

Proof of the uniform convexity lemma

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Fix $1 < p < 2$. Our goal is to prove that for every $a, b \in L_p$,

$$\|a + b\|_p^2 + (p - 1)\|a - b\|_p^2 \leq 2(\|a\|_p^2 + \|b\|_p^2). \quad (1)$$

We require the following numerical lemma:

Lemma 0.1. *For $0 \leq r \leq 1$ define*

$$\alpha(r) = (1 + r)^{p-1} + (1 - r)^{p-1} \quad \text{and} \quad \beta(r) = \frac{(1 + r)^{p-1} - (1 - r)^{p-1}}{r^{p-1}}.$$

Then for every $A, B \in \mathbb{R}$,

$$\alpha(r)|A|^p + \beta(r)|B|^p \leq |A + B|^p + |A - B|^p.$$

Proof. We may clearly assume that $A, B > 0$. Observe first of all that $\beta(r) \leq \alpha(r)$ for all $r \in [0, 1]$. Indeed, setting $h(r) = \alpha(r) - \beta(r)$ we have (since $1 < p < 2$) that $h(0) = 2$ and

$$h'(r) = (p - 1) \left(\frac{1}{r^p} - 1 \right) \left[\frac{1}{(1 - r)^{2-p}} - \frac{1}{(1 + r)^{2-p}} \right] \geq 0.$$

It follows that if $0 < A < B$ then $\alpha(r)A^p + \beta(r)B^p \leq \alpha(r)B^p + \beta(r)A^p$, which implies that it enough to prove that for $0 < B < A$, $\alpha(r)A^p + \beta(r)B^p \leq (A + B)^p + (A - B)^p$. Dividing by A , it suffices to show that for $0 \leq R \leq 1$, the function $F(r) = \alpha(r) + R^p\beta(r)$ achieves its global maximum at $r = R$. But

$$F'(r) = (p - 1)[(1 + r)^{p-2} - (1 - r)^{p-2}] \left[1 - \left(\frac{R}{r} \right)^p \right],$$

Thus, the only point in $(0, 1)$ at which F' vanishes is $r = R$, and since $1 < p < 2$, $F'(1)$ is negative. This implies that F is maximal at $r = R$. □

Corollary 0.2 (Hanner's inequality for $1 \leq p \leq 2$). *For every $f, g \in L_p$,*

$$|\|f\|_p - \|g\|_p|^p + (\|f\|_p + \|g\|_p)^p \leq \|f + g\|_p^p + \|f - g\|_p^p.$$

Proof. By symmetry we may assume that $r = \|g\|_p / \|f\|_p \leq 1$. By Lemma 0.1 the following point-wise inequality holds:

$$\alpha(r)|f|^p + \beta(r)|g|^p \leq |f + g|^p + |f - g|^p.$$

Integrating and simplifying gives the required result. □

The following numerical lemma is well known.

Lemma 0.3 (Beckner's two-point inequality). For every $a, b \in \mathbb{R}$,

$$[a^2 + (p-1)b^2]^{1/2} \leq \left(\frac{|a+b|^p + |a-b|^p}{2} \right)^{1/p}.$$

Proof. If $|a| < |b|$ then since $p < 2$, $a^2 + (p-1)b^2 \leq b^2 + (p-1)a^2$. We may therefore assume that $|a| \geq |b| > 0$. Set $x = b/a$. Our goal is to show that $[1 + (p-1)x^2]^{p/2} \leq \frac{(1+x)^p + (1-x)^p}{2}$ for every $x \in [-1, 1]$. Now, $\frac{(1+x)^p + (1-x)^p}{2} = \sum_{k=0}^{\infty} \binom{p}{2k} x^{2k} \geq 1 + \frac{p(p-1)}{2} x^2$, where we have used the fact that since $p < 2$, $\binom{p}{2k} \geq 0$. Finally, the inequality $1 + \frac{p(p-1)}{2} x^2 \geq [1 + (p-1)x^2]^{p/2}$ follows from the elementary fact that $(1+t)^\alpha \leq 1 + \alpha t$ for every $t, \alpha \in [0, 1]$. \square

Let's complete the proof of (1). Fix $x, y \in L_p$. Then

$$\begin{aligned} \left(\frac{\|x+y\|_p^2 + \|x-y\|_p^2}{2} \right)^{1/2} &\geq \left(\frac{\|x+y\|_p^p + \|x-y\|_p^p}{2} \right)^{1/p} && \text{(since } p \leq 2) \\ &\geq \left[\frac{(\|x\|_p + \|y\|_p)^p + \left| \|x\|_p - \|y\|_p \right|^p}{2} \right]^{1/p} && \text{(Hanner's inequality)} \\ &\geq [\|x\|_p^2 + (p-1)\|y\|_p^2]^{1/2} && \text{(Beckner's inequality),} \end{aligned}$$

and this is equivalent to (1) (setting $a = x + y$ and $b = x - y$).