1. Decide which of the following statements are true and which are false. Prove the true ones and provide counterexamples for the false ones.

- **(1 mark)** Suppose that \( \sum_{k=1}^{\infty} (a_k + b_k) \) converges. Then \( \sum_{k=1}^{\infty} a_k \) converges if and only if \( \sum_{k=1}^{\infty} b_k \) converges.
- **(1 mark)** Suppose that \( f(k) = a_k \) for some continuous function \( f : [1, \infty) \rightarrow [0, \infty) \) which satisfies \( f(x) \rightarrow 0 \) as \( x \rightarrow \infty \). If \( \sum_{k=1}^{\infty} a_k \) converges, then \( \int_{1}^{\infty} f(x) \, dx \) is finite.
- **(1 mark)** If \( a_k \leq b_k \) for every \( k \) and \( \sum_{k=1}^{\infty} b_k \) is absolutely convergent, then \( \sum_{k=1}^{\infty} a_k \) converges.
- **(1 mark)** If \( \sum_{k=1}^{\infty} a_k \) is absolutely convergent, then \( \sum_{k=1}^{\infty} a_k^2 \) is absolutely convergent.

2. Decide whether the following series converge or not. Justify your answer.

- **(2 marks)** \( \sum_{k=1}^{\infty} \frac{1}{k \log k} \)
- **(2 marks)** \( \sum_{k=1}^{\infty} \frac{e^k}{k!} \)

3. **(3 marks)** Assume that \( a_k \) is a positive decreasing sequence. Prove that \( \sum_{k=1}^{\infty} a_k \) converges if and only if \( \sum_{k=1}^{\infty} 2^k a_{2k} \) converges.

4. **(2 marks)** Find all real numbers \( p \) for which the series

\[
\sum_{k=1}^{\infty} \frac{1}{k (\log k)^p}
\]

converges. (Hint: you can use the result of the previous exercise.)

The following problems form the extra homework. They will not contribute to your final grade.

5. Assume that \( a_k > 0 \) and \( \sum_{k=1}^{\infty} a_k \) converges. Prove that there exists a positive sequence \( b_n \) such that \( b_n/a_n \rightarrow \infty \) and \( \sum_{k=1}^{\infty} b_k \) converges.

6. Assume that \( a_k > 0 \) and \( \sum_{k=1}^{\infty} a_k \) diverges. Prove that there exists a positive sequence \( b_n \) such that \( b_n/a_n \rightarrow 0 \) and \( \sum_{k=1}^{\infty} b_k \) diverges.