Learning with Large Number of Experts: Component Hedge Algorithm

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Regret of RWM is $O(\sqrt{T \ln N})$.
- Informative even for very large number of experts.

What if there is “overlap” between experts?
- RWM with path experts
- FPL with path experts
- → can we do better?

[Littlestone and Warmuth, 1989; Kalai and Vempala, 2004]
Better Bounds in Structured Case?

- Can “overlap” between experts lead to better regret guarantees?
- What are the lower bounds in the structured setting?
- Computationally efficient solutions that realize these bounds?
Outline

- Learning Scenario
- Component Hedge Algorithm
- Regret Bounds
- Applications & Lower Bounds
- Conclusion & Open Problems
Learning Scenario

Assumptions:

- Structured concept class $C \subseteq \{0, 1\}^d$
  - Composed of components: $C^t$ indicates which components are used for each trial $t$.

- Additive loss $\ell^t$ incurred at each trial $t$.
  - Loss of each concept $C$ is $C \cdot \ell^t \leq M := \max_{C \in \mathcal{C}} |C|$

Goal:

- minimize expected regret after $T$ trials

$$R_T = \sum_{t=1}^{T} \mathbb{E}[C^t] \cdot \ell^t - \min_{C \in \mathcal{C}} \sum_{t=1}^{T} C \cdot \ell^t$$
Component Hedge Algorithm

[Koolen, Warmuth, and Kivinen, 2010]

CH maintains weights \( w^t \in \text{conv}(C) \subseteq [0, 1]^d \) over the components at each round \( t \).

- **Update:**
  1. weights: \( \hat{w}^t_i = w^{t-1}_i e^{-\eta \ell_i^t} \)
  2. relative entropy projection:

\[
w^t := \arg\min_{w \in \text{conv}(C)} \Delta(w \| \hat{w}^t)
\]

where \( \Delta(w \| v) = \sum_{i=1}^{d} (w_i \ln \frac{w_i}{v_i} + v_i - w_i) \)
Component Hedge Algorithm

- **Prediction:**
  1. Decomposition of weights:
     \[ w^t = \sum_{C \in \mathcal{C}} \alpha_C C \]
     where \( \alpha \) is a distribution over \( \mathcal{C} \)
  2. Sample \( C^{t+1} \) according to \( \alpha \)
Efficiency

**Need** efficient implementation of:
- Decomposition (not unique) of weights over the concepts
- Entropy projection step (convex problem)

**Sufficient**: $\text{conv}(C)$ described by polynomial in $d$ constraints
Theorem: Regret Bounds for CH

Let $\ell^* = \min_{C \in C} C \cdot (\ell^1 + \ldots + \ell^T)$ be the loss of the best concept in hindsight, then

$$R_T \leq \sqrt{2\ell^* M \ln(d/M)} + M \ln(d/M)$$

by choosing $\eta = \sqrt{\frac{2M \ln(d/M)}{\ell^*}}$

- Since $\ell^* \leq MT$, regret $R_T \in O(M\sqrt{T \ln d})$.
- Matching lower bounds in applications.
Comparison of CH, RWM and FPL

1. CH has significantly **better regret bounds**: 
   - CH: $R_T \in O(M\sqrt{T \ln d})$.
   - RWM: $R_T \in O(M\sqrt{MT \ln d})$
   - FPL: $R_T \in O(M\sqrt{dT \ln d})$

2. CH is **optimal** w.r.t. regret bounds while RWM and FPL are not optimal.

3. Standard expert setting (no structure): CH, RWM and FPL reduce to the same algorithm.
Applications

- On-line shortest path problems.
- On-line PCA ($k$-sets).
- On-line ranking ($k$-permutations).
- Spanning trees.
On-line Shortest Path Problem (SPP)

- $G = (V, E)$ is a directed graph.
- $s$ is the source and $t$ is the destination.
- Each $s - t$ path is an expert.
- The loss is **additive** over edges.
Convex hull of paths cannot be captured by linear constraints

**Unit flow** polytope relaxation is used:

\[ w_{u,v} \geq 0, \quad \forall (u, v) \in E \]
\[ \sum_{v \in V} w_{s,v} = 1 \]
\[ \sum_{v \in V} w_{v,u} = \sum_{v \in V} w_{u,v}, \quad \forall u \in V \]

Relaxation does not hurt regret bounds.
Example of Unit Flow Polytope
\[
\min_w \sum_{(u,v) \in E} w_{u,v} \ln \frac{w_{u,v}}{\hat{w}_{u,v}} + \hat{w}_{u,v} - w_{u,v}
\]

subject to:

\[
w_{u,v} \geq 0, \quad \forall (u, v) \in E
\]

\[
\sum_{v \in V} w_{s,v} = 1
\]

\[
\sum_{v \in V} w_{v,u} = \sum_{v \in V} w_{u,v}, \quad \forall u \in V
\]
Dual problem

$$\max_{\lambda} \left\{ \lambda_s - \sum_{(u,v) \in E} \hat{w}_{u,v} e^{\lambda_u - \lambda_v} \right\}$$

- No constraints.
- Only $|V|$ variables.
- Primal solution: $w_{u,v} = \hat{w}_{u,v} e^{\lambda_u - \lambda_v}$
Convex Decomposition

1. Find any non-zero path from $s$ to $t$.
2. Subtract the smallest weight from each edge.
3. Repeat until no path is found.

$\Rightarrow$ At most $|E|$ iterations is needed.
Example of Convex Decomposition
Example of Convex Decomposition
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Example of Convex Decomposition
Regret Bounds for SPP

- Expected regret is bounded by

\[ 2\sqrt{\ell^* k^* \ln |V|} + 2k^* \ln |V| \in O(M \sqrt{T \ln |V|}) \]

- Bound holds for arbitrary graphs.
Lower Bounds

Any algorithm can be forced to have expected regret

$$\sqrt{l^* k^* \ln \frac{|V|}{k^*}}$$

Idea of the proof:

- Minimize the “overlap”.
- Create $|V|/k$ disjoint paths of length $k$.
- Apply lower bounds for standard expert setting.
Conclusions

- Regret of CH is often better than that of RWM or FPL in structured setting.
- Regret of CH often matches lower bounds in applications.
- Efficient solutions exist for a wide range of applications: on-line shortest path, on-line PCA, on-line ranking, spanning trees.
References

Regret Bounds

Theorem: Regret Bounds for CH

Let $\ell^* = \min_{C \in \mathcal{C}} C \cdot (\ell^1 + \ldots + \ell^T)$ be the loss of the best concept in hindsight, then

$$R_T \leq \sqrt{2\ell^* M \ln\left(\frac{d}{M}\right)} + M \ln\left(\frac{d}{M}\right)$$

by choosing $\eta = \sqrt{\frac{2M \ln\left(\frac{d}{M}\right)}{\ell^*}}$
Proof of CH Regret Bound

**1. Bound:**
\[(1 - e^{-\eta}) w^{t-1} \cdot \ell^t \leq \Delta(C \| w^{t-1}) - \Delta(C \| w^t) + \eta C \cdot \ell^t.\]
- \[1 - e^{-\eta x} \geq (1 - e^{-\eta}) x\]
- Generalized Pythagorean Theorem

**2. Sum over trials \(t\):**
\[(1 - e^{-\eta}) \sum_{t=1}^{T} w^{t-1} \cdot \ell^t \leq \Delta(C \| w^0) - \Delta(C \| w^T) + \eta C \cdot \ell_{\leq T}\]
where \[\ell_{\leq T} = \ell^1 + \ldots + \ell^T.\]

**3. Use** \(w^{t-1} \cdot \ell^t = E[C^t] \cdot \ell^t: \)
\[\sum_{t=1}^{T} E[C^t] \cdot \ell^t \leq \frac{\Delta(C \| w^0) - \Delta(C \| w^T) + \eta C \cdot \ell_{\leq T}}{(1 - e^{-\eta})}\]
Proof of CH Regret Bound

1. \( w^0 \) assumes uniform distribution over concepts
   \[ w_i^0 = \frac{M}{d} \implies \Delta(C \| w^0) = M \ln \left( \frac{d}{M} \right) \]

2. Let \( \ell^* \) best concept in hind-sight and choosing
   \[ \eta = \sqrt{\frac{2M \ln \left( \frac{d}{M} \right)}{\ell^*}} \implies \text{Regret bound } R_T. \]