Write up and turn in solutions to the highlighted problems from the list below. However, solve all other problems as well. The quizzes will cover the complete list of homework problems.

**Due date:** Friday, October 25, in the beginning of the recitation. *NO LATE HOMEWORK WILL BE ACCEPTED.*

**Problems.**

1. Let $G$ be a group and $S, T \leq G$ subgroups of $G$.
   (a) Assume that $T$ is normal in $G$. Prove that $ST \leq G$, $S \cap T \triangleleft S$ and
   $$S/(S \cap T) \cong (ST)/T.$$  
   (b) Assume that $T$ is normal in $G$ and that $T \subseteq S$. Show that the subset $S/T$ of $G/T$ is a subgroup of $G/T$ and that $S/T \triangleleft G/T$ if and only if $S \triangleleft G$.  
   (c) Assume that $S$ and $T$ are normal in $G$ and that $T \subseteq S$. prove that
   $$(G/T)/(H/T) \cong G/S.$$  
   Remark: this question is a good review of the notions that were introduced in class. Remember to check that mappings are well defined.

2. Problems 1, 5, 10, 11 on page 70 Herstein’s book.
   Remark for problem 5: Herstein denotes by $\mathcal{A}(G)$ what we denoted in class as $\text{Aut}(G)$ (the group of automorphisms of $G$). $\mathcal{A}(G)$ is the group of inner automorphisms of $G$.
   Remark for problems 10 and 11: Herstein denotes by $xT$ what we denote in class as $T(x)$ (the mapping $T$ evaluated at $x$). In problem 11, $T^2 = T \circ T$ and $I$ is the identity mapping on $G$.

3. Let $A, B$ be groups and $\theta : A \to \text{Aut}(B)$ a homomorphism. For $a \in A$ denote $\theta(a) = \theta_a \in \text{Aut}(B)$. Equip the product set $B \times A = \{(b, a) : a \in A, b \in B\}$ with the binary operation $(b, a)(b', a') = (b'', a'')$, where $a'' = aa'$ and $b'' = b\theta_a(b')$.
   (a) Show that this binary operation induces a group structure on the set $B \times A$ (i.e., it satisfies the group axioms).
   (b) Show that the group defined above is Abelian if and only if $A$ and $B$ are Abelian and for all $a \in A$ and $b \in B$ we have $\theta_a(b) = b$.
   (c) Assume that $p, q \in \mathbb{N}$ are prime and $p|(q - 1)$. Consider the case $A = \mathbb{Z}_p$, $B = \mathbb{Z}_q$. Show that there exists $\sigma \in \text{Aut}(\mathbb{Z}_q)$ which has order $p$.
   Hint: we computed $\text{Aut}(\mathbb{Z}_q)$ in class. Use Cauchy’s theorem for Abelian groups.
   (d) Fix $\sigma \in \text{Aut}(\mathbb{Z}_q)$ as above, i.e. $o(\sigma) = p$. For $a = [k] \in \mathbb{Z}_p$ and define $\theta(a) = \sigma^k \in \text{Aut}(\mathbb{Z}_q)$. Show that $\theta(a)$ is well defined and that $\theta : \mathbb{Z}_p \to \text{Aut}(\mathbb{Z}_q)$ is a homomorphism.
   (e) Deduce that for any two primes $p, q \in \mathbb{N}$ such that $p|(q - 1)$ there is a non-Abelian group of order $pq$. We shall see later in this course that is $p$ does not divide $q - 1$ then any group of order $pq$ is cyclic.

4. Let $G$ be a group and $H \leq G$ ($H$ need not necessarily be normal). For every $g \in G$ define a function $\tau_g : G/H \to G/H$ by $\tau_g(A) = Ag^{-1}$.
   (a) Show that $\tau_g$ is one to one and onto, i.e. $\tau_g \in A(G/H)$.
   (b) Define $\psi : G \to A(G/H)$ by $\psi(g) = \tau_g$. Show that $\psi$ is a homomorphism.
   (c) Let $K$ be the kernel of $\psi$. Show that $K \subseteq H$. Moreover, $K$ is the largest normal subgroup of $G$ contained in $H$, i.e. $K \triangleleft G$ and for any $N \subseteq H$ such that $N \triangleleft G$ we have $N \subseteq K$.
   Remark: compare this to problem 18 on page 48 of Herstein (from homework 4).
   (d) Deduce that if $H$ has finite index in $G$ then $H$ contains a subgroup $N$ which is normal in $G$ and has a finite index in $G$. Give a bound for $|G : N|$ in terms of $|G : H|$.