Advanced Machine Learning

Online Convex Optimization

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Outline

- Online projected sub-gradient descent.
- Exponentiated Gradient (EG).
- Mirror descent.
- Dual Averaging.

Set-Up

- Convex set C.
- For $t = 1$ to T do
  - predict $\mathbf{w}_t \in C$.
  - receive convex loss function $f_t \colon C \to \mathbb{R}$.
  - incur loss $f_t(\mathbf{w}_t)$.
- Regret of algorithm $\mathcal{A}$:

$$R_T(\mathcal{A}) = \sum_{t=1}^{T} f_t(\mathbf{w}_t) - \inf_{\mathbf{w} \in C} \sum_{t=1}^{T} f_t(\mathbf{w}).$$

Online Projected Subgrad. Desc.

- Algorithm:
  - $\mathbf{w}_1 \in C$ arbitrary.
  - $\mathbf{w}_{t+1} = \Pi_C[\mathbf{w}_t - \eta\, \delta f_t(\mathbf{w}_t)]$, where
    - Π_C is the projection over C.
    - $\delta f_t(\mathbf{w}_t) \in \partial f_t(\mathbf{w}_t)$ (sub-gradient of f_t at $\mathbf{w}_t$).
    - $\eta > 0$ parameter.

Analysis

(Zinkevich, 2003)

- **Assumptions**:
  - $\|\mathbf{w}_1 - \mathbf{w}^*\| \leq R$ where $\mathbf{w}^* \in \operatorname*{argmin}_{\mathbf{w} \in C} \sum_{t=1}^{T} f_t(\mathbf{w})$.
  - $\|\delta f_t(\mathbf{w}_t)\| \leq G$.
- **Theorem**: the regret of online projected sub-gradient descent (PSGD) is bounded as follows

$$R_T(\text{PSGD}) \leq \frac{R^2}{2\eta} + \frac{\eta G^2 T}{2}.$$

Choosing η to minimize the bound gives

$$\boxed{R_T(\text{PSGD}) \leq RG\sqrt{T}.}$$

Proof

- The proof uses the definition of subgradient and the property of projection:

$$
\begin{aligned}
R_T(\text{PSGD}) &= \sum_{t=1}^{T} \big(f_t(\mathbf{w}_t) - f_t(\mathbf{w}^*)\big) \\
&\leq \sum_{t=1}^{T} \delta f_t(\mathbf{w}_t) \cdot (\mathbf{w}_t - \mathbf{w}^*) && \text{(def. of subgrad.)} \\
&= \sum_{t=1}^{T} \frac{1}{2\eta} \Big[\|\mathbf{w}_t - \mathbf{w}^*\|^2] + \eta^2 \|\delta f_t(\mathbf{w}_t)\|^2 - \|\mathbf{w}_t - \eta \delta f_t(\mathbf{w}_t) - \mathbf{w}^*\|^2 \Big] \\
&\leq \sum_{t=1}^{T} \frac{1}{2\eta} \Big[\|\mathbf{w}_t - \mathbf{w}^*\|^2] + \eta^2 G^2 - \|\mathbf{w}_{t+1} - \mathbf{w}^*\|^2 \Big] && \text{(prop. of proj.)} \\
&\leq \frac{1}{2\eta} \Big[\|\mathbf{w}_1 - \mathbf{w}^*\|^2] + \eta^2 G^2 T - \|\mathbf{w}_{T+1} - \mathbf{w}^*\|^2 \Big] && \text{(telescop. sum)} \\
&\leq \frac{1}{2\eta} \Big[\|\mathbf{w}_1 - \mathbf{w}^*\|^2] + \eta^2 G^2 T \Big] \leq \frac{1}{2\eta} \Big[R^2 + \eta^2 G^2 T \Big].
\end{aligned}
$$

Convex Optimization

- Application: $\min_{\mathbf{w} \in C} f(\mathbf{w})$.
  - fixed loss function: $f_t = f$.
  - guarantee for average weight vector:

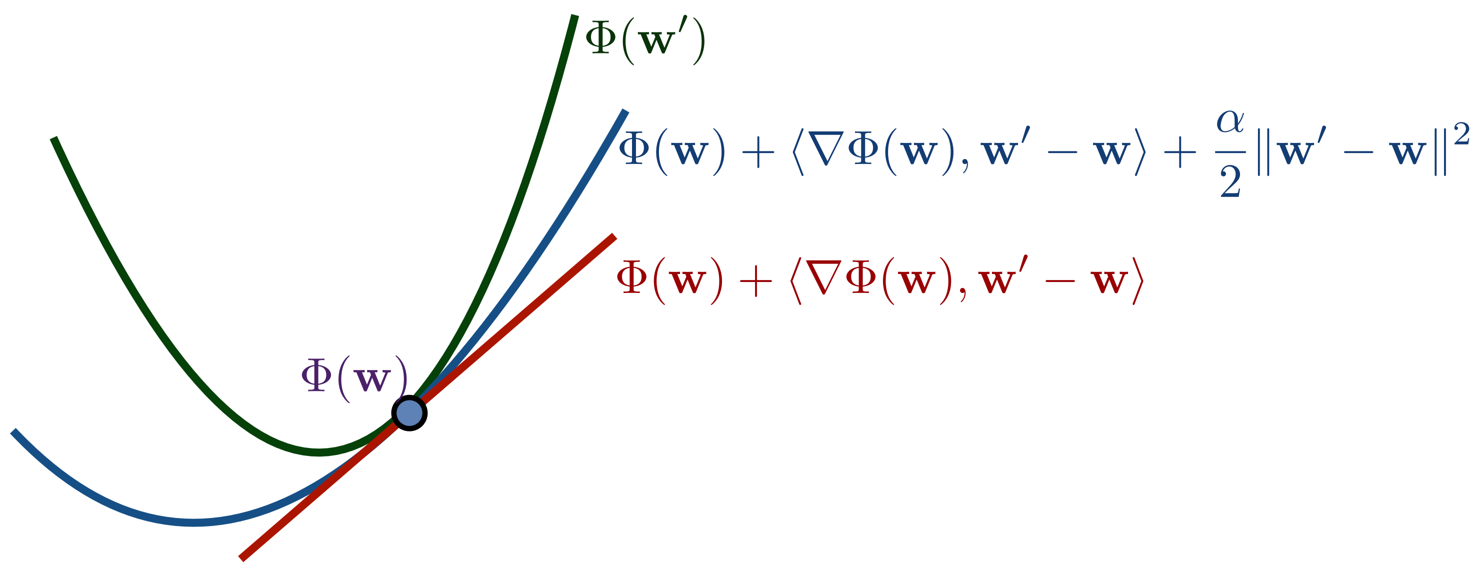
$$f\Big(\frac{1}{T}\sum_{t=1}^{T} \mathbf{w}_t\Big) - f(\mathbf{w}^*) \leq \frac{1}{T}\sum_{t=1}^{T} f(\mathbf{w}_t) - f(\mathbf{w}^*)$$

$$= \frac{R_T(\mathcal{A})}{T} = O\left(\frac{1}{\sqrt{T}}\right).$$

  - thus, convergence in $O\left(\frac{1}{\epsilon^2}\right)$.

Strong Convexity

- Definition: a convex function Φ defined over a convex set C is α-strongly convex with respect to norm $\|\cdot\|$ if the function $\mathbf{w} \mapsto \Phi(\mathbf{w}) - \frac{\alpha}{2}\|\mathbf{w}\|^2$ is convex or, equivalently,
  - for all $\mathbf{w}, \mathbf{w}'$ in C and $\delta\Phi(\mathbf{w}) \in \partial\Phi(\mathbf{w})$,
    $\Phi(\mathbf{w}') \geq \Phi(\mathbf{w}) + \delta\Phi(\mathbf{w}) \cdot (\mathbf{w}' - \mathbf{w}) + \frac{\alpha}{2}\|\mathbf{w}' - \mathbf{w}\|^2.$



Strongly Convex Objectives

(Hazan et al., 2007)

- Theorem: assume that the functions f_t are α-strongly convex and $\|\delta f_t(\mathbf{w})\| \leq G$ for all $\mathbf{w}$ and $\delta f_t \in \partial f_t(\mathbf{w})$. Then, the regret of online projected sub-gradient descent (PSGD) with parameter $\eta_{t+1} = \frac{1}{\alpha t}$ is bounded as follows

$$R_T(\text{PSGD}) \leq \frac{G^2}{2\alpha}(1 + \log T).$$

Proof

$R_T(\text{PSGD})$

$$= \sum_{t=1}^{T} \big(f_t(\mathbf{w}_t) - f_t(\mathbf{w}^*)\big)$$

$$\leq \sum_{t=1}^{T} \delta f_t(\mathbf{w}_t) \cdot (\mathbf{w}_t - \mathbf{w}^*) - \frac{\alpha}{2} \|\mathbf{w}_t - \mathbf{w}^*\|^2 \qquad \text{(strong convexity)}$$

$$= \sum_{t=1}^{T} \frac{1}{2\eta_{t+1}} \Big[\|\mathbf{w}_t - \mathbf{w}^*\|^2 + \eta_{t+1}^2 \|\delta f_t(\mathbf{w}_t)\|^2 - \|\mathbf{w}_t - \eta_{t+1}\delta f_t(\mathbf{w}_t) - \mathbf{w}^*\|^2\Big] - \frac{\alpha}{2}\|\mathbf{w}_t - \mathbf{w}^*\|^2$$

$$\leq \sum_{t=1}^{T} \frac{1}{2\eta_{t+1}} \Big[\|\mathbf{w}_t - \mathbf{w}^*\|^2 + \eta_{t+1}^2 G^2 - \|\mathbf{w}_{t+1} - \mathbf{w}^*\|^2\Big] - \frac{\alpha}{2}\|\mathbf{w}_t - \mathbf{w}^*\|^2 \qquad \text{(prop. of proj.)}$$

$$\leq \frac{\alpha}{2} \sum_{t=1}^{T} \Big[(t-1)\|\mathbf{w}_t - \mathbf{w}^*\|^2 - t\|\mathbf{w}_{t+1} - \mathbf{w}^*\|^2\Big] + \frac{G^2}{2\alpha} \sum_{t=1}^{T} \frac{1}{t} \qquad \text{(def. of } \eta_{t+1})$$

$$= \frac{\alpha}{2}\Big[-T\|\mathbf{w}_{T+1} - \mathbf{w}^*\|^2\Big] + \frac{G^2}{2\alpha}\sum_{t=1}^{T}\frac{1}{t} \leq \frac{G^2}{2\alpha}\sum_{t=1}^{T}\frac{1}{t} \leq \frac{G^2}{2\alpha}(1 + \log T). \qquad \text{(telescoping sum)}$$

Smoothness

- Definition: a continuously differentiable function f is β-smooth if its gradient is β-Lipschitz:

$$\|\nabla f(\mathbf{w}') - \nabla f(\mathbf{w})\| \leq \beta \|\mathbf{w}' - \mathbf{w}\|,$$

  for all $\mathbf{w}, \mathbf{w}'$.

- Property: if f is convex and β-smooth, then, for all $\mathbf{w}, \mathbf{w}'$,

$$0 \leq f(\mathbf{w}) - f(\mathbf{w}') - \nabla f(\mathbf{w}') \cdot (\mathbf{w} - \mathbf{w}') \leq \frac{\beta}{2} \|\mathbf{w} - \mathbf{w}'\|^2.$$

Exponentiated Gradient (EG)

(Kivinen and Warmuth, 1997)

- Convex set: simplex $C = \{\mathbf{w} \in \mathbb{R}^N : \mathbf{w} \geq 0 \wedge \|\mathbf{w}\|_1 = 1\}$.
- Algorithm:
  - $\mathbf{w}_1 = (\frac{1}{N}, \ldots, \frac{1}{N})^\top$.
  - $\mathbf{w}_{t+1,i} = \dfrac{\mathbf{w}_{t,i} \exp(-\eta \, [\delta f_t(\mathbf{w}_t)]_i)}{Z_t}$ where

    $Z_t = \sum_{i=1}^N \mathbf{w}_{t,i} e^{-\eta [\delta f_t(\mathbf{w}_t)]_i}$.

Analysis

- Assumption:
  - $\|\delta f_t(\mathbf{w}_t)\|_\infty \leq G_\infty$.
- Theorem: the regret of the Exponentiated Gradient (EG) algorithm is bounded as follows

$$R_T(\mathrm{EG}) \leq \frac{\log N}{\eta} + \frac{\eta G_\infty^2 T}{2}.$$

Choosing η to minimize the bound gives

$$\boxed{R_T(\mathrm{EG}) \leq 2G_\infty \sqrt{T \log N}.}$$

Proof

- Potential: $\Phi_t = D(\mathbf{w}^* \parallel \mathbf{w}_t) = \sum_{i=1}^N \mathbf{w}_i^* \log \frac{\mathbf{w}_i^*}{\mathbf{w}_{t,i}}$.
- $$\Phi_{t+1} - \Phi_t = \sum_{i=1}^N \mathbf{w}_i^* \log \frac{\mathbf{w}_{t,i}}{\mathbf{w}_{t+1,i}}$$
$$= \sum_{i=1}^N \mathbf{w}_i^* \Big[\log Z_t + \eta[\delta f_t(\mathbf{w}_t)]_i \Big] = \log Z_t + \eta \mathbf{w}^* \cdot \delta f_t(\mathbf{w}_t).$$
- $$\log Z_t = \log \left[\sum_{i=1}^N \mathbf{w}_{t,i} e^{-\eta [\delta f_t(\mathbf{w}_t)]_i} \right]$$
$$= \log \mathop{\mathrm{E}}_{i \sim \mathbf{w}_t} \left[e^{-\eta [\delta f_t(\mathbf{w}_t)]_i} \right]$$
$$= \log \mathop{\mathrm{E}}_{i \sim \mathbf{w}_t} \left[e^{-\eta \left([\delta f_t(\mathbf{w}_t)]_i - \mathrm{E}\left[[\delta f_t(\mathbf{w}_t)]_i \right] \right) - \eta\, \mathrm{E}\left[[\delta f_t(\mathbf{w}_t)]_i \right]} \right]$$
$$\leq \eta^2 \frac{4 G_\infty^2}{8} - \eta \mathbf{w}_t \cdot \delta f_t(\mathbf{w}_t). \qquad \text{(Hoeffding's lemma)}$$

Proof

- Combining equality and inequality:

$$\Phi_{t+1} - \Phi_t \leq \frac{\eta^2 G_\infty^2}{2} - \eta(\mathbf{w}^* - \mathbf{w}_t) \cdot \delta f_t(\mathbf{w}_t)$$

$$\Leftrightarrow \eta(\mathbf{w}^* - \mathbf{w}_t) \cdot \delta f_t(\mathbf{w}_t) \leq \frac{\eta^2 G_\infty^2}{2} + (\Phi_t - \Phi_{t+1})$$

$$\Rightarrow \sum_{t=1}^{T} (\mathbf{w}^* - \mathbf{w}_t) \cdot \delta f_t(\mathbf{w}_t) \leq \frac{\eta^2 G_\infty^2 T}{2} + \frac{\Phi_1 - \Phi_{T+1}}{\eta}$$

$$\Rightarrow \sum_{t=1}^{T} (\mathbf{w}^* - \mathbf{w}_t) \cdot \delta f_t(\mathbf{w}_t) \leq \frac{\eta^2 G_\infty^2 T}{2} + \frac{\Phi_1}{\eta}. \qquad \text{(Rel. Ent. non-neg.)}$$

- $$R_T(\text{EG}) = \sum_{t=1}^{T} \big(f_t(\mathbf{w}_t) - f_t(\mathbf{w}^*)\big)$$

$$\leq \sum_{t=1}^{T} \delta f_t(\mathbf{w}_t) \cdot (\mathbf{w}_t - \mathbf{w}^*)$$

$$\leq \frac{\eta G_\infty^2 T}{2} + \frac{\Phi_1}{\eta} = \frac{\eta G_\infty^2 T}{2} + \frac{D(\mathbf{w}^* \parallel \mathbf{w}_1)}{\eta} \leq \frac{\eta G_\infty^2 T}{2} + \frac{\log N}{\eta}.$$

Generalization

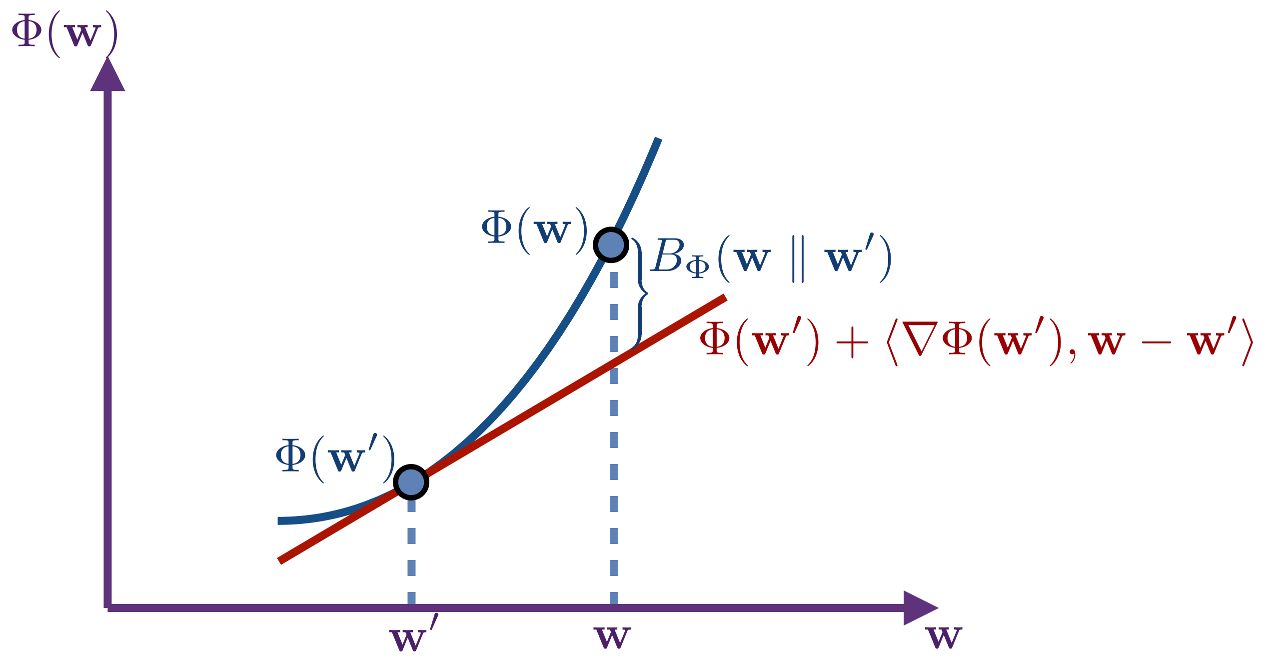
- PSGD and EG both special instances of a more general algorithm: Mirror Descent.
- Mirror Descent is based on a Bregman divergence:
  - PSGD: $B(\mathbf{w} \parallel \mathbf{w}') = \frac{1}{2}\|\mathbf{w} - \mathbf{w}'\|_2^2$.
  - EG: unnormalized relative entropy;

$$B(\mathbf{w} \parallel \mathbf{w}') = \sum_{i=1}^{N} \left[w_i \log\left[\frac{w_i}{w'_i}\right] - w_i + w'_i \right].$$

Bregman Divergence

- Definition: Φ convex differentiable over open convex set C. The Bregman divergence associated to Φ is defined by

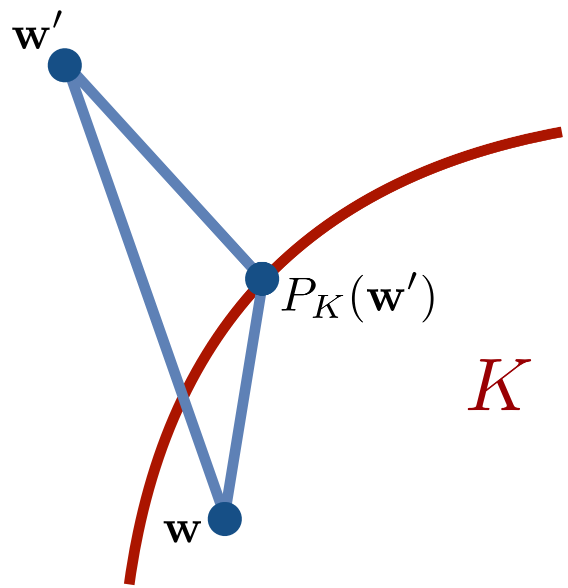
$$B_\Phi(\mathbf{w} \parallel \mathbf{w}') = \Phi(\mathbf{w}) - \Phi(\mathbf{w}') - \langle \nabla\Phi(\mathbf{w}'), \mathbf{w} - \mathbf{w}' \rangle.$$



Properties

- Proposition: the following properties hold for a Bregman divergence.
  - non-negativity: $\forall \mathbf{w}, \mathbf{w}' \in C,\ B_\Phi(\mathbf{w} \parallel \mathbf{w}') \geq 0$.
  - linearity: $B_{\alpha\Phi+\beta\Psi} = \alpha B_\Phi + \beta B_\Psi$.
  - projection: for any closed convex set $K \subseteq \overline{C}$, the projection of B_Φ-projection of $\mathbf{w}'$ over K is unique:
$$P_K(\mathbf{w}') = \operatorname*{argmin}_{\mathbf{w} \in K} B_F(\mathbf{w} \parallel \mathbf{w}').$$
  - Triangular identity:
$$(\nabla\Phi(\mathbf{w}) - \nabla\Phi(\mathbf{v})) \cdot (\mathbf{w} - \mathbf{u}) = B(\mathbf{u} \parallel \mathbf{w}) + B(\mathbf{w} \parallel \mathbf{v}) - B(\mathbf{u} \parallel \mathbf{v}).$$
  - Pythagorean theorem:
$$B_\Phi(\mathbf{w} \parallel \mathbf{w}') \geq B_\Phi(\mathbf{w} \parallel P_K(\mathbf{w}')) + B_\Phi(P_K(\mathbf{w}') \parallel \mathbf{w}').$$

Pythagorean theorem



$$B_\Phi(\mathbf{w} \parallel \mathbf{w}') \geq B_\Phi(\mathbf{w} \parallel P_K(\mathbf{w}')) + B_\Phi(P_K(\mathbf{w}') \parallel \mathbf{w}').$$

Legendre Type Functions

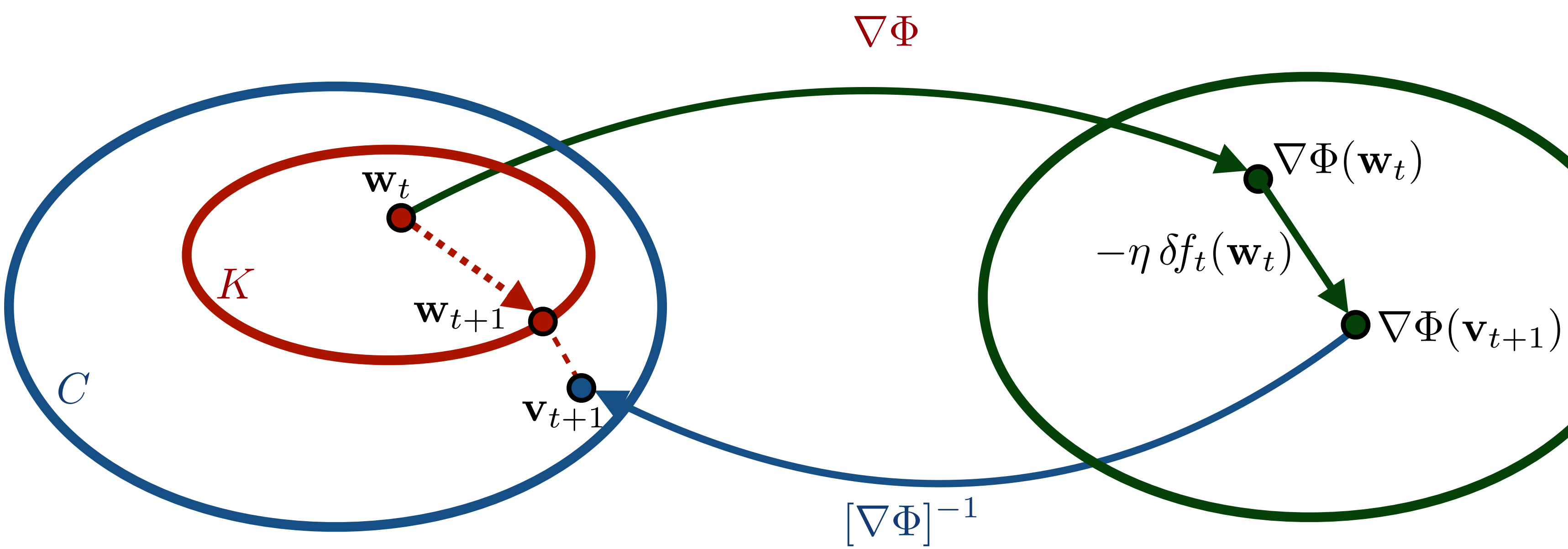
(Rockafellar, 1970)

- Definition: a real-valued function Φ defined over a non-empty open convex set C is said to be of Legendre type if it is proper closed convex and differentiable over C and if one of the following equivalent conditions holds:
  - $\nabla\Phi$ is one-to-one mapping from C to $\nabla\Phi(C)$.
  - $\lim_{\mathbf{w}\to\partial C} \|\nabla\Phi(\mathbf{w})\| = +\infty$.

  - *proper*: $(\forall x \in C, \Phi(x) > -\infty) \wedge (\exists x_0 \in C, \Phi(x_0) < +\infty)$.
  - *closed*: sublevel set $\{x \in C \colon \Phi(x) \leq t\}$ closed for any $t \in \mathbb{R}$.

Mirror Descent

(Nemirovski and Yudin, 1983)



Mirror Descent

$\textsc{Mirror-Descent}(\Phi)$

$\mathbf{w}_1 \leftarrow \operatorname{argmin}_{\mathbf{w} \in K \cap C} \Phi(\mathbf{w})$
for $t \leftarrow 1$ **to** T **do**
  $\mathbf{v}_{t+1} \leftarrow [\nabla \Phi]^{-1}\big(\nabla \Phi(\mathbf{w}_t) - \eta\, \delta f_t(\mathbf{w}_t)\big)$
  $\mathbf{w}_{t+1} \leftarrow \operatorname{argmin}_{\mathbf{w} \in K \cap C} B(\mathbf{w} \parallel \mathbf{v}_{t+1})$

MD Guarantee

- Theorem: let C be a non-empty open convex set and $K \subset \overline{C}$ a compact convex set. Assume that $\Phi\colon C \to \mathbb{R}$ is of Legendre type and α-strongly convex with respect to $\| \cdot \|$ and f_ts convex and G_*-Lipschitz with respect to $\| \cdot \|_*$. Then, the regret of Mirror Descent can be bounded as follows:
$$R_T(\text{MD}) \leq \frac{B(\mathbf{w}^* \parallel \mathbf{w}_1)}{\eta} + \frac{\eta G_*^2 T}{2\alpha}.$$
Choosing η to minimize the bound gives
$$R_T(\text{MD}) \leq D_\Phi G_* \sqrt{\frac{2T}{\alpha}},$$
with $B(\mathbf{w}^* \parallel \mathbf{w}_1) \leq D_\Phi^2$.

Proof

$$
\begin{aligned}
&R_T(\mathrm{MD}) \\
&= \sum_{t=1}^{T} \big(f_t(\mathbf{w}_t) - f_t(\mathbf{w}^*)\big) \\
&\leq \sum_{t=1}^{T} \delta f_t(\mathbf{w}_t) \cdot (\mathbf{w}_t - \mathbf{w}^*) && \text{(def. of subgrad.)} \\
&= \frac{1}{\eta} \sum_{t=1}^{T} [\nabla \Phi(\mathbf{w}_t) - \nabla \Phi(\mathbf{v}_{t+1})] \cdot (\mathbf{w}_t - \mathbf{w}^*) && \text{(def. of } \mathbf{v}_t\text{)} \\
&= \frac{1}{\eta} \sum_{t=1}^{T} \big[B(\mathbf{w}^* \parallel \mathbf{w}_t) - B(\mathbf{w}^* \parallel \mathbf{v}_{t+1}) + B(\mathbf{w}_t \parallel \mathbf{v}_{t+1})\big] && \text{(Triang. Identity)} \\
&\leq \frac{1}{\eta} \sum_{t=1}^{T} \big[B(\mathbf{w}^* \parallel \mathbf{w}_t) - B(\mathbf{w}^* \parallel \mathbf{w}_{t+1}) - B(\mathbf{w}_{t+1} \parallel \mathbf{v}_{t+1}) + B(\mathbf{w}_t \parallel \mathbf{v}_{t+1})\big] \\
& && \text{(Pythagorean ineq.)} \\
&= \frac{1}{\eta} \Big[B(\mathbf{w}^* \parallel \mathbf{w}_1) - B(\mathbf{w}^* \parallel \mathbf{w}_{T+1})\Big] + \frac{1}{\eta} \sum_{t=1}^{T} \Big[- B(\mathbf{w}_{t+1} \parallel \mathbf{v}_{t+1}) + B(\mathbf{w}_t \parallel \mathbf{v}_{t+1})\Big] \\
& && \text{(Telescoping sum)} \\
&\leq \frac{B(\mathbf{w}^* \parallel \mathbf{w}_1)}{\eta} + \frac{1}{\eta} \sum_{t=1}^{T} \Big[B(\mathbf{w}_t \parallel \mathbf{v}_{t+1}) - B(\mathbf{w}_{t+1} \parallel \mathbf{v}_{t+1})\Big]. \\
& && \text{(Non-negativity of Breg. div.)}
\end{aligned}
$$

Proof

$$
\begin{aligned}
&\Big[B(\mathbf{w}_t \parallel \mathbf{v}_{t+1}) - B(\mathbf{w}_{t+1} \parallel \mathbf{v}_{t+1})\Big] \\
&= \Phi(\mathbf{w}_t) - \Phi(\mathbf{w}_{t+1}) - \nabla\Phi(\mathbf{v}_{t+1}) \cdot (\mathbf{w}_t - \mathbf{w}_{t+1}) \\
&\leq \big(\nabla\Phi(\mathbf{w}_t) - \nabla\Phi(\mathbf{v}_{t+1})\big) \cdot (\mathbf{w}_t - \mathbf{w}_{t+1}) - \frac{\alpha}{2}\|\mathbf{w}_t - \mathbf{w}_{t+1}\|^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\alpha\text{-strong convexity}) \\
&= \eta\, \delta f_t(\mathbf{w}_t) \cdot (\mathbf{w}_t - \mathbf{w}_{t+1}) - \frac{\alpha}{2}\|\mathbf{w}_t - \mathbf{w}_{t+1}\|^2 \qquad (\text{def. of } \mathbf{v}_{t+1}) \\
&\leq \eta G_* \|\mathbf{w}_t - \mathbf{w}_{t+1}\| - \frac{\alpha}{2}\|\mathbf{w}_t - \mathbf{w}_{t+1}\|^2 \qquad (G_*\text{-Lipschitzness}) \\
&\leq \frac{(\eta G_*)^2}{2\alpha}. \qquad (\text{max. of 2nd deg. eq.})
\end{aligned}
$$

Example: PSGD

- Mirror map: $\Phi(\mathbf{w}) = \frac{1}{2}\|\mathbf{w}\|_2^2$, clearly strongly convex with respect to $\|\cdot\|_2$.
- Bregman divergence: $B(\mathbf{w} \parallel \mathbf{w}') = \frac{1}{2}\|\mathbf{w} - \mathbf{w}'\|_2^2$.

Example: EG

- Mirror map: $\Phi(\mathbf{w}) = \sum_{j=1}^{N} w_j \log w_j$, defined over $\mathbb{R}_+^N$, differentiable over $(\mathbb{R}_+^*)^N$.
  - thus, the negative entropy function.
  - 1-strongly convex with respect to $\|\cdot\|_1$ on the simplex:

$$\sum_{j=1}^{N}\left[w_j \log \frac{w_j}{w'_j} + w'_j - w_j\right] = \sum_{j=1}^{N}\left[w_j \log \frac{w_j}{w'_j}\right] \qquad (\mathbf{w} \text{ and } \mathbf{w}' \text{ in simplex})$$

$$\geq \frac{1}{2}\|\mathbf{w} - \mathbf{w}'\|_1^2. \qquad \text{(Schützenberger-Pinsker ineq.)}$$

- Bregman divergence: unnormalized relative entropy defined over $(\mathbb{R}_+^*)^N$,

$$B(\mathbf{w} \parallel \mathbf{w}') = \sum_{i=1}^{N} \left[w_i \log\left[\tfrac{w_i}{w'_i}\right] - w_i + w'_i\right].$$

Example: Spectrahedron

- Mirror map: $\Phi(\mathbf{M}) = \sum_{j=1}^{N} \lambda_j(\mathbf{M}) \log \lambda_j(\mathbf{M})$, defined over the set of semi-definite positive symmetric matrices $\mathbb{S}_+^N$:
  - thus, negative von Neumann entropy.
  - $\frac{1}{2}$-strongly convex with respect to the Shatten 1-norm

$$\|\mathbf{M}\|_{(1)} = \sum_{j=1}^{N} s_j(\mathbf{M}) = \sum_{j=1}^{N} \lambda_j(\mathbf{M}).$$

Conjugate Functions

- Definition: let $\Phi\colon C \to \mathbb{R}$ be a convex function defined over a subset $C \subseteq \mathbb{R}^N$. Then, the conjugate function Φ^* is defined by:

$$\Phi^*(u) = \sup_{x \in C} \big(\langle x, u\rangle - \Phi(x)\big).$$

- For a Legendre function Φ, $(\nabla\Phi)^{-1} = \nabla\Phi^*$.
- For a convex function Φ taking value $+\infty$ outside a convex and compact set K, Φ not necessarily Legendre but Φ^* differentiable, a variant of MD can be used.

Strongly Convex Objectives

- Theorem: assume additionally that f_ts are σ-strongly convex with respect to Φ. Then, the regret of Mirror Descent with parameter $\eta_{t+1} = \frac{1}{\sigma t}$ can be bounded as follows:

$$R_T(\text{MD}) \leq \frac{G_*^2}{2\sigma\alpha}(1 + \log T).$$

Proof

$$
\begin{aligned}
&R_T(\mathrm{MD}) \\
&= \sum_{t=1}^{T} \big(f_t(\mathbf{w}_t) - f_t(\mathbf{w}^*)\big) \\
&\leq \sum_{t=1}^{T} \delta f_t(\mathbf{w}_t) \cdot (\mathbf{w}_t - \mathbf{w}^*) - \sigma B(\mathbf{w}^* \parallel \mathbf{w}_t) && (\Phi\text{-strong convexity}) \\
&= \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} [\nabla \Phi(\mathbf{w}_t) - \nabla \Phi(\mathbf{v}_{t+1})] \cdot (\mathbf{w}_t - \mathbf{w}^*) - \sigma B(\mathbf{w}^* \parallel \mathbf{w}_t) && (\text{Def. of } \mathbf{v}_t) \\
&= \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} \big[B(\mathbf{w}^* \parallel \mathbf{w}_t) - B(\mathbf{w}^* \parallel \mathbf{v}_{t+1}) + B(\mathbf{w}_t \parallel \mathbf{v}_{t+1})\big] - \sigma B(\mathbf{w}^* \parallel \mathbf{w}_t) \\
& && (\text{Breg. div. Identity}) \\
&\leq \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} \big[B(\mathbf{w}^* \parallel \mathbf{w}_t) - B(\mathbf{w}^* \parallel \mathbf{w}_{t+1}) - B(\mathbf{w}_{t+1} \parallel \mathbf{v}_{t+1}) + B(\mathbf{w}_t \parallel \mathbf{v}_{t+1})\big] - \sigma B(\mathbf{w}^* \parallel \mathbf{w}_t) \\
& && (\text{Pyth. ineq.}) \\
&= \sigma \sum_{t=1}^{T} \big[(t-1) B(\mathbf{w}^* \parallel \mathbf{w}_t) - t B(\mathbf{w}^* \parallel \mathbf{w}_{t+1})\big] + \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} \Big[- B(\mathbf{w}_{t+1} \parallel \mathbf{v}_{t+1}) + B(\mathbf{w}_t \parallel \mathbf{v}_{t+1})\Big] \\
& && (\text{Def. of } \eta_{t+1}) \\
&\leq -\sigma T B(\mathbf{w}^* \parallel \mathbf{w}_{T+1}) + \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} \Big[B(\mathbf{w}_t \parallel \mathbf{v}_{t+1}) - B(\mathbf{w}_{t+1} \parallel \mathbf{v}_{t+1})\Big] \\
& && (\text{Telescoping sum}) \\
&\leq \sum_{t=1}^{T} \frac{1}{\eta_{t+1}} \Big[B(\mathbf{w}_t \parallel \mathbf{v}_{t+1}) - B(\mathbf{w}_{t+1} \parallel \mathbf{v}_{t+1})\Big]. && (\text{Non-negativity of Breg. div.})
\end{aligned}
$$

Proof

$$
\begin{aligned}
&\Big[B(\mathbf{w}_t \parallel \mathbf{v}_{t+1}) - B(\mathbf{w}_{t+1} \parallel \mathbf{v}_{t+1})\Big] \\
&= \Phi(\mathbf{w}_t) - \Phi(\mathbf{w}_{t+1}) - \nabla\Phi(\mathbf{v}_{t+1}) \cdot (\mathbf{w}_t - \mathbf{w}_{t+1}) && \text{(Def. of Breg. div.)} \\
&\leq \big(\nabla\Phi(\mathbf{w}_t) - \nabla\Phi(\mathbf{v}_{t+1})\big) \cdot (\mathbf{w}_t - \mathbf{w}_{t+1}) - \frac{\alpha}{2}\|\mathbf{w}_t - \mathbf{w}_{t+1}\|^2 \\
& && (\alpha\text{-strong convexity}) \\
&= \eta_{t+1}\, \delta f_t(\mathbf{w}_t) \cdot (\mathbf{w}_t - \mathbf{w}_{t+1}) - \frac{\alpha}{2}\|\mathbf{w}_t - \mathbf{w}_{t+1}\|^2 && \text{(Def. of } \mathbf{v}_{t+1}) \\
&\leq \eta_{t+1} G_* \|\mathbf{w}_t - \mathbf{w}_{t+1}\| - \frac{\alpha}{2}\|\mathbf{w}_t - \mathbf{w}_{t+1}\|^2 && (G_*\text{-Lipschitzness}) \\
&\leq \frac{(\eta_{t+1} G_*)^2}{2\alpha}. && \text{(Max. of 2nd deg. polynomial)}
\end{aligned}
$$

Thus,

$$
\sum_{t=1}^{T} \frac{1}{\eta_{t+1}} \Big[B(\mathbf{w}_t \parallel \mathbf{v}_{t+1}) - B(\mathbf{w}_{t+1} \parallel \mathbf{v}_{t+1})\Big] \leq \frac{G_*^2}{2\alpha\sigma} \sum_{t=1}^{T} \frac{1}{t} \leq \frac{G_*^2}{2\alpha\sigma}(1 + \log T).
$$

Equivalent Description

$\textsc{Mirror-Descent}(\Phi)$

1 $\mathbf{w}_1 \leftarrow \operatorname{argmin}_{\mathbf{w}\in K\cap C} \Phi(\mathbf{w})$
2 **for** $t \leftarrow 1$ **to** $(T-1)$ **do**
3 $\quad \mathbf{w}_{t+1} \leftarrow \operatorname{argmin}_{\mathbf{w}\in K\cap C} \underbrace{\delta f_t(\mathbf{w}_t)\cdot\mathbf{w}}_{\text{linearization of } f_t} + \underbrace{\frac{1}{\eta} B(\mathbf{w}\parallel \mathbf{w}_t)}_{\text{regularization}}$

- Proof:

$$\begin{aligned}
\mathbf{w}_{t+1} &= \operatorname*{argmin}_{\mathbf{w}\in K\cap C} B(\mathbf{w}\parallel \mathbf{v}_{t+1}) \\
&= \operatorname*{argmin}_{\mathbf{w}\in K\cap C} \Phi(\mathbf{w}) - \nabla\Phi(\mathbf{v}_{t+1})\cdot\mathbf{w} && \text{(def. of Breg. div.)} \\
&= \operatorname*{argmin}_{\mathbf{w}\in K\cap C} \Phi(\mathbf{w}) - \big(\nabla\Phi(\mathbf{w}_t) - \eta\,\delta f_t(\mathbf{w}_t)\big)\cdot\mathbf{w} && \text{(def. of } \mathbf{v}_{t+1}) \\
&= \operatorname*{argmin}_{\mathbf{w}\in K\cap C} \eta\,\delta f_t(\mathbf{w}_t)\cdot\mathbf{w} + B(\mathbf{w}\parallel \mathbf{w}_t). && \text{(def. of Breg. div.)}
\end{aligned}$$

Dual Averaging

(Iouditski and Nesterov, 2010)

DUAL-AVERAGING(Φ)

$\mathbf{v}_1 \leftarrow 0$
$\mathbf{w}_1 \leftarrow \operatorname{argmin}_{\mathbf{w} \in K \cap C} B(\mathbf{w} \parallel \mathbf{v}_1)$
for $t \leftarrow 1$ **to** T **do**
  $\mathbf{v}_{t+1} \leftarrow [\nabla\Phi]^{-1}\big(\underline{\nabla\Phi(\mathbf{v}_t)} - \eta\, \delta f_t(\mathbf{w}_t)\big)$
  $\mathbf{w}_{t+1} \leftarrow \operatorname{argmin}_{\mathbf{w} \in K \cap C} B(\mathbf{w} \parallel \mathbf{v}_{t+1})$

Equivalently:

$$\mathbf{v}_{t+1} \leftarrow [\nabla\Phi]^{-1}\left(\nabla\Phi(\mathbf{w}_1) - \eta \sum_{s=1}^{t} \delta f_s(\mathbf{w}_s)\right)$$

Equivalent Description

- Equivalent form:

$$
\begin{aligned}
\mathbf{w}_{t+1} &= \operatorname*{argmin}_{\mathbf{w}\in K\cap C} \; B(\mathbf{w} \,\|\, \mathbf{v}_{t+1}) \\
&= \operatorname*{argmin}_{\mathbf{w}\in K\cap C} \; \Phi(\mathbf{w}) - \nabla\Phi(\mathbf{v}_{t+1})\cdot\mathbf{w} && \text{(def. of Breg. div.)} \\
&= \operatorname*{argmin}_{\mathbf{w}\in K\cap C} \; \Phi(\mathbf{w}) - \big(\nabla\Phi(\mathbf{v}_t) - \eta\,\delta f_t(\mathbf{w}_t)\big)\cdot\mathbf{w} && \text{(def. of } \mathbf{v}_{t+1}\text{)} \\
&= \operatorname*{argmin}_{\mathbf{w}\in K\cap C} \; \eta\sum_{s=1}^{t}\delta f_t(\mathbf{w}_s) + \Phi(\mathbf{w}). && \text{(recurrence)}
\end{aligned}
$$

- In particular, for linear losses, $f_t(\mathbf{w}) = \mathbf{a}_t\cdot\mathbf{w}$, Dual Averaging coincides with regularized FL (FTRL):

$$
\mathbf{w}_{t+1} = \operatorname*{argmin}_{\mathbf{w}\in K\cap C} \; \sum_{s=1}^{t}\mathbf{a}_s\cdot\mathbf{w} + \frac{1}{\eta}\Phi(\mathbf{w}).
$$

DA Guarantee

- Theorem: under the same assumptions as for MD, the following holds for the regret of Dual Averaging,

$$R_T(\mathrm{DA}) \leq \frac{\Phi(\mathbf{w}^*) - \Phi(\mathbf{w}_1)}{\eta} + \frac{2\eta G_*^2 T}{\alpha}.$$

Choosing η to minimize the bound gives

$$R_T(\mathrm{DA}) \leq 2 D_\Phi G_* \sqrt{\frac{2T}{\alpha}},$$

with $\Phi(\mathbf{w}^*) - \Phi(\mathbf{w}_1) \leq D_\Phi^2$.

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