Blackwell Approachability

MEHRYAR MOHRI MOHRI@
GOOGLE RESEARCH & COURANT INSTITUTE

Motivation

- Extension of online learning to vectorial payoffs.
- Extension of zero-sum games theory to vectorial payoffs.
- Repeated games.

von Neumann's Theorem

(von Neumann, 1928)

- Theorem (von Neumann's minimax theorem): for any two-player zero-sum game with finite action sets,

$$\max_{\mathsf{p}\in\Delta_1(\mathcal{A}_1)} \min_{\mathsf{q}\in\Delta_1(\mathcal{A}_2)} \mathop{\mathrm{E}}_{\substack{a_1\sim\mathsf{p}\\ a_2\sim\mathsf{q}}}[u(a_1,a_2)] = \min_{\mathsf{q}\in\Delta_1(\mathcal{A}_2)} \max_{\mathsf{p}\in\Delta_1(\mathcal{A}_1)} \mathop{\mathrm{E}}_{\substack{a_1\sim\mathsf{p}\\ a_2\sim\mathsf{q}}}[u(a_1,a_2)].$$

  - single strategy good against any strategy by other player.
  - order of play does not matter.
  - mixed Nash equilibria, same payoff (value of the game).

Sion's Minimax Theorem

(Sion, 1958)

- Theorem (simplified version): let $\mathcal{X}$ and $\mathcal{Y}$ be convex and compact sets and $f\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ a function that is convex with respect to its first argument and concave with respect to its second argument. Then,

$$\inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} f(x, y) = \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} f(x, y).$$

Biaffine functions

- Definition: let $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^m$ be convex and compact sets, then biaffine is $u\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is biaffine if

$$u(\alpha \mathbf{x} + (1-\alpha)\mathbf{x}', \mathbf{y}) = \alpha u(\mathbf{x}, \mathbf{y}) + (1-\alpha) u(\mathbf{x}', \mathbf{y})$$
$$u(\mathbf{x}, \alpha \mathbf{y} + (1-\alpha)\mathbf{y}') = \alpha u(\mathbf{x}, \mathbf{y}) + (1-\alpha) u(\mathbf{x}, \mathbf{y}')$$

  - for all $\alpha \in [0,1], \mathbf{x}, \mathbf{x}' \in \mathcal{X}, \mathbf{y}, \mathbf{y}' \in \mathcal{Y}$.

Equivalent Statement

- Equivalent formulation of Sion's theorem for biaffine function: for all $c \in \mathbb{R}$,

$$\forall \mathbf{y} \in \mathcal{Y}\, \exists \mathbf{x} \in \mathcal{X} \colon u(\mathbf{x}, \mathbf{y}) \in [c, +\infty) \implies \exists \mathbf{x} \in \mathcal{X}\, \forall \mathbf{y} \in \mathcal{Y} \colon u(\mathbf{x}, \mathbf{y}) \in [c, +\infty);$$

or $$\min_{y \in \mathcal{Y}} \max_{x \in \mathcal{X}} u(\mathbf{x}, \mathbf{y}) \geq c \implies \max_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}} u(\mathbf{x}, \mathbf{y}) \geq c.$$

Vectorial Payoffs

- Vector valued games: biaffine function $u\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^d$.
- Question: can the result be extended? Given convex set S,

$$\forall \mathbf{y} \in \mathcal{Y}, \exists \mathbf{x} \in \mathcal{X}\colon u(\mathbf{x}, \mathbf{y}) \in S \implies \exists \mathbf{x} \in \mathcal{X}, \forall \mathbf{y} \in \mathcal{Y}\colon u(\mathbf{x}, \mathbf{y}) \in S.$$

- Counter-example:

$$\mathcal{X} = \mathcal{Y} = [0,1], u(\mathbf{x}, \mathbf{y}) = (x, y), S = \{(z, z)\colon z \in [0,1]\}.$$

  - clearly, $\forall \mathbf{y} \in \mathcal{Y}, \exists \mathbf{x} = \mathbf{y}, u(\mathbf{x}, \mathbf{y}) \in S$.
  - but, $\nexists \mathbf{x} \in \mathcal{X}, \forall \mathbf{y} \in \mathcal{Y}\colon u(\mathbf{x}, \mathbf{y}) \in S$.

Approachability

(Blackwell, 1956)

- Assumptions: $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^m$ convex and compact sets, biaffine vectorial payoff $u\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^d$, $S \subset \mathbb{R}^d$ a closed convex set.

- Definition: S is approachable if there exists an algorithm $\mathcal{A}$ such that for any sequence $y_1, \ldots, y_T \in \mathcal{Y}$ and $x_1, \ldots, x_T \in \mathcal{X}$ with $x_t = \mathcal{A}(y_1, \ldots, y_{t-1})$,

$$\lim_{T \to +\infty} d\left(\frac{1}{T}\sum_{t=1}^{T} u(x_t, y_t), S\right) = 0.$$

  - for any $z \in \mathbb{R}^d$, $d(z, S) = \inf_{s \in S} \|z - s\|_2$.
  - repeated game, adaptive player strategy.

Blackwell's Theorem

- **Theorem**: let $S \subset \mathbb{R}^d$ be a closed convex set with $S \subseteq \mathsf{B}(0, R)$. Assume that: $\forall y \in \mathcal{Y}, \exists x \in \mathcal{X} : u(x, y) \in S$. Then, S is approachable and there exists $\mathcal{A}$ such that

$$d\Big(\tfrac{1}{T}\textstyle\sum_{t=1}^T u(x_t, y_t), S\Big) \leq \frac{2R}{\sqrt{T}}.$$

- **Proof**: let $H = \{z \in \mathbb{R}^d \colon w \cdot z \geq c\}$ be a halfspace that contains S. By assumption: $\min_{y \in \mathcal{Y}} \max_{x \in \mathcal{X}} w \cdot u(x, y) \geq c$. By Sion's theorem, this implies:

$$\max_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}} w \cdot u(x, y) \geq c.$$

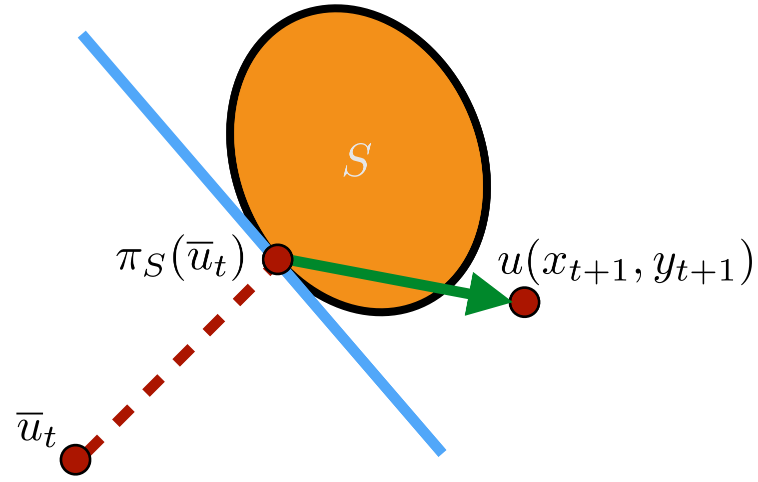
  - thus,

$$\exists x^* \in \mathcal{X} \colon \forall y \in \mathcal{Y}, u(x, y) \in H.$$

Proof

- Choice of H: $H_t = \left\{ z \in \mathbb{R}^d \colon [z - \pi_S(\overline{u}_t)] \cdot [\pi_S(\overline{u}_t) - \overline{u}_t] \geq 0 \right\}$, where $\overline{u}_t = \frac{1}{T} \sum_{i=1}^t u(x_t, y_t)$. Define x_{t+1} to be x^* for H_t .
  - since $\overline{u}_{t+1} = \frac{t}{t+1} \overline{u}_t + \frac{1}{t+1} u(x_{t+1}, y_{t+1})$,

$$
\begin{aligned}
& d(\overline{u}_{t+1}, S)^2 \\
& \leq d(\overline{u}_{t+1}, \pi_S(\overline{u}_t))^2 \\
& = \|\overline{u}_{t+1} - \pi_S(\overline{u}_t)\|_2^2 \\
& = \left\| \frac{t[\overline{u}_t - \pi_S(\overline{u}_t)]}{t+1} + \frac{[u(x_{t+1}, y_{t+1}) - \pi_S(\overline{u}_t)]}{t+1} \right\|_2^2 \\
& \leq \frac{t^2 d(\overline{u}_t, S)^2}{(t+1)^2} + \frac{\left\| u(x_{t+1}, y_{t+1}) - \pi_S(\overline{u}_t) \right\|^2}{(t+1)^2}.
\end{aligned}
$$



Proof

- This gives:

$$(t+1)^2 d(\overline{u}_{t+1}, S)^2 \leq t^2 d(\overline{u}_t, S)^2 + \left\|u(x_{t+1}, y_{t+1}) - \pi_S(\overline{u}_t)\right\|^2.$$

- assuming that $\|u(x_{t+1}, y_{t+1})\| \leq R$, this implies

$$(t+1)^2 d(\overline{u}_{t+1}, S)^2 \leq t^2 d(\overline{u}_t, S)^2 + 4R^2;$$

- Thus, $T^2 d(\overline{u}_T, S)^2 \leq 4TR^2$ and

$$d(\overline{u}_T, S) \leq \frac{2R}{\sqrt{T}}.$$

External Regret

- K actions, $\mathcal{X} = \Delta_K$, losses bounded by one.
- Regret per round of algorithm $\mathcal{A}$:

$$\frac{\text{Reg}_T(\mathcal{A})}{T} = \max_{a \in [K]} \frac{1}{T} \sum_{t=1}^{T} \big[\mathsf{p}_t \cdot \ell_t - \ell_t(a)\big].$$

- Vectorial payoff:

$$u(\mathsf{p}_t, \ell_t) = \begin{bmatrix} \mathsf{p}_t \cdot \ell_t - \ell_t(a_1) \\ \vdots \\ \mathsf{p}_t \cdot \ell_t - \ell_t(a_K) \end{bmatrix}.$$

- Approachable set: $S = [-1, 0]^K$.
- Assumption holds: for any $\ell \in [0,1]^K$,

$$u(\mathsf{p}_t, \ell_t) \in S \text{ for } \mathsf{p}_t = \mathbf{e}_k, \text{ with } k \in \operatorname*{argmin}_{k \in [K]} \ell(a_k).$$

- Regret bound: $O\Big(\sqrt{\frac{K}{T}}\Big)$ (suboptimal).

Approachability Using OCO

(Abernethy et al., 2011; Dann et al., 2023)

- Fenchel duality:

$$
\begin{aligned}
d(x, S) &= \min_{s \in S} \|x - s\|_2 \\
&= \min_{s \in \mathbb{R}^d} \|x - s\|_2 + I_S(s) && \text{(def. of } I_S\text{)} \\
&= \max_{\theta \in \mathbb{R}^d} -\{I_{\mathsf{B}_2(0,1)}(\theta) + \theta \cdot x\} - \sup_{s \in S}\{-\theta \cdot s\} && \text{(Fenchel duality theorem)} \\
&= \max_{\theta \in \mathsf{B}_2(0,1)} -\theta \cdot x - \sup_{s \in S}\{-\theta \cdot s\} && \text{(def. of } I_{\mathsf{B}_2(0,1)}\text{)} \\
&= \max_{\theta \in \mathsf{B}_2(0,1)} \left\{\theta \cdot x - \underbrace{\sup_{s \in S} \theta \cdot s}_{I_S^*(\theta)}\right\}.
\end{aligned}
$$

OCO-Based Algorithm

OCO2APP$(\mathcal{A}, S)$

$\theta_1 \leftarrow \textsc{Random}(\mathsf{B}(0,1))$

for $t \leftarrow 1$ **to** T **do**

  $\mathsf{p}_t \leftarrow \mathsf{p} \in \Delta_K \colon \forall y \in \mathcal{Y}, \theta_t \cdot u(\mathsf{p}, y) \leq \max_{s \in S} \theta_t \cdot s$ ▷ exists by sep. assumption

  $y_t \leftarrow \textsc{Receive}(\mathcal{Y})$

  $f_t \leftarrow \theta \mapsto \max_{s \in S} \theta \cdot s - \theta \cdot u(\mathsf{p}_t, y_t)$

  $\theta_{t+1} \leftarrow \mathcal{A}(f_1^t, \theta_1^t)$

- Guarantee:

$$
\begin{aligned}
d(\overline{u}_T, S) &= \max_{\theta \in \mathsf{B}_2(0,1)} \left\{ \theta \cdot \overline{u}_T - \max_{s \in S} \theta \cdot s \right\} \\
&= \max_{\theta \in \mathsf{B}_2(0,1)} \left\{ -\frac{1}{T} \sum_{t=1}^{T} f_t(\theta) \right\} = - \min_{\theta \in \mathsf{B}_2(0,1)} \left\{ \frac{1}{T} \sum_{t=1}^{T} f_t(\theta) \right\} \\
&= \frac{\mathsf{Reg}_T(\mathcal{A})}{T} - \frac{1}{T} \sum_{t=1}^{T} \underbrace{f_t(\theta_t)}_{\geq 0} \leq \frac{\mathsf{Reg}_T(\mathcal{A})}{T}.
\end{aligned}
$$

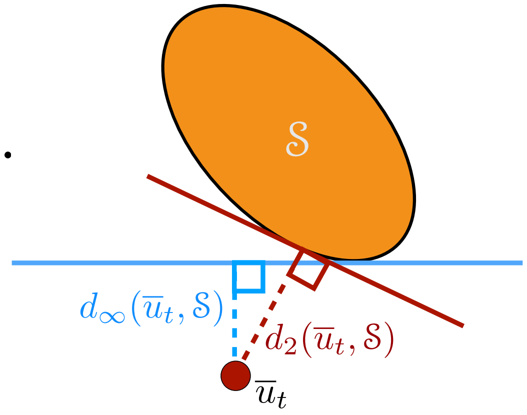
ℓ_∞-Approachability

(e.g., Dann et al., 2023)

- Assumptions: $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^m$ convex and compact sets, biaffine vectorial payoff $u\colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^d$, $S \subset \mathbb{R}^d$ a closed convex set.

- Definition: S is ℓ_∞-approachable if there exists an algorithm $\mathcal{A}$ such that for any sequence $y_1, \ldots, y_T \in \mathcal{Y}$ and $x_1, \ldots, x_T \in \mathcal{X}$ with $x_t = \mathcal{A}(y_1, \ldots, y_{t-1})$,

$$\lim_{t \to +\infty} d_\infty\left(\frac{1}{T} \sum_{t=1}^{T} u(x_t, y_t), S\right) = 0.$$



  - for any $z \in \mathbb{R}^d$, $d(z, S) = \inf_{s \in S} \|z - s\|_\infty$.
  - repeated game, adaptive player strategy.

ℓ_∞-Approachability Using OCO

(Abernethy et al., 2011; Dann et al., 2023)

- Fenchel duality:

$$
\begin{aligned}
d_\infty(x, S) &= \min_{s \in S} \|x - s\|_\infty \\
&= \min_{s \in \mathbb{R}^d} \|x - s\|_\infty + I_S(s) && \text{(def. of } I_S\text{)} \\
&= \max_{\theta \in \mathbb{R}^d} -\{I_{\mathsf{B}_1(0,1)}(\theta) + \theta \cdot x\} - \sup_{s \in S}\{-\theta \cdot s\} \\
& && \text{(Fenchel duality theorem)} \\
&= \max_{\theta \in \mathsf{B}_1(0,1)} -\theta \cdot x - \sup_{s \in S}\{-\theta \cdot s\} && \text{(def. of } I_{\mathsf{B}_1(0,1)}\text{)} \\
&= \max_{\theta \in \mathsf{B}_1(0,1)} \left\{\theta \cdot x - \sup_{s \in S} \theta \cdot s\right\}.
\end{aligned}
$$

ℓ_∞-OCO-Based Algorithm

OCO2APP($\mathcal{A}, S$)

```
θ_1 ← RANDOM(B(0, 1))
for t ← 1 to T do
    p_t ← p ∈ Δ_K : ∀y ∈ 𝒴, θ_t·u(p, y) ≤ max_{s∈S} θ_t·s   ▷ exists by sep. assumption
    y_t ← RECEIVE(𝒴)
    f_t ← θ ↦ max_{s∈S} θ·s − θ·u(p_t, y_t)
    θ_{t+1} ← 𝒜(f_1^t, θ_1^t)
```

- **Guarantee:**

$$\begin{aligned}
d(\overline{u}_T, S) &= \max_{\theta \in \mathsf{B}_1(0,1)} \left\{ \theta \cdot \overline{u}_T - \max_{s \in S} \theta \cdot s \right\} \\
&= \max_{\theta \in \mathsf{B}_1(0,1)} \left\{ -\frac{1}{T} \sum_{t=1}^{T} f_t(\theta) \right\} = - \min_{\theta \in \mathsf{B}_1(0,1)} \left\{ \frac{1}{T} \sum_{t=1}^{T} f_t(\theta) \right\} \\
&= \frac{\mathsf{Reg}_T(\mathcal{A})}{T} - \frac{1}{T} \sum_{t=1}^{T} \underbrace{f_t(\theta_t)}_{\geq 0} \leq \frac{\mathsf{Reg}_T(\mathcal{A})}{T}.
\end{aligned}$$

External Regret

- K actions, $\mathcal{X} = \Delta_K$, losses bounded by one.
- Vectorial payoff and regret per round of algorithm $\mathcal{A}$:

$$u(\mathsf{p}_t, \ell_t) = \begin{bmatrix} \mathsf{p}_t \cdot \ell_t - \ell_t(a_1) \\ \vdots \\ \mathsf{p}_t \cdot \ell_t - \ell_t(a_K) \end{bmatrix} \qquad \left| \frac{\mathrm{Reg}_T(\mathcal{A})}{T} \right| = \left\| \frac{1}{T} \sum_{t=1}^{T} u(\mathsf{p}_t, y_t) \right\|_\infty .$$

- Approachable set: $S = [-1, 0]^K$.
- Choice of p_t: $\mathsf{p}_t = \theta_t \in \Delta_K$.
- Loss function: $f_t(\theta) = -\theta \cdot u(\mathsf{p}_t, \ell_t) = -\theta \cdot u(\theta_t, \ell_t)$.
- Algorithm $\mathcal{A}$: use EG algorithm.
- Regret bound: $2\sqrt{T \log K}$.

References

- (Blackwell, 1956): paper introducing approachability with constructive proof.
- (Abernethy et al., 2011): reduction of approachability to online linear optimization.
- (Foster and Vohra, 1999): internal regret and relationship with approachability; see also references therein.
- (Hart and Mas-Colell, 2000): internal regret as approachability, with explicit strategies.
- (Hart and Mas-Colell, 2001): class of approachability algorithms, related to projection in some norm.

References

- (Shimkin, 2016): approachability for an arbitrary norm, general duality result using Sion's minimax theorem.
- (Kwon, 2021): duality theorem similar to (Shimkin, 2016), FTRL algorithm for general norm.
- (Perchet, 2015): ℓ_∞-approachability, exponential weight algorithm + several other interesting approachability publications from this author.
- (Dann, Mansour, MM, Schneider, and Sivan, 2023): emphasizes importance of ℓ_∞-approachability, general conversion to lower-dim space of vectorial payoffs, general Fenchel duality result and pseudonorm approachability.