Mathematical Thinking, Fall/2003 - Review problems

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1 Problem 1

The following formula expresses the value \( v \) of a certain car \( t \) years after it has been bought, where \( 0 \leq t \leq 10 \).

\[
v(t) = (20,000) \times (0.7)^t
\]

1. What was the value of this car when it was bought?

2. What is this formula saying? What happens to the value of the car from one year to the next? Express this in terms of percentages.

3. Sketch the graph of \( v(t) \) for \( 0 \leq t \leq 10 \).

4. Find the inverse to this function and sketch its graph.

2 Problem 2

Galileo noticed that, when an object is dropped from a height \( h_0 \), it moves 5 meters during the first second of its fall, 15 meters during the second second, 25 meters during the third second, 35 during the fourth second, 45 seconds during the fifth, and so on. That is, during the \( n \)th second, the object moves for 10 more meters than it did during the \( (n - 1) \)th second. We will assume in what follows that \( h_0 \) is so big that in none of the discussed situations the object hits the ground.

1. Let \( a_1 = 5, a_2 = 15, a_3 = 25, a_4 = 35, a_5 = 45, \ldots \) be the sequence of values described above. Why is this an arithmetic sequence? Write down a formula for its general term \( a_n \).

2. Classical Newtonian Mechanics tells us that the object described above will be approximately \( d(n) = 5n^2 \) meters away from its starting point \( n \) seconds after it has been dropped. On the other hand, according to the previous item, the total distance from the starting point \( n \) seconds after the object is dropped off should be \( \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n \). Therefore, one should have

\[
5n^2 = \sum_{i=1}^{n} a_i \tag{1}
\]

Prove that this is true for \( n = 100 \).

3. (Harder) Starting out from the formula for \( d(n) \), prove that the difference in distance between consecutive seconds \( n - 1 \) and \( n \) is precisely \( 10(n - 1) + 5 \), for any \( n \) integer bigger than 1. How does that tie in with your answer to the first item?

4. (Harder) Prove that \( ?? \) holds for any value \( n \in \{1, 2, 3, 4, \ldots \} \).
3 Problem 3

A often-cited law about computer microprocessors is Moore’s law: the clock speed of the fastest microchip in the market doubles every 18 months (notice that that’s 3/2 years). Assume that the fastest chip currently in the market runs at a clock speed of 3 GHz, and that the speed of $s(t)$ of the fastest chip in the market $t$ years from now is an exponential function of $t$.

1. Write down a formula for $s(t)$. Remember, saying that $s$ is an exponential function of $t$ means that
   
   $$s(t) = (\text{some number}) \times (\text{base})^t$$

2. Sketch the graph of $s$. What is its value at $t = 0$?

3. Write down the formula for the inverse function $t = t(s)$ of $s$. Use this formula to figure out how long we will have to wait until 24 GHz chips become available.

4 Problem 4

Each month of the fiscal year, Social Security holds $10.00 of my earnings. At the end of that period they give me a $120.00 refund, thus returning me all the money they had retained. Suppose that they would deposit my $10.00 in a savings account with 1% monthly interest, and that they did so at each month of the year, without ever withdrawing money from it. How much money would they have in that account by the end of the fiscal year? How much ”profit” would they earn over my $120.00?

5 Problem 5

(An old friend from the midterm) The force of gravity $F$ exerted by the Earth on a rocket in space is inversely proportional to the rocket’s mass $m$ and inversely proportional to the square of distance $d$ between the rocket and the center of the Earth.

1. Write down the relationship between $F$, $m$ and $d$ in terms of an unknown numerical constant $K$.

2. Find $K$ given that the force acting on a 1000 kg rocket that is 10,000 m away from the center of the Earth is $3.6 \times 10^9$ Newtons (“Newtons” are just the unit of force in the International System).

3. Now use $K$ to write down a formula relating $F$ to $m$ when $d$ is fixed to 10,000 m. This formula should be $F = 10 \times m$.

4. High-school physics teaches us that the force of gravity exerted by the Earth on a body of mass $m$ on its surface is approximately $10 \times m$. Why does this enable us to conclude that the radius of the Earth is close to 10,000 m?
6 Problem 6

Argue that the following statements cannot be true.

1. My aunt Clara has $n$ parrots and $m$ dogs. One day, she put all those animals together in the same room, and she counted 10 animal heads and 38 legs.

2. If one kicks a soccer ball at a certain angle $\alpha$ and with a certain force $F$, its height in meters after $t$ seconds is $h(t) = 4t^2 - 2t + 10$, for all $t \geq 1$. (Hint: for very large $t$, where will the ball be, according to the formula? Where should it be, according to common sense?)

3. Each year, my dad’s car is devalued by 25%. Therefore, if I wait long enough, I’ll be able to buy his car for a negative amount of money. That is, he’ll actually pay me to have his car!

4. (Harder) I live on $x$ Street, and my best friend lives on $y$th Avenue ($x, y$ are just numbers), both in Manhattan. As we were talking the other day, we realized that the difference between the number of the street immediately north of mine - which is $(x + 1)$ - and the number of my street is exactly the number of the avenue he lives on. Also, four times the number of my street is two less than three times the number of his avenue.

7 Problem 7

Find the inverses to the following functions. Assume that $x > 0$ whenever this makes a difference. You should also note that there are input values for which the inverse function will not be well-defined. For instance, since $a(x) = 2^x$ is never $\leq 0$, the inverse function $b(y) = \log_2 y$ is not well-defined for $y \leq 0$ (otherwise, say that $b(-100) = x$; then it would be true that $a(x) = 2^x = -100$, which is impossible).

1. $f(x) = \sqrt{x+1}$
2. $g(x) = 4 + \log_2(x + 2)$
3. $h(x) = \frac{x-1}{2+x}$
4. $p(x) = 10x - 2$
5. $q(x) = \sqrt{\log_{10} x + 20}$ (assume that $x \geq 10^{-20}$ when inverting.)
6. $a(x) = 466 \times (1.035)^x$
7. $b(x) = 2 + \log_7 x$

8 Problem 8

Express the following inequalities in terms of one or more expressions of the form $x \geq$ (some number), or $x < \ldots$, etc. E.g.

$x^2 \geq 9 \Rightarrow x \geq 3 \text{ or } x \leq -3$

$5 \geq \log_2 x \Rightarrow x \leq 2^5 = 32$

Also plot the corresponding regions on the real line.
9 Problem 9

Plot the graphs of the following quadratic functions. Be sure to point out the location of the 
\( x \)-intercepts (i.e. the roots of the corresponding equation), the \( y \)-intercept and the vertex, if they 
exist.

1. \( f(x) = 2x^2 - 6x + 2 \)
2. \( g(x) = x^2 - 7x + 12 \)
3. \( h(x) = -5x^2 + 4x - \frac{1}{3} \)
4. \( p(x) = -9 + 4x - x^2 \)
5. \( q(x) = x^2 - 7 \)

10 Problem 10

In what follows two terms in an arithmetic sequence are given. Deduce the values of the first term 
\( a_1 \) and of the (constant) difference \( d \) between consecutive terms. Then find the sum of the first 10 
terms in the sequence.

1. \( a_7 = 9 \) and \( a_{10} = 10 \) (that is, the seventh term in the sequence is 9 and the tenth term is 10)
2. \( a_4 = 10 \) and \( a_8 = 34 \)
3. \( a_{32} = 9 \) and \( a_{64} = -119 \)
4. \( a_2 = 15 \) and \( a_5 = 21 \)
5. \( a_5 = 11 \) and \( a_9 = -1 \)

11 Problem 11

Compute the following logarithms without using a calculator. E.g.

\[ \log_3 \sqrt[3]{81} = \log_3 3^4 = \log_3 3^\frac{4}{3} = \frac{4}{5} \]

1. \( \log_{229} 1 \)
12 Problem 12

Carbon dating. Scientists use Carbon-14 dating to find the age of fossils and other artifacts. The amount of Carbon-14 in an organism will yield information concerning its age. A formula used in Carbon-14 dating is

\[ A(t) = A_0 \times 2^{-\frac{t}{5600}} \]

where \( A_0 \) is the amount in grams of Carbon-14 originally in the organism, \( t \) is time in years, and \( A \) is the amount of carbon remaining after \( t \) years. Determine the number of years since an organism died if it originally contained 1,000 grams of Carbon-14 and it currently contains 600 grams of Carbon-14.

13 Problem 13

Bankruptcy Model. In 1997, there were a total of 1,316,999 bankruptcies filed under the Bankruptcy Reform Act. The model for the number of bankruptcies filed is \( B(t) = 0.798 \times 1.164^t \), where \( t \) is the number of years since 1994 and \( B \) is the number of bankruptcies filed in terms of millions (Administrative Office of the U.S. Courts, Statistical Tables for the Federal Judiciary. In what year is it predicted that 12 million bankruptcies will be filed?

14 Problem 14

A ball is dropped from a height of 20 feet. Each time it bounces it returns to \( \frac{7}{8} \) of the height it fell from. If the ball is allowed to bounce 30 times, find the total vertical distance that the ball travels.
15 Problem 15

The lengths of the sides of a right triangle are in arithmetic progression. The length of the shortest side of a right triangle is 3. Find the other two lengths. You must write down the right equation and solve it appropriately. Moreover, you must notice that the equation actually provides two candidate solutions, but only one of them works.

16 Problem 16

The area of a certain triangle is 3 times the length of its shorter side. Moreover, its shorter side measures 1 meter less than its longer side. Find the area of the rectangle.

17 Problem 17

Solve the following equations involving radicals and rational functions. Don’t forget to check if your potential solutions work!

1. \( \sqrt{x + 3} = \sqrt{x + 4} - 1 \)
2. \( \sqrt{3x + 1} = x + 3 \)
3. \( \sqrt[3]{2x + 7} = -1 \)
4. \( x - 2\sqrt{x} - 3 = 0 \)
5. \( 3y^{2/7} - 4y^{4/7} + 1 = 0 \)
6. \( 3x^{2/3} + 5x^{1/3} - 2 = 0 \)
7. \( 9x^{2/3} - 49 = 0 \)
8. \( \frac{a}{a^2 - 9} + \frac{3}{a^2 - 9} = 1 \)
9. \( 1 - \frac{1}{x} = \frac{6}{x^2} \)
10. \( \frac{3}{x - 1} = \frac{3}{5} \)
11. \( \frac{5}{y + 1} = \frac{4}{y + 2} \)
12. \( \frac{x + 1}{x^2 + 5x} + \frac{2}{x^2 - 25} = 0 \)

18 Problem 18

Use whatever means you can think of to simplify the following expressions as much as possible. You can assume that no argument of log is non-positive, that all denominators are not 0, etc.

1. \( \frac{125x^4y^2z^3}{35x^2y^3z^7} \)
2. \( \frac{x^4 - 2}{x^2 - 7} \)
3. \( \frac{2y^3 - 9y^2 - 17y + 39}{2y - 3} \)
19 Problem 19

Chaudra is tossing a softball into the air with an underhand motion. The distance of the ball above her hand is given by the function

\[ h(t) = 32t - 16t^2 \quad \text{for } 0 \leq t \leq 2 \]

where \( h(t) \) is the height of the ball (in feet) and \( t \) is the time (in seconds). Find the times at which the ball is in her hand, and the maximum height of the ball.

20 Problem 20

Use the binomial formula to write down the first four terms in the expansion of.

1. \((x - 1)^{20}\)
2. \((2x - 3y)^{8}\)
3. \((1 - 8x)^{7}\)
4. \((10 + y)^{5}\)
21  Problem 21

Solve the following equations.

1. $\log_3(x - 3) - \log_3(x + 2) = 1$
2. $3^x = 1$
3. $\log_2(x^2 - 1) = 4$
4. $\log_7 \sqrt{x} = -12$
5. $9 \times 4^t = 144$

22  Problem 22

In each item that follows two terms in a geometric sequence are given. Deduce the values of the first term $a_1$ and of the (constant) ratio $q$ between consecutive terms. Then find the sum of the first 10 terms in the sequence.

1. $a_7 = 9$ and $a_{10} = 72$ (that is, the seventh term in the sequence is 9 and the tenth term is 72)
2. $a_4 = 10$ and $a_8 = 10$
3. $a_{32} = 9$ and $a_{35} = -81$
4. $a_2 = 10$ and $a_6 = 6250$
5. $a_5 = -11$ and $a_8 = 88$

23  Problem 23

Each year, the world population increases by 1%. Suppose that on every year after 2004 each newborn would be photographed right after birth by Camera X. How many photographs would have been taken by Camera X in 12 years?

24  Problem 24

A pendulum swings 10 feet left to right on its first swing. On each subsequent swing, the pendulum swings $\frac{4}{5}$ of the previous swing.

1. Write a sequence for the distance travelled by the pendulum on the first, second and third swing.
2. Write a general term for the sequence, with $n$ representing the number of the swing.
3. How far will the pendulum swing on its tenth swing? (Round to the nearest hundredth. )
25  Problem 25

Find the slope and $y$-intercept of the line through

1. $(2, -3)$ and $(-1, -4)$
2. $(2, 1)$ and $(4, -1)$
3. $(-5, 1)$ and $(-4, 3)$
4. $(-2, 2)$ and $(3, 4)$