**Completely optional non-review extra “fun” differentiation problems**

1. Consider the function \( f(x, y) = \frac{1}{2}y^2 + \cos x \).

   (a) Find the critical points of \( f \).

   (b) Use the second derivative test to classify the critical points you found.

   (c) Sketch some level curves of \( f \) (feel free to use a computer, but this is doable by hand).

Consider a pendulum under the influence of gravity, and let \( \theta \) be the angle the pendulum makes with the vertical (so that \( \theta = 0 \) when the pendulum is at rest). Then (up to some constants), \( \theta \) satisfies the differential equation \( \theta'' = \sin \theta \).

(d) Use the chain rule to show that

\[
\frac{d}{dt} f(\theta'(t), \theta(t)) = 0 \tag{1}
\]

where \( f \) is the same function from parts (a), (b), and (c). Physically, \( f(\theta'(t), \theta(t)) \) represents the energy of the pendulum at time \( t \), so \((1)\) says that the energy of the pendulum is conserved.

(e) Interpret the level curves you found in part (b) in terms of the possible motions of the pendulum.

(f) When \( \theta \) is small, we often use the approximate equation \( \theta'' = \theta \) (what sort of approximation is this?). Show that the energy for this approximate equation,

\[
g(\theta', \theta) = \frac{1}{2}((\theta')^2 + \theta^2),
\]

is conserved, \( \frac{dg}{dt} = 0 \). How do the level curves of \( g(x, y) \) and \( f(x, y) \) compare for \( x \) and \( y \) small? What does this say about our approximation?

References:
- [en.wikipedia.org/wiki/Pendulum](en.wikipedia.org/wiki/Pendulum)
- [en.wikipedia.org/wiki/Pendulum_(mathematics)](en.wikipedia.org/wiki/Pendulum_(mathematics))

2. Let \( f(x, y) \) be a function of two variables.

(a) Define a polynomial \( p(x, y) \) by

\[
p(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2.
\]

Check that

\[
p(0, 0) = f(0, 0), \quad p_x(0, 0) = f_x(0, 0), \quad p_y(0, 0) = f_y(0, 0),
\]

\[
p_{xx}(0, 0) = f_{xx}(0, 0), \quad p_{xy}(0, 0) = f_{xy}(0, 0), \quad p_{yy}(0, 0) = f_{yy}(0, 0).
\]

We call \( p \) the **second degree Taylor approximation of \( f \) at \( 0, 0 \).**
(b) Fix \( h \) and \( k \), and define \( g(t) = f(th, tk) \). Use the chain rule to compute \( g'(t) \) and \( g''(t) \).

(c) Show that
\[
g(0) + g'(0) \cdot 1 + \frac{1}{2} g''(0) \cdot 1^2 = p(h, k).
\]

In what sense does the left hand side approximate \( g(1) = f(h, k) \)? How could you get error bounds?

(d) What should the formula for the second degree Taylor approximation of \( f \) at \((x_0, y_0)\) be?

3. Suppose we have data points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) and that we want to find a linear approximation \( y = ax + b \) that minimizes the error
\[
E(a, b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2.
\]

(a) Compute \( \frac{\partial E}{\partial a} \) and \( \frac{\partial E}{\partial b} \).

(b) Show that the critical point equations
\[
\frac{\partial E}{\partial a} = 0, \quad \frac{\partial E}{\partial b} = 0
\]
are
\[
\begin{align*}
\{ & (\sum x_i^2) a + (\sum x_i) b, = \sum x_i y_i \\
& (\sum x_i) a + nb, = \sum y_i.
\end{align*}
\]

(c) Show that the solution to (2) is
\[
a = \frac{\sum x_i y_i - \frac{1}{n}(\sum x_i)(\sum y_i)}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}, \quad b = \frac{1}{n} \sum y_i - \frac{a}{n} \sum x_i.
\]

References
- en.wikipedia.org/wiki/Least_squares
- en.wikipedia.org/wiki/Linear_least_squares_(mathematics)
- en.wikipedia.org/wiki/Ordinary_least_squares

4. Suppose I’m a fruit stand which sells apples and bananas. The price of bananas is \( p \), and the price of apples is \( q \), and I’m going to spend \( w \) on apples and bananas. If \( x \) is how many bananas I buy, and \( y \) the number of apples, then this means
\[
px + qy = w. \tag{3}
\]

Let \( 0 < \alpha < 1 \) be a parameter (describing my fruit preferences), and that
\[
u(x, y) = x^\alpha y^{1-\alpha} \tag{4}
\]
describes how happy I am with \( x \) bananas and \( y \) apples.

(a) Let \((x^*, y^*)\) be the banana-apple combination that maximizes my happiness \( u \) subject to the constraint (3). Use Lagrange multipliers to compute \((x^*, y^*)\).

(b) What’s the economic interpretation of \( \alpha \)?

(c) Thinking of \( x^*, y^* \) from part (a) as functions of \((p, q, w)\), compute the first partials
\[
\begin{align*}
\frac{\partial x^*}{\partial p}, & \quad \frac{\partial x^*}{\partial q}, & \quad \frac{\partial x^*}{\partial w}, & \quad \frac{\partial y^*}{\partial p}, & \quad \frac{\partial y^*}{\partial q}, & \quad \frac{\partial y^*}{\partial w}.
\end{align*}
\]
Can you interpret these economically?

(d) Set up and do parts (a), (b), and (c) when there are three kinds of fruit to choose from.

References:
- en.wikipedia.org/wiki/Cobb-Douglas_production_function#Some_applications
- en.wikipedia.org/wiki/Utility