1. **(3 marks)** The positive part of \( a \in \mathbb{R} \) is defined by
\[
a^+ = \frac{|a| + a}{2}
\]
and the negative part by
\[
a^- = \frac{|a| - a}{2}.
\]
Prove that \( a = a^+ - a^- \) and \( |a| = a^+ + a^- \).
Prove that if \( a \geq 0 \), then \( a^+ = a \) and \( a^- = 0 \); and similarly, if \( a < 0 \), then \( a^+ = 0 \)
and \( a^- = -a \).

2. **(3 marks)** Prove that for any \( a, b \in \mathbb{R} \) with \( 0 \leq a \leq b \), the following inequalities hold
\[
a \leq \sqrt{ab} \leq \frac{a + b}{2} \leq b.
\]
(You can assume that non-negative reals have a unique non-negative square root.)

3. **(4 marks)** Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples to the false ones.

- If \( A \) and \( B \) are nonempty bounded subsets of \( \mathbb{R} \), then \( \sup (A \cup B) = \max \{ \sup A, \sup B \} \).
- If \( A \) and \( B \) are nonempty bounded subsets of \( \mathbb{R} \), then \( \sup (A \cap B) \leq \sup A \).
- If \( A + B = \{a + b : a \in A \text{ and } b \in B \} \), where \( A \) and \( B \) are nonempty bounded subsets of \( \mathbb{R} \), then \( \sup (A + B) = \sup A + \sup B \).
- If \( A - B = \{a - b : a \in A \text{ and } b \in B \} \), where \( A \) and \( B \) are nonempty bounded subsets of \( \mathbb{R} \), then \( \sup (A - B) = \sup A - \sup B \).

4. **(3 marks)** The set \( E \subset \mathbb{R} \) is said to be bounded from below if there exists some \( m \in \mathbb{R} \) such that \( m \leq a \) for all \( a \in E \). In this case, \( m \) is called a lower bound. If \( s \) is a lower bound such that for any other lower bound \( t \), \( s \geq t \) holds, then \( s \) is an infimum of \( E \), \( s = \inf E \).
Prove that if \( E \) has an infimum, then it is unique and for all \( \varepsilon > 0 \) there is an \( a \in E \) such that \( \inf E \leq a < \inf E + \varepsilon \).

The following problems form the extra homework. They will not contribute to your final grade.

5. Prove that the additive and the multiplicative inverses are unique, that is without postulating uniqueness in Axiom \( \text{vii)} \) and \( \text{viii)} \) derive the uniqueness from these new axioms.

6. Construct a field of four elements. Hint: try to extend the example given on the lecture to a field where each polynomial of order two has a root.