1. (10 points) Let \( \mathbf{a} = \langle 1, 1, 0 \rangle \) and \( \mathbf{b} = \langle 0, 1, 1 \rangle \).

   (a) (3 points) Find the angle \( \theta \) between \( \mathbf{a} \) and \( \mathbf{b} \).
   We know that
   \[
   \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.
   \]
   Solving for \( \cos \theta \), we get
   \[
   \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{1 \cdot 0 + 1 \cdot 1 + 0 \cdot 0}{\sqrt{1^2 + 1^2}\sqrt{1^2 + 1^2}} = \frac{1}{2}
   \]
   and hence
   \[
   \theta = \frac{\pi}{3}.
   \]

   (b) (4 points) Find the area of the parallelogram determined by \( \mathbf{a} \) and \( \mathbf{b} \).
   The area of this parallelogram is
   \[
   |\mathbf{a} \times \mathbf{b}|.
   \]
   Since
   \[
   \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} = \mathbf{i} - \mathbf{j} + \mathbf{k},
   \]
   The area is therefore
   \[
   |\mathbf{a} \times \mathbf{b}| = |\mathbf{i} - \mathbf{j} + \mathbf{k}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}.
   \]

   (c) (3 points) Compute \( (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} \).
   For any vectors \( \mathbf{u} \) and \( \mathbf{v} \), the cross product \( \mathbf{u} \times \mathbf{v} \) is perpendicular to both \( \mathbf{u} \) and \( \mathbf{v} \). In particular, this means that \( (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} \) must be zero.

   Since we’ve computed \( \mathbf{a} \times \mathbf{b} \) in part (b), this is also easy to check directly:
   \[
   (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \langle -1, -1, 1 \rangle \cdot \langle 0, 1, 1 \rangle = 1 \cdot 0 + (-1) \cdot 1 + 1 \cdot 1 = 0.
   \]

3. (10 points) Suppose that a particle moving in the plane has initial position \( \mathbf{r}(0) = \langle 1, 0 \rangle \), initial velocity \( \mathbf{v}(0) = \langle 0, 2 \rangle \), and acceleration \( \mathbf{a}(t) = \langle -1, t \rangle \).

   (a) (5 points) Find the velocity vector \( \mathbf{v}(t) \) of the particle at time \( t \).
   Since \( \mathbf{v}'(t) = \mathbf{a}(t) \), we know
   \[
   \mathbf{v}(t) = \int \mathbf{a}(t) \, dt = \int \langle -1, t \rangle \, dt = \langle -t, \frac{1}{2}t^2 \rangle + \mathbf{c}_1,
   \]
   where \( \mathbf{c}_1 \) is a constant vector. Plugging in \( t = 0 \) and using the initial condition \( \mathbf{v}(0) = \langle 0, 2 \rangle \), we find
   \[
   \langle 0, 2 \rangle = \mathbf{v}(0) = \langle 0, 0 \rangle + \mathbf{c}_1
   \]
   and hence that \( \mathbf{c}_1 = \langle 0, 2 \rangle \). Therefore
   \[
   \mathbf{v}(t) = \langle -t, \frac{1}{2}t^2 + 2 \rangle.
   \]
(b) (5 points) Find the position vector \( \mathbf{r}(t) \) of the particle at time \( t \).

Since \( \mathbf{r}'(t) = \mathbf{v}(t) \), we know

\[
\mathbf{r}(t) = \int \mathbf{v}(t) \, dt = \int (-t, \frac{1}{2}t^2 + 2) \, dt = (-\frac{1}{2}t^2, \frac{1}{6}t^3 + 2t) + \mathbf{c}_2,
\]

where \( \mathbf{c}_2 \) is a constant vector. Plugging in \( t = 0 \) and using the initial condition \( \mathbf{r}(0) = (1, 0) \), we find

\[
(1, 0) = \mathbf{r}(0) = (0, 0) + \mathbf{c}_2
\]

and hence that \( \mathbf{c}_2 = (1, 0) \). Therefore

\[
\mathbf{r}(t) = (1 - \frac{1}{2}t^2, \frac{1}{6}t^3 + 2t).
\]

In this problem our constants of integration were the same as our initial conditions, \( \mathbf{v}(0) = \mathbf{c}_1 \) and \( \mathbf{r}(0) = \mathbf{c}_2 \). This isn’t always the case. Consider for instance the differential equation

\[
\frac{dy}{dx} = \sin x, \quad y(0) = 0.
\]

We know that \( y \) is an antiderivative of \( \sin x \), so

\[
y(x) = \int \sin x \, dx = -\cos x + c
\]

for some constant \( c \). Plugging in \( x = 0 \), we get

\[
0 = y(0) = -\cos 0 + c = -1 + c,
\]

and hence that \( c = 1 \neq y(0) \).

4. (10 points) Suppose that \( f(t) \) is a scalar function and that \( \mathbf{u}(t) \) and \( \mathbf{v}(t) \) are vector-valued functions.

(a) (5 points) Express \( \frac{d}{dt} \left( \mathbf{u}(t) \times \mathbf{v}(t) \right) \) in terms of \( \mathbf{u}(t), \mathbf{u}'(t), \mathbf{v}(t), \mathbf{v}'(t) \).

From the book and from lecture we know that there’s a sort of “product rule” for differentiating cross products:

\[
\frac{d}{dt} \left( \mathbf{u}(t) \times \mathbf{v}(t) \right) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)
\]

To make sure we’ve got our signs right, we could look at a special case. Suppose

\[
\mathbf{u}(t) = f(t)\mathbf{i}, \quad \mathbf{v}(t) = g(t)\mathbf{j}.
\]

Then

\[
\mathbf{u}(t) \times \mathbf{v}(t) = f(t)g(t)(\mathbf{i} \times \mathbf{j}) = f(t)g(t)\mathbf{k},
\]

and we can compute \( \frac{d}{dt} \left( \mathbf{u}(t) \times \mathbf{v}(t) \right) \) using the usual product rule,

\[
\frac{d}{dt} \left( \mathbf{u}(t) \times \mathbf{v}(t) \right) = (f(t)g(t))'\mathbf{k} = (f'(t)g(t) + f(t)g'(t))\mathbf{k} = f'(t)g(t)\mathbf{k} + f(t)g'(t)\mathbf{k}.
\]

Since

\[
\mathbf{u}'(t) \times \mathbf{v}(t) = f'(t)g(t)\mathbf{k}, \quad \mathbf{u}(t) \times \mathbf{v}'(t) = f(t)g'(t)\mathbf{k}
\]

we have

\[
\frac{d}{dt} \left( \mathbf{u}(t) \times \mathbf{v}(t) \right) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)
\]
in our special case, and not, for instance,
\[ \mathbf{u}'(t) \times \mathbf{v}(t) - \mathbf{u}(t) \times \mathbf{v}'(t). \]

(b) (5 points) Express \( \frac{d}{dt}(f(t)\mathbf{u}(t) + \mathbf{v}(t)) \) in terms of \( f(t), f'(t), \mathbf{u}(t), \mathbf{u}'(t), \mathbf{v}(t), \mathbf{v}'(t) \).

First we can split up the derivative across the sum,
\[ \frac{d}{dt}(f(t)\mathbf{u}(t) + \mathbf{v}(t)) = \frac{d}{dt}(f(t)\mathbf{u}(t)) + \mathbf{v}'(t). \]

Then we can use the “product rule” for scalar multiplication,
\[ \frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t). \]

Putting it all together we have
\[ \frac{d}{dt}(f(t)\mathbf{u}(t) + \mathbf{v}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) + \mathbf{v}'(t). \]

5. (15 points) Consider the parametric curve \( \mathbf{r}(t) = e^t(\cos t, \sin t) \).

(a) (6 points) Find the velocity \( \mathbf{v}(t) = \mathbf{r}'(t) \) and speed \( v(t) = |\mathbf{v}(t)| \).

To compute the velocity we differentiate \( \mathbf{r}(t) \),
\[ \mathbf{v}(t) = \mathbf{r}'(t) = \left( \frac{d}{dt}e^t \right)\langle \cos t, \sin t \rangle + e^t \frac{d}{dt}\langle \cos t, \sin t \rangle \\
= e^t\langle \cos t, \sin t \rangle + e^t\langle -\sin t, \cos t \rangle \\
= e^t\langle \cos t - \sin t, \sin t + \cos t \rangle. \]

The speed \( v = |\mathbf{v}(t)| \) is then given by
\[ v(t) = |\mathbf{v}(t)| = e^t|\langle \cos t - \sin t, \sin t + \cos t \rangle| \\
= e^t \sqrt{\cos^2 t - 2 \cos t \sin t + \sin^2 t + \cos^2 t + 2 \cos t \sin t} \\
= e^t \sqrt{2 \cos^2 t + \sin^2 t - 2 \cos t \sin t}. \]

The cross terms \( \pm 2 \cos t \sin t \) cancel, and we’re left with
\[ v(t) = e^t \sqrt{2(\cos^2 t + \sin^2 t)} = e^t \sqrt{2}. \]

(b) (3 points) Find the acceleration \( \mathbf{a}(t) = \mathbf{r}''(t) \).

We differentiate the formula for \( \mathbf{v}(t) \) from part (a),
\[ \mathbf{a}(t) = \mathbf{v}'(t) = \left( \frac{d}{dt}e^t \right)\langle \cos t - \sin t, \sin t + \cos t \rangle + e^t \frac{d}{dt}\langle \cos t - \sin t, \sin t + \cos t \rangle \\
= e^t\langle \cos t - \sin t, \sin t + \cos t \rangle + e^t\langle -\sin t - \cos t, \cos t - \sin t \rangle \\
= e^t\langle -2 \sin t, 2 \cos t \rangle \\
= [2e^t\langle -\sin t, \cos t \rangle]. \]

(c) (6 points) Find the curvature \( \kappa(t) \).

If \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \), then the curvature \( \kappa(t) \) is given by
\[ \kappa(t) = \frac{|x''(t)y'(t) - y''(t)x'(t)|}{v(t)^3}. \]
Plugging in $v(t) = \langle x'(t), y'(t) \rangle$ and $a(t) = v(t)$ from part (a), and $a(t) = \langle x''(t), y''(t) \rangle$ from part (b), we get

$$\kappa(t) = \frac{|(−2e^t \sin t)(e^t(sin t + \cos t)) − (2e^t \cos t)(e^t(cos t − \sin t))|}{(\sqrt{2e^t})^3}$$

$$= \frac{2e^{2t}}{2^{3/2}e^{3t}}|−(−\sin t)(\sin t + \cos t) − (\cos t)(\cos t − \sin t)|$$

$$= \frac{e^{-t}}{\sqrt{2}}|−\sin^2 t − \sin t \cos t − \cos^2 t + \cos t \sin t|$$

$$= \frac{e^{-t}}{\sqrt{2}}|−1|$$

$$= \frac{e^{-t}}{\sqrt{2}}$$

Here we first factored out all of the $e^t$'s, then expanded the products of the trig functions, and finally cancelling the cross terms and used $\cos^2 t + \sin^2 t = 1$.

We could also use the formula for 3-dimensional (space) curves,

$$\kappa = \frac{|a \times v|}{v^3}.$$

But in order for $a \times v$ to make sense, we need to think of $a$ and $v$ as the 3-dimensional vectors

$$a = \langle −2e^t \sin t, 2e^t \cos t, 0 \rangle, \quad v = \langle e^t(\cos t − \sin t), e^t(\sin t + \cos t), 0 \rangle.$$

Their cross product is then

$$a \times v = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
−2e^t \sin t & 2e^t \cos t & 0 \\
e^t(\cos t − \sin t) & e^t(\sin t + \cos t) & 0
\end{vmatrix}$$

which gives

$$\kappa = \frac{|a \times v|}{v^3} = \frac{2e^{2t}}{2^{3/2}e^{3t}} = \frac{e^{-t}}{\sqrt{2}}$$

as before.

6. (10 points) Figures (a) through (e) show the graphs $z = f(x, y)$ of five different functions $f$. Figures I through V show the same functions, but in a different order; the level curves in each figure correspond to contours at equally spaced heights on the surface $z = f(x, y)$. Match each surface with its level curves.

Each correct match is worth 2 points. The correct matches are

$$\begin{array}{c|c|c|c|c}
a & b & c & d & e \\
III & I & IV & V & II
\end{array}$$

Since (c) and (d) each have one valley and no peaks, they must be either IV or V. In IV the level curves are closer together at the edge of the plot than they are at the center. The spacing between level curves in V, on the other hand, is more regular. Since the level curves correspond to equally spaced values of $f$, this means that the graph corresponding IV is flatter near the center and steeper at the edges, while the graph for V has more uniform slopes. Since (c) is flatter in the center and gets steeper at the edges, (c) must be IV and (d) must be V.
Next we look at II, which seems to have a total of 2 peaks and valleys. It also has a lot of level curves touching the edges of the plotting window, which means that the corresponding graph must have a lot of variation in its height along these same edges. Since (a) has 2 peaks and 2 valleys and (b) is flat near the edges, II must be (e). Finally, since (a) has 2 peaks and 2 valleys while (b) only has 1 peak and 1 valley, we conclude that (a) is III and (b) is I.

The actual functions used were

(a) \( f(x, y) = 10 + xye^{-x^2-y^2} \)  
(b) \( f(x, y) = 1 + xe^{-x^2-y^2} \)  
(c) \( f(x, y) = (x^2 + y^2)^{1.2} \)  
(d) \( f(x, y) = (1 + x^2 + y^2)^{0.2} \)  
(e) \( f(x, y) = 1 + \sin(1.3x) \cos(1.3y) \)

all on the same rectangle \(-2 \leq x \leq 2, -2 \leq y \leq 2\).

7. (15 points) Let \( f(x, y) = 1 + x - y + xy^2 \).

(a) (5 points) Compute the partial derivatives \( f_x(x, y) \) and \( f_y(x, y) \).

To compute \( f_x \), we treat \( y \) as a constant and differentiate with respect to \( x \):

\[
 f_x(x, y) = 1 - y^2 .
\]

To compute \( f_y \), we treat \( x \) as a constant and differentiate with respect to \( y \):

\[
 f_y(x, y) = -1 + 2xy .
\]
(b) (5 points) Write the tangent plane to \( z = f(x, y) \) at \((0,0,1)\) in the form \( z = ax + by + c \).

The general equation for the tangent plane to \( z = f(x, y) \) at \((x_0, y_0, f(x_0, y_0))\) is

\[
z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

Plugging in \((x_0, y_0) = (0, 0)\) and

\[
f_x(0, 0) = 1 - (0)^2 = 1, \quad f_y(0, 0) = -1 + 2(0)(0) = -1, \quad f(0, 0) = 1,
\]

we get

\[
z - 1 = x - y.
\]

Finally, we put this in the form \( z = ax + by + c \):

\[
[ z = 1 + x - y ]
\]

(c) (5 points) With \(a, b, c\) from part (b), use polar coordinates to show

\[
\lim_{(x,y) \to (0,0)} \frac{f(x, y) - (ax + by + c)}{\sqrt{x^2 + y^2}} = 0. \tag{\star}
\]

This is not a coincidence: the tangent plane \( z = ax + by + c \) to a graph \( z = f(x, y) \) at \((0,0, f(0,0))\) is the unique plane satisfying \(\star\).

Taking the tangent plane \( z = ax + by + c \) from part (b), we have

\[
ax + by + c = 1 + x - y,
\]

and hence

\[
f(x, y) - (ax + by + c) = 1 + x - y + xy^2 - (1 + x - y) = xy^2.
\]

Thus, in polar coordinates \( x = r \cos \theta, y = r \sin \theta \),

\[
\frac{f(x, y) - (ax + by + c)}{\sqrt{x^2 + y^2}} = \frac{xy^2}{\sqrt{x^2 + y^2}} = \frac{(r \cos \theta)(r^2 \sin^2 \theta)}{\sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta}} = \frac{r^3 \cos \theta \sin^2 \theta}{r} = r^2 \cos \theta \sin^2 \theta.
\]

Since \(|\cos \theta \sin^2 \theta| \leq 1\) for any \(\theta\), we conclude

\[
\lim_{(x,y) \to (0,0)} \frac{f(x, y) - (ax + by + c)}{\sqrt{x^2 + y^2}} = \lim_{r \to 0} r^2 \cos \theta \sin^2 \theta = 0.
\]

Here we’re using the fact that \(r = \sqrt{x^2 + y^2} \to 0\) as \((x,y) \to (0,0)\).

It’s not true that \(\theta \to 0\) (or any other value) as \((x,y) \to (0,0)\). Indeed, sending \((r, \theta) \to (0, \theta_0)\) means sending \((x, y) \to (0, 0)\) along a curve through the origin which makes an angle \(\theta_0\) with the \(x\)-axis.

8. (20 points) Let \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) be the parametric curve

\[
x(t) = \cosh t = \frac{e^t + e^{-t}}{2}, \quad y(t) = \sinh t = \frac{e^t - e^{-t}}{2},
\]

and consider the function \( f(x, y) = y^2 - x^2 \).

(a) (5 points) Show that \( f(x(t), y(t)) \) is a constant independent of \(t\), i.e. that \( \mathbf{r}(t) \) describes a level curve of \( f \).
We compute
\[
f(x(t), y(t)) = y(t)^2 - x(t)^2
\]
\[
= \left( \frac{e^t - e^{-t}}{2} \right)^2 - \left( \frac{e^t + e^{-t}}{2} \right)^2
\]
\[
= \frac{e^{2t} - 2 + e^{-2t}}{4} - \frac{e^{2t} + 2 + e^{-2t}}{4}
\]
\[
= -\frac{2 - 2}{4}
\]
\[
= -1,
\]
which is a constant independent of \( t \).

(b) (6 points) Find parametric equations for the line \( L \) tangent to \( r(t) \) at the point \( (\frac{5}{4}, \frac{3}{4}) \) where \( t = \ln 2 \).

The tangent vector to a curve \( r(t) \) through \( r(t_0) \) is the line passing through \( r(t_0) \) with direction vector \( r'(t_0) \). Thus the tangent line \( L \) we’re interested in is the line through \( r(\ln 2) \) with direction vector \( r'(\ln 2) \).

We’re given in the problem that \( r = (\frac{5}{4}, \frac{3}{4}) \) when \( t = \ln 2 \), but this is also easy to check
\[
r(\ln 2) = \frac{1}{2} (e^{\ln 2} + e^{-\ln 2}, e^{\ln 2} - e^{-\ln 2}) = \frac{1}{2} (2 + \frac{1}{2}, 2 - \frac{1}{2}) = (\frac{5}{4}, \frac{3}{4}).
\]

Our next step is to compute
\[
r'(t) = (x'(t), y'(t)) = \frac{d}{dt} \left( \frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2} \right) = \left( \frac{e^t - e^{-t}}{2}, \frac{e^t + e^{-t}}{2} \right).
\]
Incidentally, we’ve just shown that \( [\sinh(t)]' = \cosh(t) \) and \( [\cosh(t)]' = \sinh(t) \). Plugging in \( t = \ln 2 \), we get
\[
r'(\ln 2) = \left( \frac{e^{\ln 2} - e^{-\ln 2}}{2}, \frac{e^{\ln 2} + e^{-\ln 2}}{2} \right) = \left( \frac{2 - \frac{1}{2}}{2}, \frac{2 + \frac{1}{2}}{2} \right) = (\frac{3}{4}, \frac{5}{4}).
\]
Thus \( L \) is the line through \( (\frac{5}{4}, \frac{3}{4}) \) with direction vector \( (\frac{3}{4}, \frac{5}{4}) \). In particular, \( L \) is described by the parametric equations
\[
\begin{align*}
x &= \frac{5}{4} + \frac{3}{4}t, \\
y &= \frac{3}{4} + \frac{5}{4}t.
\end{align*}
\]

(c) (5 points) Compute the vector \( u = (f_x(\frac{5}{4}, \frac{3}{4}), f_y(\frac{5}{4}, \frac{3}{4})) \). We call \( u \) the gradient of \( f \) at \( (\frac{5}{4}, \frac{3}{4}) \).

We compute
\[
f_x(x, y) = -2x, \quad f_y(x, y) = 2y.
\]
So
\[
u = (f_x(\frac{5}{4}, \frac{3}{4}), f_y(\frac{5}{4}, \frac{3}{4})) = (-2(\frac{5}{4}), 2(\frac{3}{4})) = (-\frac{5}{2}, \frac{3}{2}).
\]

(d) (4 points) Show that vector \( u \) from part (c) is orthogonal to the line \( L \) from part (b). This is not a coincidence: the gradient of a function \( f \) at a point \( (a, b) \) is always normal to the level curve passing through \( (a, b) \).

The vector \( u \) is orthogonal to the line \( L \) if it is orthogonal to \( L \)’s direction vector \( r'(\ln 2) \), i.e. if \( u \cdot [r'(\ln 2)] = 0 \). Thus it’s enough to compute
\[
u \cdot [r'(\ln 2)] = (-\frac{5}{2}, \frac{3}{2}) \cdot (\frac{3}{4}, \frac{5}{4}) = -\frac{5}{2} \cdot \frac{3}{4} + \frac{3}{2} \cdot \frac{5}{4} = 0.
\]