}_c, p^0, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in X_I \times Q \times W_r \times W_c \times \Pi^0 \times U \times M$ such that

$$\begin{pmatrix} A_{II} & G_{II} & A_{Ir} & A_{Ic} & G_{I0} & \mathbf{0} & \mathbf{0} \\ G_{II}^t & \mathbf{0} & G_{rI}^t & G_{cI}^t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{rI} & G_{rI} & A_{rr} & A_{rc} & G_{r0} & B_r^t R & B_r^t \\ A_{cI} & G_{cI} & A_{cr} & A_{cc} & G_{c0} & B_c^t R & B_c^t \\ G_{I0}^t & \mathbf{0} & G_{r0}^t & G_{c0}^t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & R^t B_r & R^t B_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_r & B_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_I \\ p_I \\ \boldsymbol{u}_r \\ \boldsymbol{u}_c \\ p^0 \\ \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_I \\ \mathbf{0} \\ \boldsymbol{f}_r \\ \boldsymbol{f}_c \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{7.10}$$

where

$$\begin{pmatrix} A_{II} & A_{Ir} & A_{Ic} \\ A_{rI} & A_{rr} & A_{rc} \\ A_{cI} & A_{cr} & A_{cc} \end{pmatrix} \text{ is a stiffness matrix induced from } \sum_{i=1}^{N} (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{\Omega_i},$$

$$\begin{pmatrix} G_{II} \\ G_{rI} \\ G_{cI} \end{pmatrix} \text{ is a matrix induced from } \sum_{i=1}^{N} (-\nabla \cdot \boldsymbol{v}, p_I)_{\Omega_i},$$

$$\begin{pmatrix} G_{I0} \\ G_{r0} \\ G_{c0} \end{pmatrix} \text{ is a matrix induced from } \sum_{i=1}^{N} (-\nabla \cdot \boldsymbol{v}, p^0)_{\Omega_i}$$

and the subscripts I, r and c denote the interior, edges and corners, respectively. Since $p^0|_{\Omega_i}$ is constant, we have $G_{I0} = \mathbf{0}$. Let

$$\boldsymbol{z}_r = \begin{pmatrix} \boldsymbol{u}_I \\ p_I \\ \boldsymbol{u}_r \end{pmatrix}, \boldsymbol{z}_c = \begin{pmatrix} \boldsymbol{u}_c \\ p^0 \\ \boldsymbol{\mu} \end{pmatrix}.$$

We regard $\boldsymbol{z}_c$ as a primal variable. Then (7.10) can be written as

$$\begin{pmatrix} K_{rr} & K_{rc} & \widetilde{B}_r^t \\ K_{rc}^t & K_{cc} & \widetilde{B}_c^t \\ \widetilde{B}_r & \widetilde{B}_c & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{z}_r \\ \boldsymbol{z}_c \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \widetilde{\boldsymbol{f}}_r \\ \widetilde{\boldsymbol{f}}_c \\ \mathbf{0} \end{pmatrix}.$$

After eliminating $\boldsymbol{z}_r$, we obtain the following equation for $\boldsymbol{z}_c$ and $\boldsymbol{\lambda}$:

$$\begin{pmatrix} -F_{cc} & F_{cl} \\ F_{cl}^t & F_{ll} \end{pmatrix} \begin{pmatrix} \boldsymbol{z}_c \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -\mathbf{d}_c \\ \mathbf{d}_l \end{pmatrix}$$

where

$$\begin{aligned} F_{ll} &= \widetilde{B}_r K_{rr}^{-1} \widetilde{B}_r^t, \\ F_{cl} &= K_{rc}^t K_{rr}^{-1} \widetilde{B}_r^t - \widetilde{B}_c^t, \\ F_{cc} &= K_{cc} - K_{rc}^t K_{rr}^{-1} K_{rc}, \\ \mathbf{d}_l &= \widetilde{B}_r K_{rr}^{-1} \widetilde{\boldsymbol{f}}_r, \\ \mathbf{d}_c &= \widetilde{\boldsymbol{f}}_c - K_{rc}^t K_{rr}^{-1} \widetilde{f}_r. \end{aligned}$$

Note that $\begin{pmatrix} G_{r0} & B_r^t R \\ G_{c0} & B_c^t R \end{pmatrix} \begin{pmatrix} p^0 \\ \mu \end{pmatrix} = \mathbf{0}$ implies that $\begin{pmatrix} p^0 \\ \mu \end{pmatrix} = \mathbf{0}$. Using this it can be shown easily that F_{cc} is invertible. Hence eliminating $\boldsymbol{z}_c$, we obtain the following equation for $\boldsymbol{\lambda}$:

$$(F_{ll} + F_{cl}^t F_{cc}^{-1} F_{cl})\boldsymbol{\lambda} = \mathbf{d}_l - F_{cl}^t F_{cc}^{-1} \mathbf{d}_c. \tag{7.11}$$

Let $F_{DP} = F_{ll} + F_{cl}^t F_{cc}^{-1} F_{cl}$ and call it the FETI-DP operator. Since we add the redundant constraints to the coarse problem, $\boldsymbol{\lambda}$ is not uniquely determined in M. Let us define

$$M_R = \left\{ \boldsymbol{\lambda} \in M \;:\; R^t \boldsymbol{\lambda} = \mathbf{0} \right\}. \tag{7.12}$$

In Section 7.3, we will show that F_{DP} is s.p.d. on M_R and $\boldsymbol{\lambda} \in M_R$ is uniquely determined. In the following section, we define several norms on the finite element function spaces and propose a preconditioner for the operator F_{DP}.

7.2.2 Preconditioner

For $\boldsymbol{w}_i \in W_i$, we define $S_i \boldsymbol{w}_i$ by

$$\begin{pmatrix} A_{II}^i & G_{II}^i & A_{Ir}^i & A_{Ic}^i \\ {G_{II}^i}^t & \mathbf{0} & {G_{rI}^i}^t & {G_{cI}^i}^t \\ A_{rI}^i & G_{rI}^i & A_{rr}^i & A_{rc}^i \\ A_{cI}^i & G_{cI}^i & A_{cr}^i & A_{cc}^i \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_I^i \\ p_I^i \\ \boldsymbol{w}_r^i \\ \boldsymbol{w}_c^i \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ S_i \begin{pmatrix} \boldsymbol{w}_r^i \\ \boldsymbol{w}_c^i \end{pmatrix} \end{pmatrix},$$

where the superscript i denotes submatrices corresponding to the subdomain Ω_i.

Let us define

$$S := \mathrm{diag}(S_1, \cdots, S_N)$$

and it can be seen easily that S is s.p.d. on W. Hence, we define

$$\|\boldsymbol{w}\|_W := \left(\sum_{i=1}^N < S_i \boldsymbol{w}_i, \boldsymbol{w}_i > \right)^{1/2} \tag{7.13}$$

as a norm for $\boldsymbol{w} \in W$. For a function $\boldsymbol{w}_{ij} \in W_{ij}^0$ with $j \in m_i$, let $\widetilde{\boldsymbol{w}}_{ij}$ be the zero extension of $\boldsymbol{w}_{ij}$ into W_i. Using this, for $\boldsymbol{w} \in W^0$ we define an extension $\widetilde{\boldsymbol{w}} \in W$ by

$$\widetilde{\boldsymbol{w}} = (\widetilde{\boldsymbol{w}}_1, \cdots, \widetilde{\boldsymbol{w}}_N) \text{ with } \widetilde{\boldsymbol{w}}_i = \sum_{j \in m_i} \widetilde{\boldsymbol{w}}_{ij} \; \forall i = 1, \cdots, N,$$

and define a norm on W^0 as

$$\|\boldsymbol{w}\|_{W^0} := \|\widetilde{\boldsymbol{w}}\|_W. \tag{7.14}$$

We introduce the following subspaces with the norms induced from the spaces W and W^0:

$$\begin{aligned} W_R &:= \left\{\boldsymbol{w} \in W \ : \ R^t(B_r\boldsymbol{w}_r + B_c\boldsymbol{w}_c) = \boldsymbol{0}\right\}, \\ W_{R,G} &:= \left\{\boldsymbol{w} \in W_R \ : \ G_{r0}^t\boldsymbol{w}_r + G_{c0}^t\boldsymbol{w}_c = \boldsymbol{0}\right\} \\ W_R^0 &:= \left\{\boldsymbol{w} \in W^0 \ : \ \widetilde{\boldsymbol{w}} \in W_R\right\}. \end{aligned}$$

Recall the definition of M_R in (7.12) and let $< \cdot, \cdot >_m$ be a duality pairing between M_R and W_R^0 defined as

$$< \boldsymbol{\lambda}, \boldsymbol{w} >_m = \sum_{i=1}^N \sum_{j \in m_i} \int_{\Gamma_{ij}} \boldsymbol{\lambda}_{ij} \cdot \boldsymbol{w}_{ij} \, ds.$$

Then we define a dual norm for $\boldsymbol{\lambda} \in M_R$ by

$$\|\boldsymbol{\lambda}\|_{M_R}^2 := \max_{\boldsymbol{w} \in W_R^0 \setminus \{0\}} \frac{< \boldsymbol{\lambda}, \boldsymbol{w} >_m^2}{\|\boldsymbol{w}\|_{W^0}^2}. \tag{7.15}$$

Now, we will find an operator $\widehat{F}_{DP}$ which gives

$$< \widehat{F}_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} > = \|\boldsymbol{\lambda}\|_{M_R}^2 \tag{7.16}$$

and propose $\widehat{F}_{DP}^{-1}$ as a preconditioner for the FETI-DP operator in (7.11). Define $R_{ij} : W^0 \to W_{ij}^0$ as a restriction operator and $E_{ij}^i : W_{ij}^0 \to W_i$ as an extension operator by zero. Then for $\boldsymbol{w} \in W_R^0$,

$$\begin{aligned} \|\boldsymbol{w}\|_{W^0}^2 &= \|\widetilde{\boldsymbol{w}}\|_W^2 \\ &= \sum_{i=1}^N < S_i\widetilde{\boldsymbol{w}}_i, \widetilde{\boldsymbol{w}}_i > \\ &= \sum_{i=1}^N < S_i(\sum_{j \in m_i} E_{ij}^i R_{ij}\boldsymbol{w}), \sum_{j \in m_i} E_{ij}^i R_{ij}\boldsymbol{w} > . \end{aligned}$$

Let $\widehat{S} = \sum_{i=1}^{N} (\sum_{j\in m_i} E_{ij}^i R_{ij})^t S_i (\sum_{j\in m_i} E_{ij}^i R_{ij})$. Moreover, we have

$$< \boldsymbol{\lambda}, \boldsymbol{w} >_m = < \widehat{B}\boldsymbol{w}, \boldsymbol{\lambda} > \tag{7.17}$$

where $\widehat{B} = \text{diag}_{i=1,\cdots,N}\left(\text{diag}_{j\in m_i} \widehat{B}_i^{ij}\right)$ and $\widehat{B}_i^{ij}$ is a matrix obtained from B_i^{ij} after deleting the columns corresponding to the d.o.f. at the end points of Γ_{ij}. Note that $\widehat{B}_i^{ij}$ is invertible. Since, we restrict $\boldsymbol{\lambda} \in M_R$ and $\boldsymbol{w} \in W_R^0$, to find $\widehat{F}_{DP}$ in a matrix form we need the following l^2-orthogonal projections:

$$P_{W_R^0} : W^0 \to W_R^0,$$
$$P_{M_R} : M \to M_R.$$

For $\boldsymbol{\lambda} \in M_R$ and $\boldsymbol{w} \in W_R^0$, we may write

$$< \boldsymbol{\lambda}, \boldsymbol{w} >_m = < \widehat{B}_p\boldsymbol{w}, \boldsymbol{\lambda} >, \quad \|\boldsymbol{w}\|_{W^0}^2 = < \widehat{S}_p\boldsymbol{w}, \boldsymbol{w} >, \tag{7.18}$$

where

$$\widehat{S}_p = P_{W_R^0}^t \widehat{S} P_{W_R^0}, \quad \widehat{B}_p = P_{M_R}^t \widehat{B} P_{W_R^0}.$$

Then it can be shown that the operators

$$\widehat{S}_p : W^0 \to W_R^0,$$
$$\widehat{B}_p : W^0 \to M_R$$

are invertible on W_R^0 and $\widehat{S}_p$ is s.p.d. on W_R^0. Hence, using (7.18), the maximum in (7.15) occurs when $\boldsymbol{w} \in W_R^0$ satisfies $\widehat{S}_p\boldsymbol{w} = \widehat{B}_p^t\boldsymbol{\lambda}$. Therefore, we have

$$\|\boldsymbol{\lambda}\|_{M_R}^2 = < \widehat{B}_p\widehat{S}_p^{-1}\widehat{B}_p^t\boldsymbol{\lambda}, \boldsymbol{\lambda} > .$$

Let

$$\widehat{F}_{DP}^{-1} = (\widehat{B}_p\widehat{S}_p^{-1}\widehat{B}_p^t)^{-1} = (\widehat{B}_p^t)^{-1}\widehat{S}_p\widehat{B}_p^{-1}.$$

and we call it a Neumann-Dirichlet preconditioner for the operator F_{DP}. Define l^2-orthogonal projections

$$P_{W_R^0}^{ij} : W^0|_{\Gamma_{ij}} \to W_R^0|_{\Gamma_{ij}},$$
$$P_{M_R}^{ij} : M|_{\Gamma_{ij}} \to M_R|_{\Gamma_{ij}}.$$

Then the projection operators $P_{W_R^0}$ and P_{M_R} are composed of diagonal blocks of $P_{W_R^0}^{ij}$'s and $P_{M_R}^{ij}$'s, respectively. Moreover, it can be shown easily that

$$(P_{M_R}^{ij})^t \widehat{B}_i^{ij} P_{W_R^0}^{ij} : W_R^0|_{\Gamma_{ij}} \to M_R|_{\Gamma_{ij}}$$

is invertible. Hence, it follows that

$$\widehat{B}_p^{-1} = \text{diag}_{i=1,\cdots,N}\text{diag}_{j\in m_i}\left(\widehat{B}_{ij}^{-1}\right),$$

where $\widehat{B}_{ij} = (P_{M_R}^{ij})^t \widehat{B}_i^{ij} P_{W_R^0}^{ij}$ and

$$\widehat{F}_{DP}^{-1} = \sum_{i=1}^{N} \left(\sum_{j\in m_i} E_{ij}^i \widehat{B}_{ij}^{-1} R_{ij} \right)^t S_i \left(\sum_{j\in m_i} E_{ij}^i \widehat{B}_{ij}^{-1} R_{ij} \right).$$

Therefore, the computation of $\widehat{F}_{DP}^{-1}\boldsymbol{\lambda}$ can be done parallely in each subdomain.

7.3 Condition number bound estimation

Lemma 7.1 *We have*

$$B(W_{R,G}) = B(W_R) = M_R.$$

Proof. Since $W_{R,G} \subset W_R$, $B(W_{R,G}) \subset B(W_R)$.

Now, we will show that $B(W_R) \subset B(W_{R,G})$. Let $\widetilde{\boldsymbol{w}} = (\widetilde{\boldsymbol{w}}_1, \cdots, \widetilde{\boldsymbol{w}}_N) \in W$ be the zero extension of $\boldsymbol{w} \in W^0$. Since $\widetilde{\boldsymbol{w}}_j|_{\Gamma_{ij}} = 0$ for $j \in m_i$ and $\widetilde{\boldsymbol{w}}$ is zero at subdomain corners, we have

$$B\widetilde{\boldsymbol{w}} = \widehat{B}\boldsymbol{w}, \tag{7.19}$$

with $\widehat{B}$ as defined in (7.17). From the fact that $\widehat{B}$ is a $1-1$ mapping from W^0 onto M and the definitions of W_R^0 and M_R, we get

$$\widehat{B}(W_R^0) = M_R. \tag{7.20}$$

For $\boldsymbol{w} \in W_R^0$, the zero extension $\widetilde{\boldsymbol{w}} = (\widetilde{\boldsymbol{w}}_1, \cdots, \widetilde{\boldsymbol{w}}_N)$ satisfies

$$\int_{\partial\Omega_i} \widetilde{\boldsymbol{w}}_i \, ds = \mathbf{0} \quad \forall\, i = 1, \cdots, N$$

and then applying the divergence theorem

$$G_{r0}^t \widetilde{\boldsymbol{w}}_r + G_{c0}^t \widetilde{\boldsymbol{w}}_c = \mathbf{0}$$

holds for $\widetilde{\boldsymbol{w}}$. Hence, for $\boldsymbol{w} \in W_R^0$, we have $\widetilde{\boldsymbol{w}} \in W_{R,G}$ and from (7.19) we obtain

$$\widehat{B}(W_R^0) \subset B(W_{R,G}). \tag{7.21}$$

From the definitions of W_R and M_R,

$$B(W_R) = M_R. \tag{7.22}$$

Combining (7.22), (7.20) and (7.21), we have $B(W_R) \subset B(W_{R,G})$. ■

Remark 7.2 *For $\boldsymbol{w} \in W_R^0$, we have $\widetilde{\boldsymbol{w}} \in W_{R,G}$.*

Lemma 7.3 *For $\boldsymbol{\lambda} \in M_R$, we have*

$$< F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} >= \max_{\boldsymbol{w}\in W_{R,G}\setminus\{0\}} \frac{< B\boldsymbol{w}, \boldsymbol{\lambda} >^2}{\|\boldsymbol{w}\|_W^2}.$$

Proof. The problem (7.10) is equivalent to solving the following min-max problem:

$$\max_{\boldsymbol{\lambda}\in B(W_{R,G})} \min_{\boldsymbol{w}\in W_{R,G}} \left\{ \sum_{i=1}^{N} \left(\frac{1}{2} < S_i \boldsymbol{w}_i, \boldsymbol{w}_i > - < \mathbf{d}_i, \boldsymbol{w}_i > \right) + < B\boldsymbol{w}, \boldsymbol{\lambda} > \right\}, \tag{7.23}$$

where $\mathbf{d}_i$ is the Schur complement forcing vector obtained from $\begin{pmatrix} \boldsymbol{f}_I^t & \mathbf{0}^t & \boldsymbol{f}_r^t & \boldsymbol{f}_c^t \end{pmatrix}^t$ after solving Stokes problem in each subdomain Ω_i.

Let $P_{W_{R,G}}$ be the l^2-orthogonal projection from W onto $W_{R,G}$. Recall that $B(W_{R,G}) = M_R$ from Lemma 7.1 and P_{M_R} is the projection operator from M onto M_R introduced in Section 7.2.2. Then taking Euler-Lagrangian in (7.23), we obtain

$$\begin{pmatrix} S_p & B_p^t \\ B_p & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} P_{W_{R,G}}^t \mathbf{d} \\ \mathbf{0} \end{pmatrix}, \tag{7.24}$$

where

$$S_p = P_{W_{R,G}}^t S P_{W_{R,G}}, \quad B_p = P_{M_R}^t B P_{W_{R,G}},$$
$$\mathbf{d} = \begin{pmatrix} \mathbf{d}_1^t & \cdots & \mathbf{d}_N^t \end{pmatrix}^t.$$

Since S_p is s.p.d. on $W_{R,G}$, the equation for $\boldsymbol{\lambda}$ follows by eliminating $\boldsymbol{w}$ in (7.24):

$$B_p S_p^{-1} B_p^t \boldsymbol{\lambda} = B_p S_p^{-1} \mathbf{d}, \tag{7.25}$$

which is the same as (7.11). Therefore we have

$$B_p S_p^{-1} B_p^t = F_{DP}. \tag{7.26}$$

For $\boldsymbol{\lambda} \in M_R$, we consider

$$\max_{\boldsymbol{w} \in W_{R,G} \setminus \{0\}} \frac{< B\boldsymbol{w}, \boldsymbol{\lambda} >^2}{\|\boldsymbol{w}\|_W^2}. \tag{7.27}$$

From (7.13), the definition of $\| \cdot \|_W$, we may write

$$\|\boldsymbol{w}\|_W^2 = < S_p \boldsymbol{w}, \boldsymbol{w} > \quad \text{for } w \in W_{R,G}.$$

Since S_p is s.p.d. on $W_{R,G}$, the maximum in (7.27) occurs when $\boldsymbol{w} \in W_{R,G}$ satisfies $S_p \boldsymbol{w} = B_p^t \boldsymbol{\lambda}$. Hence, we have

$$\max_{\boldsymbol{w} \in W_{R,G} \setminus \{0\}} \frac{< B\boldsymbol{w}, \boldsymbol{\lambda} >^2}{\|\boldsymbol{w}\|_W^2} = < B_p S_p^{-1} B_p^t \boldsymbol{\lambda}, \boldsymbol{\lambda} > . \tag{7.28}$$

Combining (7.26) and (7.28), we complete the proof. ■

Remark 7.4 *For $\boldsymbol{\lambda} \in M_R$, $B_p^t \boldsymbol{\lambda} = \mathbf{0}$ gives $\boldsymbol{\lambda} = \mathbf{0}$ and S_p is s.p.d. on $W_{R,G}$. Hence, from* (7.26)*, we can see that F_{DP} is s.p.d. on M_R.*

Lemma 7.5 *For $\boldsymbol{\lambda} \in M_R$, we have*

$$\max_{\boldsymbol{w} \in W_{R,G} \setminus \{0\}} \frac{< B\boldsymbol{w}, \boldsymbol{\lambda} >^2}{\|\boldsymbol{w}\|_W^2} \geq \|\boldsymbol{\lambda}\|_{M_R}^2.$$

Proof. By definition, we have

$$\|\boldsymbol{\lambda}\|_{M_R}^2 = \max_{\boldsymbol{w} \in W_R^0 \setminus \{0\}} \frac{< \boldsymbol{\lambda}, \boldsymbol{w} >_m^2}{\|\boldsymbol{w}\|_{W^0}^2}. \tag{7.29}$$

Let $\widetilde{\boldsymbol{w}} \in W$ be the zero extension of $\boldsymbol{w} \in W_R^0$. Then, $\widetilde{\boldsymbol{w}} \in W_{R,G}$. Moreover, we get

$$< \boldsymbol{\lambda}, \boldsymbol{w} >_m = < B\widetilde{\boldsymbol{w}}, \boldsymbol{\lambda} > . \tag{7.30}$$

From (7.29) and (7.30), we prove the lemma. ■

Let us define a notation $| \cdot |_{S_i} := < S_i \cdot, \cdot >^{1/2}$. Then the following lemma can be found in Bramble and Pasciak [16].

Lemma 7.6 *For $\boldsymbol{w}_i \in W_i$, we have*

$$C_1\beta|\boldsymbol{w}_i|_{S_i} \leq |\boldsymbol{w}_i|_{1/2,\partial\Omega_i} \leq C_2|\boldsymbol{w}_i|_{S_i},$$

where β is the inf-sup constant for the finite elements of subdomain Ω_i and the constants C_1 and C_2 are independent of h_i and H_i.

Since we have chosen inf-sup stable $P_1(h) - P_0(2h)$ finite elements for each subdomain, the constant β is independent of h_i and H_i. Therefore, we have

$$C_1|\boldsymbol{w}_i|_{S_i} \leq |\boldsymbol{w}_i|_{1/2,\partial\Omega_i} \leq C_2|\boldsymbol{w}_i|_{S_i}, \tag{7.31}$$

where C_1 and C_2 are constants independent of h_i and H_i.

From Lemma 2.11, we have the following result for the space W.

Lemma 7.7 *For $\boldsymbol{w} \in W$, we have*

$$\|\boldsymbol{w}_i - \boldsymbol{w}_j\|^2_{H^{1/2}_{00}(\Gamma_{ij})} \leq C \max_{l\in\{i,j\}} \left\{ \left(1 + \log\frac{H_l}{h_l}\right)^2 \right\} \left(|\boldsymbol{w}_i|^2_{1/2,\partial\Omega_i} + |\boldsymbol{w}_j|^2_{1/2,\partial\Omega_j} \right),$$

where $\boldsymbol{w}_i$ is the restriction of $\boldsymbol{w}$ onto $\partial\Omega_i$ for $i = 1, \cdots, N$ and C is a constant independent of h_i's and H_i's.

Definition 7.8 *We define a projection $\pi_{ij} : [H^{1/2}_{00}(\Gamma_{ij})]^2 \to W^0_{ij}$ for $\boldsymbol{v} \in [H^{1/2}_{00}(\Gamma_{ij})]^2$ by*

$$\int_{\Gamma_{ij}} (\boldsymbol{v} - \pi_{ij}\boldsymbol{v}) \cdot \boldsymbol{\lambda}_{ij}\, ds \; = 0 \quad \forall \boldsymbol{\lambda}_{ij} \in M_{ij}.$$

From (4.11), π_{ij} is a continuous operator on $H^{1/2}_{00}(\Gamma_{ij})$. By extending the result to the product space $[H^{1/2}_{00}(\Gamma_{ij})]^2$, we obtain

$$\|\pi_{ij}\boldsymbol{v}\|_{H^{1/2}_{00}(\Gamma_{ij})} \leq C\|\boldsymbol{v}\|_{H^{1/2}_{00}(\Gamma_{ij})} \quad \forall \boldsymbol{v} \in [H^{1/2}_{00}(\Gamma_{ij})]^2, \tag{7.32}$$

with the constant C independent of H_i's and h_i's.

Lemma 7.9 *For $\boldsymbol{\lambda} \in M_R$, we have*

$$\max_{\boldsymbol{w}\in W_{R,G}\setminus\{0\}} \frac{< B\boldsymbol{w}, \boldsymbol{\lambda} >^2}{\|\boldsymbol{w}\|^2_W} \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log\frac{H_i}{h_i}\right)^2 \right\} \|\boldsymbol{\lambda}\|^2_{M_R},$$

where C is a constant independent of h_i's and H_i's.

Proof. Note that

$$< B\boldsymbol{w}, \boldsymbol{\lambda} >= \sum_{i=1}^{N} \sum_{j \in m_i} \int_{\Gamma_{ij}} (\boldsymbol{w}_i - \boldsymbol{w}_j) \cdot \boldsymbol{\lambda}_{ij} \, ds.$$

Since $\boldsymbol{w}_i - \boldsymbol{w}_j \in [H_{00}^{1/2}(\Gamma_{ij})]^2$, from the definition of π_{ij}, we have

$$< B\boldsymbol{w}, \boldsymbol{\lambda} >= \sum_{i=1}^{N} \sum_{j \in m_i} \int_{\Gamma_{ij}} \pi_{ij}(\boldsymbol{w}_i - \boldsymbol{w}_j) \cdot \boldsymbol{\lambda}_{ij} \, ds. \tag{7.33}$$

Let $\boldsymbol{z}_{ij} = \pi_{ij}(\boldsymbol{w}_i - \boldsymbol{w}_j)$ and $\boldsymbol{z} \in W^0$ with $\boldsymbol{z}|_{\Gamma_{ij}} = \boldsymbol{z}_{ij}$. Since $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in M_{ij}$ and $\boldsymbol{w} \in W_{R,G}$,

$$\int_{\Gamma_{ij}} \boldsymbol{z}_{ij} \, ds = \int_{\Gamma_{ij}} (\boldsymbol{w}_i - \boldsymbol{w}_j) \, ds = 0. \tag{7.34}$$

From (7.34), we can see that $R^t B \widetilde{\boldsymbol{z}} = 0$ with $\widetilde{\boldsymbol{z}} \in W$ as the zero extension of $\boldsymbol{z}$. Hence, $\boldsymbol{z} \in W_R^0$ and (7.33) is the duality pairing between $\boldsymbol{z} \in W_R^0$ and $\boldsymbol{\lambda} \in M_R$. From (7.15), we get

$$< B\boldsymbol{w}, \boldsymbol{\lambda} >^2 = < \boldsymbol{\lambda}, \boldsymbol{z} >_m^2 \leq \|\boldsymbol{\lambda}\|_{M_R}^2 \|\boldsymbol{z}\|_{W^0}^2. \tag{7.35}$$

From (7.14), (7.13), (7.31), (2.1), (7.32) and Lemma 7.7, we obtain

$$\begin{aligned} \|\boldsymbol{z}\|_{W^0}^2 &= \|\widetilde{\boldsymbol{z}}\|_W^2 \\ &= \sum_{i=1}^{N} |\widetilde{\boldsymbol{z}}_i|_{S_i}^2 \\ &\leq C \sum_{i=1}^{N} |\widetilde{\boldsymbol{z}}_i|_{1/2,\partial\Omega_i}^2 \\ &\leq C \sum_{i=1}^{N} \sum_{j \in m_i} \|\boldsymbol{w}_i - \boldsymbol{w}_j\|_{H_{00}^{1/2}(\Gamma_{ij})}^2 \\ &\leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \sum_{i=1}^{N} |\boldsymbol{w}_i|_{1/2,\partial\Omega_i}^2 \\ &\leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \|\boldsymbol{w}\|_W^2. \end{aligned} \tag{7.36}$$

Here, C is a generic constant which is independent of h_i's and H_i's. Combining (7.35) and (7.36), we complete the proof. ■

From Lemma 7.3, Lemma 7.5 and Lemma 7.9, we have

Theorem 7.10 *For $\boldsymbol{\lambda} \in M_R$,*

$$\|\boldsymbol{\lambda}\|_{M_R}^2 \leq < F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} > \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \|\boldsymbol{\lambda}\|_{M_R}^2,$$

where C is a constant independent of h_i's and H_i's.

Consequently, from (7.16) we obtain the following condition number estimate:

Corollary 7.11

$$\kappa(\widehat{F}_{DP}^{-1} F_{DP}) \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\}.$$

8. Numerical results

In this chapter, we provide numerical tests for the FETI-DP formulation developed in this dissertation. The numerical tests are done for elliptic problems in $2D$ and Stokes problem in $2D$. Especially for the elliptic problems, we compare our results with the previously developed FETI-DP formulation and FETI-DP preconditioners. We present the approximate errors as well as the number of iterations in CGM.

8.1 Elliptic problems in 2D

Let $\Omega = [0,1] \times [0,1] \in \mathbb{R}^2$. We consider the following model problem:

$$\begin{aligned} -\nabla \cdot (\alpha(x,y)\nabla u) &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{8.1}$$

where $\alpha(x,y) > 0$ and $f \in L^2(\Omega)$. As mentioned in Section 5.2.1, our formulation is different from that of Dryja and Widlund [21, 22]. We compare these two formulations for the same problem on matching and nonmatching discretizations both. To compare them, we consider the elliptic problem with continuous coefficients. Further, we show the efficiency of the Neumann-Dirichlet preconditioner compared with the existing FETI preconditioners for the elliptic problems with highly discontinuous coefficients.

To distinguish our FETI-DP formulation from that of Dryja and Widlund, we denote them by F_{KL} and F_{DW}, respectively. Also, for the preconditioners, we use the notation $\widehat{F}_{KL}^{-1}$ for our preconditioner, that is, the Neumann-Dirichlet preconditioner, and $\widehat{F}_{DW}^{-1}$ for Dryja and Widlund's. The preconditioner $\widehat{F}_{DW}^{-1}$ has the form

$$\widehat{F}_{DW}^{-1} = (B_r \widetilde{B}_r^t)^{-1} \widetilde{B}_r S_{rr} \widetilde{B}_r^t (\widetilde{B}_r B_r^t)^{-1},$$

where $\widetilde{B}_r$ is the scaled matrix of B_r divided by the mesh parameters of each subdomains (see (3.13) in [22]). We also consider a preconditioner $\widehat{F}_{KW}^{-1}$ by Klawonn and

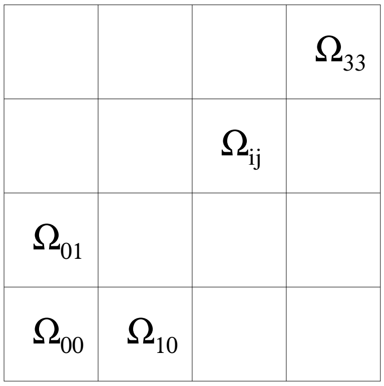

Figure 8.1: Partition of subdomains when $N = 4 \times 4$

Widlund [31], which was developed for solving the heterogeneous coefficient elliptic problems with FETI formulation. Stefanica [48] observed that this preconditioner is the most efficient one for the FETI formulation with mortar constraints. We adapt the preconditioner to the FETI-DP formulation with mortar methods and compare it with the preconditioners $\widehat{F}_{KL}^{-1}$ and $\widehat{F}_{KW}^{-1}$ for elliptic problems with highly discontinuous coefficients. The preconditioner $\widehat{F}_{KW}^{-1}$ is given by

$$\widehat{F}_{KW}^{-1} = (B_r D_r^{-1} B_r^t)^{-1} B_r D_r^{-1} S_{rr} D_r^{-1} B_r^t (B_r D_r^{-1} B_r^t)^{-1}, \qquad (8.2)$$

where D_r is a diagonal matrix whose entries are determined by the coefficients of the elliptic problem. The matrix D_r will be described later.

8.1.1 An elliptic problem with smooth coefficients

We consider an elliptic problem with smooth coefficients. Simply, we take $\alpha(x, y) = 1$ and the exact solution $u(x, y) = y(1-y) \sin \pi x$ in (8.1). In CG(Conjugate Gradient) iteration, the stopping criterion is when the relative residual is less than 10^{-6}. We use n to denote the number of nodes on edges including end points and N to denote the number of subdomains. In this problem, we use the same n for all subdomains, divide Ω into rectangular subdomains as Figure 8.1 and denote each subdomain by Ω_{ij}.

To make nonmatching grids across subdomain interfaces, we generate triangulations in each subdomain in the following way: For each subdomain, we have chosen

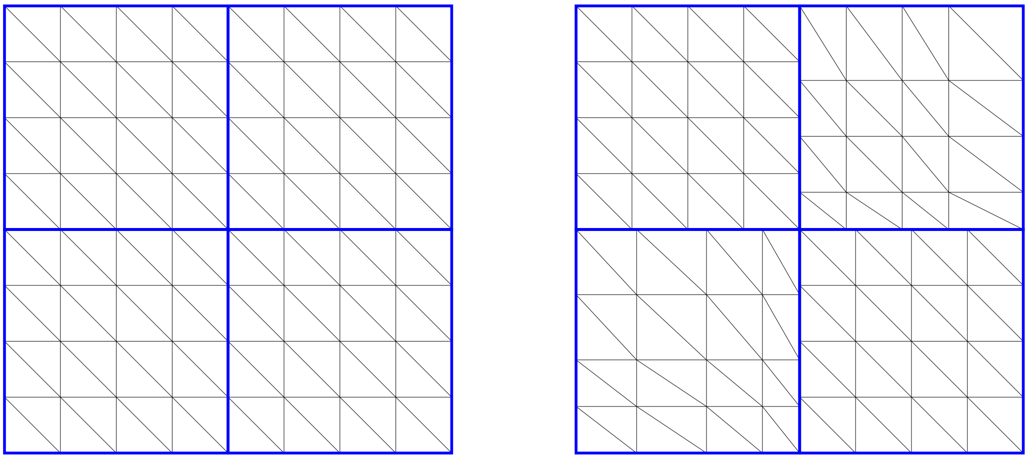

Figure 8.2: Matching grids(left) and nonmatching grids(right) when $n = 5$

n random quasi-uniform nodes on each horizontal and vertical edges. Using these nodes, we generate nonuniform structured grids in each subdomain. Since we use the same n for all subdomains, the sizes of meshes between neighboring subdomains are comparable. For matching grids, we use uniform meshes. Figure 8.2 shows examples of matching and nonmatching grids.

First, we divide Ω into $N = 4\times4$ subdomains and increase the number of nodes n. Table 8.1 shows L^2 and H^1-errors and the number of CG iterations between those two formulations on both matching and nonmatching discretizations. For the H^1-error, we compute the broken H^1-norm of errors over all subdomains. Table 8.2 shows the numerical results when we fix $n - 1 = 4$ and increase the number of subdomains N. For the cases $N = 8 \times 8$, 16×16 and 32×32, we divide Ω into subdomains as the same manner with $N = 4 \times 4$. In the case of matching grids, $B_c = 0$ in the FETI-DP formulation. Hence, two formulations are the same. However, they are different on nonmatching grids. From the Tables 8.1 and 8.2, we can see that in F_{DW}-formulation the approximated solution does not converge to the exact solution under nonmatching grids as n and N increase. Since the mortar matching condition is imposed incorrectly, F_{DW}-formulation dose not give the correct approximation. In F_{KL}-formulation, $O(h^2)$ and $O(h)$ convergences are observed for L^2 and H^1-errors, respectively. Furthermore, we can see that both preconditioners seem to give $\log^2$-

$n-1$	F_{KL}, F_{DW}-formulation			
	L^2-error	H^1-error	$\widehat{F}_{KL}^{-1}$	$\widehat{F}_{DW}^{-1}$
4	4.1293e-4	5.7497e-2	10	5
8	1.0399e-4	2.8798e-2	12	6
16	2.6046e-5	1.4405e-2	14	6
32	6.5127e-6	7.2036e-3	15	7

$n-1$	F_{KL}-formulation				F_{DW}-formulation		
	L^2-error	H^1-error	$\widehat{F}_{KL}^{-1}$	$\widehat{F}_{DW}^{-1}$	L^2-error	H^1-error	$\widehat{F}_{DW}^{-1}$
4	5.0850e-4	6.0126e-2	10	7	8.2409e-3	1.4987e-1	7
8	1.2865e-4	3.0128e-2	13	8	9.4588e-3	1.5738e-1	8
16	3.2235e-5	1.5072e-2	15	10	9.6715e-3	1.5766e-1	9
32	8.0627e-6	7.5374e-3	16	10	9.5528e-3	1.5599e-1	10

Table 8.1: Comparison between F_{KL} and F_{DW} on matching(up) and nonmatching(down) grids when $N = 4 \times 4$

growth of the condition number bound and the CG iteration of $\widehat{F}_{DW}^{-1}$ is smaller than $\widehat{F}_{KL}^{-1}$.

8.1.2 Elliptic problems with highly discontinuous coefficients

We consider the problem (8.1) when $\alpha(x, y)$ is highly discontinuous across subdomain interfaces and the mesh sizes are not comparable between subdomains. Under this situation, we will compare preconditioners $\widehat{F}_{KL}^{-1}$, $\widehat{F}_{DW}^{-1}$ and $\widehat{F}_{KW}^{-1}$ in F_{KL}-formulation. Recall the preconditioner $\widehat{F}_{KW}^{-1}$ in (8.2). The diagonal matrix D_r consists of diagonal matrices D_r^i's:

$$D_r = \mathrm{diag}_{i=1,\cdots,N}(D_r^i).$$

Here, we describe the matrix D_r^i precisely. For each subdomain Ω_i, let N_i be the set of nodes on the boundary of Ω_i except $\partial\Omega$. Let us define

$$\mu_i(x) = \sum_{\partial\Omega_j \ni x} \rho_j^\gamma \text{ for } x \in N_i,$$

N	F_{KL}, F_{DW}-formulation			
	L^2-error	H^1-error	$\widehat{F}_{KL}^{-1}$	$\widehat{F}_{DW}^{-1}$
4×4	4.1293e-4	5.7497e-2	10	5
8×8	1.0399e-4	2.8798e-2	11	6
16×16	2.6045e-5	1.4405e-2	11	6
32×32	6.5144e-6	7.2036e-3	11	6

N	F_{KL}-formulation				F_{DW}-formulation		
	L^2-error	H^1-error	$\widehat{F}_{KL}^{-1}$	$\widehat{F}_{DW}^{-1}$	L^2-error	H^1-error	$\widehat{F}_{DW}^{-1}$
4×4	5.0850e-4	6.0126e-2	10	7	8.2409e-3	1.4987e-1	7
8×8	1.1744e-4	2.9900e-2	11	8	2.5171e-2	2.5418e-1	8
16×16	2.9743e-5	1.4980e-2	12	8	6.8789e-2	4.2452e-1	9
32×32	7.4318e-6	7.4917e-3	12	8	1.0531e-1	5.2951e-1	12

Table 8.2: Comparison between F_{KL} and F_{DW} on matching(up) nonmatching(down) grids when $n-1=4$

where $\alpha(x) = \rho_j (> 0)$ for $x \in \Omega_j$ and $\gamma \in [1/2, \infty)$. Then the matrix D_r^i is given by

$$D_r^i = \mathrm{diag}_{x \in N_i} \left(\frac{\rho_i^\gamma}{\mu_i(x)} \right).$$

We consider the cases of $N = 2\times 2$, 4×4, 8×8 subdomains. For each subdomain Ω_{ij}, we choose the coefficient $\alpha(x, y)$ in the following way:

$$\alpha(x,y) = \begin{cases} 1 & \text{if both } i \text{ and } j \text{ are even,} \\ 250 & \text{if } i \text{ is odd and } j \text{ is even,} \\ 5000 & \text{if } i \text{ is even and } j \text{ is odd,} \\ 10 & \text{if both } i \text{ and } j \text{ are odd,} \end{cases}$$

and denote them by ρ_{ij}. In addition, we consider the exact solution $u(x, y)$, which belongs to $H^1(\Omega)$, according to the partition of the domain:

$$u(x,y) = \begin{cases} p_1(x,y)\sin(\pi x)\sin(\pi y)/\alpha(x,y) & \text{when } N = 2\times 2, \\ p_2(x,y)\sin(2\pi x)\sin(2\pi y)/\alpha(x,y) & \text{when } N = 4\times 4, \\ \sin(8\pi x)\sin(8\pi y)/\alpha(x,y) & \text{when } N = 8\times 8, \end{cases}$$

where

$$p_1(x,y) = (x - 1/2)(y - 1/2),$$
$$p_2(x,y) = (x - 1/4)(x - 3/4)(y - 1/4)(y - 3/4).$$

Following [54] (see Section 1.5.3), we have chosen different mesh size in each subdomain according to the ratio of coefficients between neighboring subdomains, that is,

$$\frac{h_{ij}}{h_{kl}} \simeq \sqrt[4]{\frac{\rho_{ij}}{\rho_{kl}}},$$

where h_{ij} is the mesh size of the subdomain Ω_{ij}. Using the mesh sizes of these ratios, we divide each subdomain into uniform meshes. Let H_{ij} be the size of the subdomain Ω_{ij}. When $N = 2\times 2$ and $\max(H_{ij}/h_{ij}) = 16$, we obtain triangulations as Figure 8.3 and the triangulations are not comparable between neighboring subdomains.

In Section 1.5.3 of [54], it was shown that a good approximation of the solution is obtained when the slave side is chosen to give a Lagrange multiplier space of higher

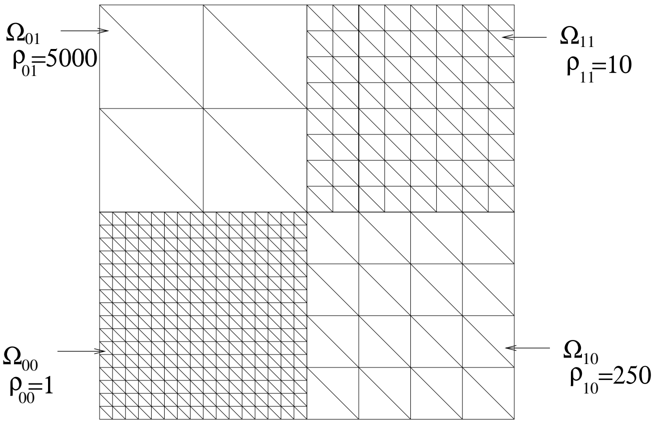

Figure 8.3: Triangulations for the case $N = 2 \times 2$ and $\max(H_{ij}/h_{ij}) = 16$

dimension. Hence, choosing the subdomain with smaller h_{ij} (smaller ρ_{ij}) as the slave side, we can approximate the exact solution more accurately. This observation coincides with the choice of master and slave sides in Remark 5.11.

Table 8.3 shows L^2 and H^1-errors and CG iterations with $\widehat{F}_{KL}^{-1}$, $\widehat{F}_{DW}^{-1}$ and $\widehat{F}_{KW}^{-1}$ as preconditioners under the F_{KL}-formulation. In CG iteration, we use the same stopping criterion 10^{-6} as before. Increasing $\max(H_{ij}/h_{ij})$, we observe the $O(h^2)$ and $O(h)$ convergences of L^2 and H^1-errors, respectively, for all cases of N. Furthermore, we see that the CG iterations of $\widehat{F}_{KL}^{-1}$ and $\widehat{F}_{KW}^{-1}$ are much smaller than $\widehat{F}_{DW}^{-1}$. The number of iterations of $\widehat{F}_{KL}^{-1}$ and $\widehat{F}_{KW}^{-1}$ show similar behaviors in Table 8.3.

In Table 8.4, we compare the number of iterations and condition numbers of $\widehat{F}_{KL}^{-1}$ and $\widehat{F}_{KW}^{-1}$ with various γ. From the results, we can observe that as γ goes to the infinity, the number of iterations and condition numbers of $\widehat{F}_{KW}^{-1}$ converge to those of $\widehat{F}_{KL}^{-1}$. Moreover, we can show that

$$\widehat{F}_{KW}^{-1} \to \widehat{F}_{KL}^{-1} \text{ as } \gamma \to \infty.$$

Since the nonmortar sides have smaller ρ_i's on the interfaces, the followings hold:

$$\begin{aligned} &(D_r^i)^{-1}|_{\Gamma_{ij}} \to \infty \text{ as } \gamma \to \infty, \text{ if } j \in m_i, \\ &(D_r^i)^{-1}|_{\Gamma_{ij}} \to 0 \text{ as } \gamma \to \infty, \text{ otherwise.} \end{aligned} \tag{8.3}$$

We rewrite

$$B_r D_r^{-1} B_r^t = \begin{pmatrix} B_{r,n} & B_{r,m} \end{pmatrix} \begin{pmatrix} D_{r,n}^{-1} & 0 \\ 0 & D_{r,m}^{-1} \end{pmatrix} \begin{pmatrix} B_{r,n} & B_{r,m} \end{pmatrix}^t,$$

where the subscripts n and m represent submatrices on nonmortar and mortar sides, respectively. From (8.3), it holds

$$B_r D_r^{-1} B_r^t \to B_{r,n} D_{r,n}^{-1} B_{r,n}^t \quad \text{as } \gamma \to \infty.$$

Hence, we have

$$(B_r D_r^{-1} B_r^t)^{-1} \to (B_{r,n}^{-1})^t D_{r,n}^{-1} B_{r,n}^{-1} \quad \text{as } \gamma \to \infty. \tag{8.4}$$

Similarly, we obtain

$$B_r D_r^{-1} = \begin{pmatrix} B_{r,n} & B_{r,m} \end{pmatrix} \begin{pmatrix} D_{r,n}^{-1} & 0 \\ 0 & D_{r,m}^{-1} \end{pmatrix} \to \begin{pmatrix} B_{r,n} D_{r,n}^{-1} & 0 \end{pmatrix} \quad \text{as } \gamma \to \infty. \tag{8.5}$$

Therefore, from (8.4) and (8.5), it follows that

$$\widehat{F}_{KW}^{-1} \to \begin{pmatrix} B_{r,n}^{-1} & 0 \end{pmatrix} S_{rr} \begin{pmatrix} (B_{r,n}^{-1})^t \\ 0 \end{pmatrix} (= \widehat{F}_{KL}) \quad \text{as } \gamma \to \infty.$$

From our numerical results, we conclude that our formulation gives the correct approximation of the model problem with nonmatching grids. For the case of continuous coefficients and comparable meshes between subdomain interfaces, the preconditioner $\widehat{F}_{DW}^{-1}$ by Dryja and Widlund gives smaller number of iterations than our preconditioner $\widehat{F}_{KL}^{-1}$. However, the preconditioner $\widehat{F}_{KL}^{-1}$ turns out to be much more efficient than $\widehat{F}_{DW}^{-1}$ for the problem with highly discontinuous coefficients on noncomparable meshes. Furthermore, the preconditioner $\widehat{F}_{KL}^{-1}$ is the limit of $\widehat{F}_{KW}^{-1}$ as γ goes to the infinity.

8.2 Stokes problem in 2D

In this section, we present numerical results for the FETI-DP formulation of the Stokes problem. Let $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ and consider the following Stokes

N	$\max(H_{ij}/h_{ij})$	L^2-error	H^1-error	$\widehat{F}_{DW}^{-1}$	$\widehat{F}_{KL}^{-1}$	$\widehat{F}_{KW}^{-1}$
2×2	16	3.0571e-5	7.6362e-3	17	3	3
	32	7.8276e-6	3.8249e-3	26	3	3
	64	1.9747e-6	1.9133e-3	39	4	3
	128	4.9571e-7	9.5675e-4	50	4	4
	256	1.2421e-7	4.7839e-4	60	4	4
4×4	16	2.1574e-6	1.0939e-3	75	4	3
	32	5.4460e-7	5.4805e-4	81	4	4
	64	1.3799e-7	2.7415e-4	111	4	4
	128	3.4810e-8	1.3709e-4	130	4	4
8×8	16	1.0262e-3	8.8753e-1	113	3	3
	32	2.4870e-4	4.4462e-1	136	4	4
	64	6.4579e-5	2.2240e-1	168	4	4

Table 8.3: Comparison of preconditioners $\widehat{F}_{KL}^{-1}$, $\widehat{F}_{DW}^{-1}$ and $\widehat{F}_{KW}^{-1}(\gamma = 2.0)$ for the problem with highly discontinuous coefficients

N	$\max(H_{ij}/h_{ij})$	$\widehat{F}_{KW}^{-1}$				$\widehat{F}_{KL}^{-1}$
		$\gamma = 0.5$	$\gamma = 1.0$	$\gamma = 2.0$	$\gamma = 10.0$	
2×2	16	5.26e+1(12)	1.09(4)	1.03(3)	1.04(3)	1.04(3)
	32	7.48e+1(17)	1.15(4)	1.04(3)	1.04(3)	1.04(3)
	64	9.79e+1(21)	1.22(4)	1.05(3)	1.05(4)	1.05(4)
	128	1.24e+2(28)	1.30(4)	1.06(4)	1.07(4)	1.07(4)
	256	1.54e+2(32)	1.39(5)	1.08(4)	1.08(4)	1.08(4)
4×4	16	1.31e+1(33)	1.25(5)	1.05(3)	1.06(4)	1.06(4)
	32	2.06e+2(38)	1.42(5)	1.08(4)	1.09(4)	1.09(4)
	64	2.84e+2(51)	1.62(6)	1.12(4)	1.13(4)	1.13(4)
	128	3.44e+2(56)	1.85(6)	1.17(4)	1.17(4)	1.17(4)
8×8	16	1.42e+2(45)	1.28(5)	1.05(3)	1.05(3)	1.05(3)
	32	2.16e+2(56)	1.48(6)	1.08(4)	1.09(4)	1.09(4)
	64	2.94e+2(65)	1.72(7)	1.12(4)	1.12(4)	1.12(4)

Table 8.4: Condition numbers (number of iterations) of $\widehat{F}_{KW}^{-1}(\gamma = 0.5,\ 1.0,\ 2.0,\ 10.0)$ and $\widehat{F}_{KL}^{-1}$ for the problems with highly discontinuous coefficients

problem:

$$\begin{aligned} -\triangle \mathbf{u} + \nabla p &= \mathbf{f} \ \text{ in } \Omega, \\ -\nabla \cdot \mathbf{u} &= 0 \ \text{ in } \Omega, \\ \mathbf{u} &= \mathbf{0} \ \text{ on } \partial\Omega, \end{aligned} \tag{8.6}$$

where $\mathbf{f}$ is chosen so that the exact solution of the problem becomes

$$\mathbf{u} = \begin{pmatrix} \sin^3(\pi x)\sin^2(\pi y)\cos(\pi y) \\ -\sin^2(\pi x)\sin^3(\pi y)\cos(\pi x) \end{pmatrix} \ \text{ and } \ p = x^2 - y^2.$$

Let N denote the number of subdomains. We only consider the uniform partition of Ω as mentioned in Section 8.1.1. With this partition, we triangulate each subdomain in the following manner. For all subdomains, we take the same number of nodes n, including end points, in horizontal and vertical edges with $n = 4k + 1$ for some positive integer k. We solve (8.6) on matching and nonmatching grids both. For matching grids, we make uniform triangulations in each subdomain with $(n-1)/2+1$ nodes on horizontal and vertical edges of the subdomain and denote it by $\Omega_i^{2h_i}$, a triangulation for the pressure. After bisecting each edge of triangles in $\Omega_i^{2h_i}$, we obtain $\Omega_i^{h_i}$, a triangulation for the velocity. For nonmatching grids, we take $(n-1)/2+1$ random quasi-uniform nodes on each horizontal and vertical edges of subdomain, and generate nonuniform structured triangulations. We denote it by $\Omega_i^{2h_i}$. The triangulation $\Omega_i^{h_i}$ is obtained from $\Omega_i^{2h_i}$ similarly to matching grids. For example, see Figure 8.4.

Now, we solve the FETI-DP operator with and without preconditioner varying N and n. Those cases are denoted by PFETI-DP and FETI-DP, respectively. The CG(Conjugate Gradient) iteration is stopped when the relative residual is less than 10^{-6}.

In Tables 8.5-8.7, the number of CG iterations and condition numbers are shown varying N and n. In Table 8.5, $N = 4 \times 4$ and $n - 1$ increases by double. On both matching and nonmatching grids, PFETI-DP performs well and the condition numbers seem to behave $\log^2$-growth as n increases. Especially on nonmatching grids, the CG iteration stops quickly with the preconditioner. In Tables 8.6 and 8.7, N increases with $n = 5$ and $n = 9$. For both cases of FETI-DP and PFETI-DP, the

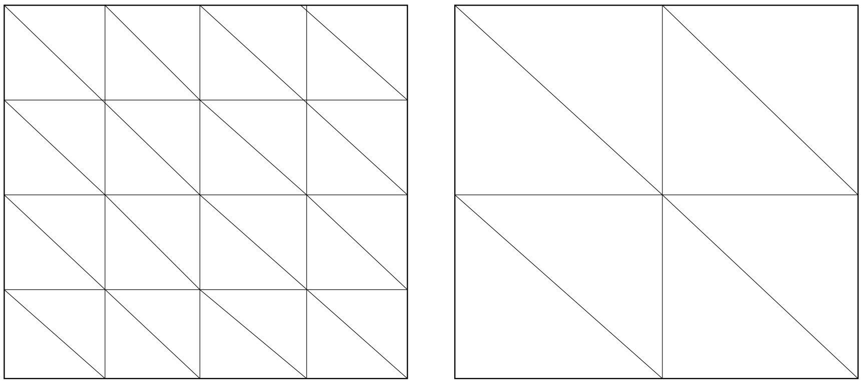

Figure 8.4: Triangulations $\Omega_i^{h_i}$(left) and $\Omega_i^{2h_i}$(right) when $n = 5$

n	Matching		Nonmatching	
	FETI-DP	PFETI-DP	FETI-DP	PFETI-DP
5	12(5.23)	9(2.62)	16(8.35)	12(3.75)
9	24(2.50e+1)	13(4.39)	50(1.15e+2)	15(5.79)
17	37(6.68e+1)	15(5.94)	86(5.01e+2)	17(7.93)
33	45(1.45e+2)	17(7.75)	119(1.31e+3)	20(9.88)
65	58(2.69e+2)	19(9.85)	153(3.29e+3)	22(1.20e+1)

Table 8.5: CG iterations(condition number) when $N = 4 \times 4$

CG iteration becomes stable as N increases. From the results, we can see that the developed preconditioner gives the condition number bound as confirmed in theory.

Moreover, we have observed the convergent behaviors of the approximated solutions. The H^1 and L^2-errors for velocity and pressure are examined. $\boldsymbol{u}^h$ and p^h denote the approximated solutions for the velocity and pressure, respectively, and $\|\boldsymbol{u} - \boldsymbol{u}^h\|_{1,*}$ means the square root of $\sum_{i=1}^{N} \|\boldsymbol{u} - \boldsymbol{u}^h\|_{1,\Omega_i}^2$. The errors and reduction factors are shown in Table 8.8 for various N and n with matching grids. Three cases are considered: when $n-1$ increases by double with $N = 4 \times 4$, when N increases by double in both edges of Ω with $n = 5$, and when N increases by double in both edges of Ω with $n = 9$. For all cases, we can see that the H^1-error for velocity, $\|\boldsymbol{u} - \boldsymbol{u}^h\|_{1,*}$, and L^2-error for pressure, $\|p - p^h\|_0$, reduce by half and L^2-error for

N	Matching		Nonmatching	
	FETI-DP	PFETI-DP	FETI-DP	PFETI-DP
4×4	12(5.23)	9(2.62)	16(8.35)	12(3.75)
8×8	12(5.42)	9(2.62)	16(9.18)	12(3.68)
16×16	10(5.54)	9(2.55)	16(9.57)	11(3.42)
32×32	10(5.61)	9(2.53)	16(10.88)	12(3.78)

Table 8.6: CG iterations(condition number) when $n = 5$

N	Matching		Nonmatching	
	FETI-DP	PFETI-DP	FETI-DP	PFETI-DP
4×4	24(2.50e+1)	13(4.39)	50(1.15e+2)	15(5.79)
8×8	25(2.60e+1)	13(4.35)	53(1.19e+2)	15(6.21)
16×16	24(2.62e+1)	12(4.27)	57(1.34e+2)	16(6.27)
32×32	23(2.62e+1)	12(4.27)	56(1.25e+2)	16(6.24)

Table 8.7: CG iterations(condition number) when $n = 9$

$N = 4\times 4$	$n=5$	$n=9$	$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{1,*}$	$\|\boldsymbol{u}-\boldsymbol{u}^h\|_0$	$\|p-p^h\|_0$
n	N	N			
5	4×4		3.37e-1	3.75e-3	1.07e-1
9	8×8	4×4	1.72e-1 (0.510)	1.02e-3 (0.272)	5.99e-2 (0.559)
17	16×16	8×8	8.64e-2 (0.502)	2.64e-4 (0.258)	3.08e-2 (0.514)
33	32×32	16×16	4.32e-2 (0.500)	6.65e-5 (0.258)	1.55e-2 (0.503)
65		32×32	2.16e-2 (0.500)	1.66e-5 (0.249)	7.79e-3 (0.502)

Table 8.8: H^1 and L^2-errors(factor) on matching grids

n	$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{1,*}$	$\|\boldsymbol{u}-\boldsymbol{u}^h\|_0$	$\|p-p^h\|_0$
5	3.41e-1	3.79e-3	1.05e-1
9	1.78e-1 (0.521)	1.10e-3 (0.290)	6.08e-2 (0.579)
17	8.95e-2 (0.502)	2.85e-4 (0.259)	3.16e-2 (0.517)
33	4.48e-2 (0.500)	7.21e-5 (0.252)	1.58e-2 (0.500)
65	2.24e-2 (0.500)	1.81e-5 (0.251)	7.93e-3 (0.501)

Table 8.9: H^1 and L^2-errors(factor) on nonmatching grids: $N = 4\times 4$

velocity, $\|\boldsymbol{u}-\boldsymbol{u}^h\|_0$, reduces by quarter. For the finite elements $P_1(h)-P_0(2h)$, these convergent behaviors are optimal.

For the case of nonmatching grids, the errors and reduction factors are shown in Tables 8.9-8.11 with various N and n. In Table 8.9, we observe that the error $\|\boldsymbol{u}-\boldsymbol{u}^h\|_{1,*}$ and $\|p-p^h\|_0$ reduce by half and the error $\|\boldsymbol{u}-\boldsymbol{u}^h\|_0$ reduces by quarter as $n-1$ increases by double with $N = 4\times 4$. When $n = 5$ and $n = 9$, as N increases, the errors also show the optimal convergent behaviors in Tables 8.10 and 8.11. These numerical results confirm that the stopping criterion for CG iteration in Tables 8.5-8.7 is sufficient.

As mentioned in Section 7.1, if the inf-sup constant for the space $V\times P$ is independent of N and n, then the optimality of approximation can be shown. Let β^* and β be the inf-sup constants for the space $V\times P$ and the $P_1(h)-P_0(2h)$

N	$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{1,*}$	$\|\boldsymbol{u}-\boldsymbol{u}^h\|_0$	$\|p-p^h\|_0$
4×4	1.78e-1	1.10e-3	6.08e-2
8×8	8.95e-2 (0.502)	2.94e-4 (0.269)	3.28e-2 (0.539)
16×16	4.49e-2 (0.501)	7.33e-5 (0.249)	1.63e-2 (0.496)
32×32	2.25e-2 (0.501)	1.84e-5 (0.251)	8.18e-3 (0.501)

Table 8.10: H^1 and L^2-errors(factor) on nonmatching grids: $n = 5$

N	$\|\boldsymbol{u}-\boldsymbol{u}^h\|_{1,*}$	$\|\boldsymbol{u}-\boldsymbol{u}^h\|_0$	$\|p-p^h\|_0$
4×4	3.37e-1	3.75e-4	1.07e-1
8×8	1.72e-1 (0.510)	1.02e-3 (0.272)	5.99e-2 (0.559)
16×16	8.64e-2 (0.502)	2.64e-4 (0.258)	3.08e-2 (0.514)
33×32	4.32e-2 (0.500)	6.65e-5 (0.258)	1.55e-2 (0.503)

Table 8.11: H^1 and L^2-errors(factor) on nonmatching grids: $n = 9$

finite elements, respectively, and β_0 be the inf-sup constant for the space $V \times \Pi^0$. Then the constant β^* depends on β and β_0 from the trick conceived by Boland and Nicolaides [12]. Hence, if the constant β_0 is independent of n and N, then the same holds for β^*. In [6], for $V \times \Pi^0$ which is obtained from the Hood-Taylor finite elements, it was shown that the constant β_0 is independent of n, but not shown for N. Following the proofs in [6], we can obtain the same results for the space $V \times \Pi^0$ of the $P_1(h) - P_0(2h)$ finite elements. We have no proof that β_0 is independent of N. Instead, we compute the constant β_0 numerically as N increases. The results are given in Table 8.12 both for matching and nonmatching grids when $n = 5$ and $n = 9$. We observe that the constant β_0 becomes stable as N increases. Table 8.13 gives the constant β_0 as n increases with $N = 4\times 4$. This confirms that the constant β_0 is independent of n.

N	$n = 5$		$n = 9$	
	Nonmatching	Matching	Nonmatching	Matching
4×4	0.5780	0.5785	0.5921	0.5924
8×8	0.5293	0.5294	0.5352	0.5353
16×16	0.5008	0.5010	0.5041	0.5042
32×32	0.4827	0.4828	0.4854	0.4848

Table 8.12: Inf-sup constant β_0 when $n = 5$ and $n = 9$

n	Nonmatching	Matching
5	0.5780	0.5785
9	0.5921	0.5294
17	0.5966	0.5967
33	0.5973	0.5979
65	0.5983	0.5983

Table 8.13: Inf-sup constant β_0 when $N = 4 \times 4$

Appendix

In the following, we show that how we approximate the inf-sup constant β_0 of the space $V \times \Pi^0$. By definition, the inf-sup constant β_0 is

$$\inf_{q\in\Pi^0} \sup_{\boldsymbol{v}\in V} \frac{\left(\int_\Omega \nabla \cdot \boldsymbol{v} q \, dx\right)^2}{\left(\sum_{i=1}^N \|v\|_{1,\Omega_i}^2\right) \|q\|_{0,\Omega}^2} \geq \beta_0. \tag{A.1}$$

Since $v \in V$, there exist a constant C not depending on h_i's and H_i's, such that

$$\sum_{i=1}^N \|v\|_{1,\Omega_i}^2 \leq C \sum_{i=1}^N |v|_{1,\Omega_i}^2. \tag{A.2}$$

Using the above relation, we replace the H^1-norm in (A.1) by the semi H^1-norm and we will compute the constant β_0 such that

$$\inf_{q\in\Pi^0} \sup_{\boldsymbol{v}\in V} \frac{\left(\int_\Omega \nabla \cdot \boldsymbol{v} q \, dx\right)^2}{\left(\sum_{i=1}^N |\boldsymbol{v}|_{1,\Omega_i}^2\right) \|q\|_{0,\Omega}^2} \geq \beta_0. \tag{A.3}$$

Our objective is to see that the constant β is independent of h_i's and H_i's. Hence it suffices to consider the above inequality to estimate the inf-sup constant. For this purpose, we will give a matrix whose second smallest eigenvalue is the inf-sup constant β_0.

For $\boldsymbol{v} \in X$, we split it into four parts, that is, the interior parts of subdomains, the mortar sides of interfaces without end points, the nonmortar sides of interfaces without end points and the global corners, and denote them by $\boldsymbol{v}_I$, $\boldsymbol{v}_m$, $\boldsymbol{v}_n$ and $\boldsymbol{v}_c$, respectively. Since $q \in \Pi^0$ is constant in each subdomain, the denominator of L.H.S. in (A.3) is independent of $\boldsymbol{v}_I$. We eliminate $\boldsymbol{v}_I$ using

$$\inf_{\boldsymbol{v}_I} \sum_{i=1}^N |\boldsymbol{v}_i|_{1,\Omega_i}^2 = \sum_{i=1}^N < S^i \boldsymbol{w}_i, \boldsymbol{w}_i >,$$

$$\int_\Omega \nabla \cdot \boldsymbol{v} q \, dx = \sum_{i=1}^N \int_{\partial\Omega_i} \boldsymbol{w}_i \cdot \boldsymbol{n}_i q \, ds,$$

where $\boldsymbol{v}_i = \boldsymbol{v}|_{\Omega_i}$ and $\boldsymbol{w}_i = \boldsymbol{v}_i|_{\partial\Omega_i}$.

Let us define

$$Z := \{\boldsymbol{w} = (\boldsymbol{w}_1, \cdots, \boldsymbol{w}_N) \,:\, \boldsymbol{w}_i = \boldsymbol{v}_i|_{\partial\Omega_i} \quad \text{for } i = 1, \cdots, N, \ \forall \boldsymbol{v} \in V\}.$$

Similarly, we define $\boldsymbol{w}_m$, $\boldsymbol{w}_c$ and $\boldsymbol{w}_n$ for $\boldsymbol{w} \in Z$. Then, we rewrite (A.3) into

$$\inf_{q \in \Pi^0} \sup_{\boldsymbol{w} \in Z} \frac{\left(\sum_{i=1}^N \int_{\partial\Omega_i} \boldsymbol{w} \cdot \boldsymbol{n}_i q \, ds\right)^2}{\left(\sum_{i=1}^N < S^i \boldsymbol{w}_i, \boldsymbol{w}_i >\right) \|q\|_{0,\Omega}^2} \geq \beta_0. \tag{A.4}$$

The space $W_{m,c}$ is defined as a space with vectors $\boldsymbol{w}_{m,c} = \begin{pmatrix} \boldsymbol{w}_m \\ \boldsymbol{w}_c \end{pmatrix}$.

Since $\boldsymbol{w} \in Z$ satisfies

$$\int_{\Gamma_{ij}} (\boldsymbol{w}_i - \boldsymbol{w}_j) \cdot \boldsymbol{\lambda} \, ds = 0 \quad \forall i = 1, \cdots, N, \ \forall j \in m_i, \tag{A.5}$$

we can see that $\boldsymbol{w}_n$ is determined by $\boldsymbol{w}_{m,c} \in W_{m,c}$. We rewrite (A.5) into

$$B_n \boldsymbol{w}_n = B_m \boldsymbol{w}_m + (B_{m,c} - B_{n,c}) \boldsymbol{w}_c. \tag{A.6}$$

Let

$$B_{mc} = \begin{pmatrix} B_m & B_{m,c} - B_{n,c} \end{pmatrix}.$$

Using (A.6), $\boldsymbol{w} \in V$ is obtained from $\boldsymbol{w}_{m,c} \in W_{m,c}$

$$\begin{pmatrix} \boldsymbol{w}_{m,c} \\ \boldsymbol{w}_n \end{pmatrix} = \begin{pmatrix} I \\ B_n^{-1} B_{mc} \end{pmatrix} \boldsymbol{w}_{m,c}.$$

More precisely, we have

$$\boldsymbol{w}|_{\Omega_i} = \begin{pmatrix} L^i_{m,c} \boldsymbol{w}_{m,c} \\ L^i_n \boldsymbol{w}_n \end{pmatrix},$$

where the maps $L^i_{m,c}$ and L^i_n restrict $\boldsymbol{w}_{m,c}$ and $\boldsymbol{w}_n$ on the subdomain Ω_i. Let us define $E^i_m : W_{m,c} \to Z|_{\partial\Omega_i}$ by

$$E^i_m = \begin{pmatrix} L^i_{m,c} & 0 \\ 0 & L^i_n \end{pmatrix} \begin{pmatrix} I \\ B_n^{-1} B_{mc} \end{pmatrix}.$$

We may write

$$S^i = \begin{pmatrix} S^i_{mm} & S^i_{mn} \\ S^i_{nm} & S^i_{nn} \end{pmatrix},$$

where m and n denote the d.o.f. on mortar sides and corners ,and the d.o.f. on nonmortar sides without end points, respectively. Then, we have

$$\sum_{i=1}^{N} < S^i \boldsymbol{w}_i, \boldsymbol{w}_i >=< S_m \boldsymbol{w}_{m,c}, \boldsymbol{w}_{m,c} >, \tag{A.7}$$

with

$$S_m = \sum_{i=1}^{N} (E^i_m)^t S^i E^i_m.$$

Let G^i be a matrix that gives

$$< G^i \boldsymbol{w}_i, q >= \int_{\partial\Omega_i} \boldsymbol{w}_i \cdot \boldsymbol{n}_i q \, ds.$$

We may consider the matrix G^i to be ordered as in S^i and write

$$< G_m \boldsymbol{w}_{m,c}, q >= \sum_{i=1}^{N} \int_{\partial\Omega_i} \boldsymbol{w}_i \cdot \boldsymbol{n}_i q \, ds, \tag{A.8}$$

with

$$G_m = \sum_{i=1}^{N} G^i E^i_m.$$

In addition, a matrix M is defined as

$$< Mq, q >= \|q\|^2_{0,\Omega}. \tag{A.9}$$

Since $q \in \Pi^0$ is constant in each subdomain, the matrix M is diagonal.

From (A.7), (A.8) and (A.9), we have the following identity:

$$\frac{\left(\sum_{i=1}^{N} \int_{\partial\Omega_i} \boldsymbol{w} \cdot \boldsymbol{n}_i q \, ds\right)^2}{\left(\sum_{i=1}^{N} < S^i \boldsymbol{w}_i, \boldsymbol{w}_i >\right) \|q\|^2_{0,\Omega}} = \frac{< G_m \boldsymbol{w}_{m,c}, q >^2}{< S_m \boldsymbol{w}_{m,c}, \boldsymbol{w}_{m,c} >< Mq, q >}.$$

Hence, we consider

$$\min_{q \in \Pi^0} \max_{\boldsymbol{w}_{m,c} \in W_{m,c}} \frac{< G_m \boldsymbol{w}_{m,c}, q >^2}{< S_m \boldsymbol{w}_{m,c}, \boldsymbol{w}_{m,c} >< Mq, q >} \tag{A.10}$$

to estimate the inf-sup constant β_0. From (A.2), we can see that S_m is a s.p.d. operator on $W_{m,c}$. Therefore, in (A.10), the maximum occurs when $S_m \boldsymbol{w}_{m,c} = G_m^t q$ and (A.10) is reduced into

$$\min_{q \in \Pi^0} \frac{< G_m S_m^{-1} G_m^t q, q >}{< Mq, q >}.$$

Since $q \in \Pi^0$ is constant in each subdomain and $\int_\Omega q\,dx = 0$, the d.o.f. of Π^0 is exactly $N-1$ and $1 \perp \Pi^0$. We may assume that there exist constants C_1 and C_2 independent of the number of subdomains and meshes such that

$$C_1 H^2 q^t q \le < Mq, q > \le C_2 H^2 q^t q,$$

where $H = \max_{i=1,\cdots,N} H_i$. Let

$$C_m = \frac{1}{H^2} G_m S_m^{-1} G_m^t.$$

Hence, we consider

$$\min_{q \in \Pi^0} < C_m q, q > \tag{A.11}$$

to estimate the constant β_0. From the fact that S_m is s.p.d. and $\mathrm{Null}(G_m^t) = \{\mathbf{1}\}$ (see [6]), the matrix C_m is symmetric and semi-positive definite and it has 0 eigenvalue associated with the eigenvector $\mathbf{1}$. Therefore, to estimate the inf-sup constant β_0 of the space $V \times \Pi^0$, we compute the second smallest eigenvalue of the matrix C_m.

요 약 문

모르타르 방법으로 이산화된 FETI-DP 형식의 preconditioner에 관한 연구

FETI(-DP) 영역 분할법은 순응 유한요소로 이산화된 미분 방정식을 풀기 위하여 개발되었으며, 타원형 문제들을 병렬 계산으로 푸는데 있어서 가장 효율적이라고 알려져 있다. 이 방법의 특징은 기존의 영역 분할법에서와 같이 영역의 경계에서 해의 연속성을 직접적으로 구현하지 않고 라그랑지 승수를 도입하여 간접적으로 맞추어 준다는 것에 있다. 이러한 라그랑지 승수를 도입함으로서 FETI 방법에서는 mixed problem으로부터 도출된 선형방정식을 얻게되며 이 선형방정식에서 라그랑지 승수 이외의 다른 미지수들을 소거하여 라그랑지 승수와 관련된 선형방정식을 얻는다. 이 선형방정식은 일반적으로 ill-conditioned 방정식이므로, 수치적으로 그 해를 효율적으로 찾기 위해서는 적절한 preconditioner가 반드시 필요하다. 다른 영역 분할법에 의한 선형방정식에 비해, 이 선형방정식은 수학적으로 분석하기가 쉽다. 이러한 수학적 분석으로부터 preconditiner를 비교적 쉽게 도출해 낼 수가 있다. 또한, 이 preconditioner는 기존의 영역분할법에서 연구되어진 것과 달리 coarse problem을 가지고 있지 않으므로, 완전히 병렬화할 수 있으며 실제로 대용량 문제를 푸는데 있어서 그 효율성이 입증 되었다.

최근에 FETI(-DP) 방법은 비순응 유한요소로 이산화된 문제를 푸는데 적용되었다. 특히, 각각의 영역에서는 순응 유한요소로 구성이 되어 있지만 인접하는 영역들간의 유한요소들이 독립적으로 주어져서 생겨난 비순응 유한요소에 대해 연구가 이루어져 왔다. 이와 같은 비순응 요소는 특이점이 있는 미분방정식을 이산화 하는 경우, 3차원 영역에서와 같이 유한요소를 만드는 데 많은 시간이 소요되는 경우, 접합문제, multi-physics 문제 등을 다루는데 있어서 효과적이기 때문에 많이 연구되어 왔다. 이러한 비순응 유한요소를 이산화 하는 과정에서 영역의 경계에서 유한요소들에 적절한 조건을 주는데, 이것을 모르타르 일치 조건이라고 하며 이와 같은 이산화 방법을 모르타르 방법이라고 한다. 이 방법은 동일한 차수의 유한요소를 사용한 경우 그 근사 해의 정확도가 순응 유한요소와 동일하다. 모르타르 일치조건이

라그랑지 승수를 통하여 구현되면 이러한 모르타르 방법으로부터 mixed problem이 도출되며 이 문제는 FETI 방법들에서 도출되는 mixed problem과 동일하다.

지금까지 모르타르 방법으로부터 도출된 mixed problem에 관한 preconditioner의 연구는 수학적 해석의 어려움으로 인해 여러 가지 한계점이 있었다. FETI(-DP) 방법을 이용하여 이러한 mixed problem을 푸는 것은 preconditioner를 구성하는 것이 용이하므로 이에 관한 수치적인 계산은 Stefanica [48], Rapetti [42]등에 의해 연구되었으며, 최근에 Widlund와 Dryja [21, 22]에 의해 여러 가지 형태의 preconditioner들에 대한 수학적인 해석이 이루어졌다. 그러나 그들의 수학적인 해석은 유한요소에 제약이 있고 일반적인 타원형 문제에 적용할 수 없다는 한계를 가지고 있었다. 이 논문에서는 이와 같은 문제를 다른 형태의 preconditioner를 도입하여 해결하였으며, 3차원 타원문제 그리고, 스톡스 문제에도 그 결과를 확장하였다. 특히, 계수가 불연속적인 타원문제의 경우, 기존에 개발된 다른 preconditioner들 보다 효율적이라는 것을 수치적 계산으로 입증하였다.

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감 사 의 글

제가 학문의 길에 발을 내딛고 이 논문을 완성하기까지 도움을 주신 분들께 감사의 마음을 전하고자 합니다.

먼저, 이창옥 선생님께 무한한 감사를 드립니다. 제가 포기하지 않고 문제를 해결할 수 있도록 항상 같이 고민해 주셨고, 진정한 학자의 자세를 깨우쳐 주셨습니다. 무엇보다도 연구를 하는 중간 중간 저의 사소한 의견과 생각들을 존중해 주신 선생님의 배려로 이 논문을 완성할 수 있었습니다. 선생님과 공부한 2년 반의 기간은 짧았지만, 제 학문적 일생에서 가장 소중하고 즐거웠던 시간이 될 것입니다.

바쁘신 중에 논문의 심사를 해 주신 권길헌 교수님과 김홍오 교수님께도 감사드립니다. 그리고, 실질적인 측면에서 제 연구의 방향을 제시해 주셨던 임세영 교수님께 감사드립니다. 제가 3차원 문제를 계속 고민하고 해결하는데 있어서 강한 동기를 얻을 수 있었습니다. 제 연구에 많은 조언을 해주신 신동우 교수님께 감사드립니다. 교수님의 조언으로 인해 좀 더 폭 넓은 시각으로 연구에 임할 수 있었습니다.

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제가 학업과 가정 생활에 바빠서 많은 시간을 같이 하지 못한 것이 아쉬운 선후배들과 친구들에게 감사의 마음을 전하고자 합니다. 여러가지 면에서 많은 도움을 주셨던 가장 기억에 남는 범자 언니, 연구실 생활을 하는데 있어 많은 배려를 해 주신 용식 선배와 기복 선배께 감사드립니다. 연구실의 든든한 후배들 주영, 은희, 길수씨, 기완, 정현, 그레고리, 졸업한 연경, 진희, 연우, 그리고 막내들 유라, 제욱이 모두 기억에 많이 남을 것입니다. 소중한 친구들 세연, 정연, 희화에게도 감사의 마음을 전하고 싶습니다.

공부하는 며느리의 부족함을 감싸주시고, 많은 도움을 주셨던 시부모님들께 머리 숙여 감사드립니다. 무엇보다도, 어린 무성이를 돌봐주셨던 어머님께 깊이 감사드립니다. 그리고, 저를 항상 지켜봐 주시고 믿어주셨던, 가장 존경하는 제 부모님께 감사드리며, 이 작은 결실을 드립니다. 마지막으로, 사랑하는 아들 무성이와 든든한 후원자인 남편에게 이 논문으로 감사의 마음과 미안함을 전하고자합니다.

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