Integral Equations and Fast Algorithms
with applications

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1 Introduction

Before describing the current lay of the land in modern computational science (at least with regard to PDEs and classical physics-driven problems), it is worth pointing out what is necessary to understand the following lecture notes. We will assume that the reader has a strong grasp of linear algebra and multi-variable calculus, a working knowledge of the basic theorems and ideas in complex analysis, and at least some exposure to ordinary and partial differential equations. Of course, proficiency in computer programming (preferably C, Fortran, or Matlab) will prove invaluable if any of the algorithms in these notes are to be implemented. Remember, the level of work required to implement a fast algorithm is not an excuse to avoid implementing it!

Keep in mind that these notes are far from finished! Please send suggestions and errors to oneil@cims.nyu.edu.

1.1 Motivation: electrostatics

As motivation for the use of integral equations and fast algorithms over the usual PDE formulations (and solvers) of many physical systems, we first examine a special case of Maxwell’s equations: electrostatics. In an isotropic medium with constant magnetic permeability $\mu$ and electric permittivity $\epsilon$, the fully time-dependent Maxwell equations governing the propagation of electric and magnetic fields, $\mathbf{E}$ and $\mathbf{H}$, are given by:

\begin{align*}
\nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t}, \\
\nabla \times \mathbf{H} &= \epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}, \\
\nabla \cdot \mathbf{E} &= \rho \epsilon, \\
\nabla \cdot \mathbf{H} &= 0,
\end{align*}

(1.1)

where $\mathbf{J}$ and $\rho$ are the electric current and charge, respectively. For various important physical devices, it suffices to study the above equations in the time harmonic case. Assuming an implicit time dependence of $e^{-i\omega t}$ on $\mathbf{E}$, $\mathbf{H}$, $\mathbf{J}$, and $\rho$, we have what are known as the time-harmonic Maxwell’s equations:

\begin{align*}
\nabla \times \mathbf{E} &= i\omega \mu \mathbf{H}, \\
\nabla \times \mathbf{H} &= -i\omega \epsilon \mathbf{E} + \mathbf{J}, \\
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon}, \\
\nabla \cdot \mathbf{H} &= 0,
\end{align*}

(1.2)

where script quantities have been replaced by bold quantities to show only spatial dependence. Furthermore, letting $\omega \to 0$ we arrive at the equations of electro- and magnetostatics:

\begin{align*}
\nabla \times \mathbf{E} &= 0, \\
\nabla \times \mathbf{H} &= \mathbf{J}, \\
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon}, \\
\nabla \cdot \mathbf{H} &= 0.
\end{align*}

(1.3)
In this regime, we see that the electric and magnetic fields have completely decoupled. Examining the equations for $E$ for a moment, the first implies that the electric field must be a gradient. We then write:

$$E = -\nabla \varphi,$$  \hspace{1cm} (1.4)

where $\varphi$ will be referred to as the electric potential. Inserting this representation into the divergence equation, we have a Poisson problem for the potential function:

$$-\Delta \varphi = \frac{\rho}{\varepsilon}.$$  \hspace{1cm} (1.5)

That is to say, for a known charge distribution $\rho$, the calculation of the electric field requires the solution to the above Poisson problem.

Standard PDE discretizations of the Poisson problem (finite difference, finite element, etc.) result in sparse linear systems which can be solved relatively efficiently using sparse linear algebra methods (nested dissection, etc.) or iterative methods (e.g. conjugate gradients). Often solvers which scale near linearly in the number of unknowns can be derived. However, if the domain of interest is unbounded (i.e. all of $\mathbb{R}^3$) then a prohibitive amount of discretization is required, or such methods as Perfectly Matched Layers (PMLs) must be used. Lastly, derivatives of the solution $\varphi$ must be calculated numerically, which can lead to severe loss of precision if a very fine discretization mesh was used.

On the other hand, using a little bit of physics and mathematics, we can directly write down the solution to equation (1.5) using Coulomb’s Law:

$$\varphi(x) = \int \frac{\rho(x')}{4\pi \varepsilon |x - x'|} \, dV(x').$$  \hspace{1cm} (1.6)

Mathematically, this corresponds to the use of the Green’s function for the Laplace operator:

$$\Delta g = \delta,$$  \hspace{1cm} (1.7)

with $g(x, x') = -1/4\pi |x - x'|$. Note that by writing down the solution $\varphi$ in integral form, we have eliminated the solve step! Only an evaluation of the integral in (1.6) is required. In avoiding having to actually solve a linear system, however, we have introduced various numerical analysis and algorithmic issues that need to be overcome. First, the singular kernel in this expression needs to be integrated accurately. This requires the development of specialized quadrature rules for singular integrals. And second, if support of $\rho$ is discretized at $N$ points $x_j$, and the potential $\varphi$ is required at each of these points, denoting our quadrature weights by $w_j$, we are left with the dense sum:

$$\varphi(x_i) \approx \frac{1}{\varepsilon} \sum_{j \neq i} \frac{w_j \rho(x_j)}{4\pi |x_i - x_j|}.$$  \hspace{1cm} (1.8)

It is the rapid evaluation of sums such as these that precipitated the development, originally, of what are known as tree codes. These algorithms are almost always low-order accurate, and scale as $O(N \log N)$. Later on, in the late 80s, more efficient algorithms known as fast multipole methods (FMMs) which scale linearly in $N$ were developed to optimally accelerate the computation of $N$-body sums such as (1.8).

Despite these numerical and algorithmic requirements in order to efficiently use integral methods, the benefits far outweigh work required to implement them. For example, derivatives
of $\varphi$ can be computed analytically, and the potential can be computed at locations $y$ outside the support of $\rho$ straightforwardly – that is to say, merely the sum above is evaluated with $bx_i$ replaced by $y$. Lastly, any numerical conditioning issues in the evaluation of the solution are usually tied to underlying physical considerations, and not artifacts of numerical differentiation or other various instabilities.

It is with examples such as these that the following lecture notes were developed. While fast multipole methods are most commonly associated with the evaluation of $N$-body sums, their true power lies in their ability to accelerated integral equation methods. This feature has revolutionized the field of computational electromagnetics, in particular. Analogous methods are used every day in large-scale calculations in fluid flow, acoustic wave propagation, and elastodynamics simulations. We will address many of these applications in the following chapters, with a focus on modern developments and thorough treatment of both the integral equation theory and the implementation of the related algorithms.

1.2 The state of computing

For the past several decades, Moore’s Law regarding the growth of computational power has approximately held true: the number of transistors in integrated circuits has roughly doubled every two years. This behavior has resulted in unprecedented growth in computational power, allowing for computations that previously required entire buildings of computers to now be performed on laptops. While Moore’s Law cannot continue to hold true indefinitely, it will most likely persist for some time.

This being said, it is more common today for computing environments to obtain more flops per second by adding parallelism in the form of networked machines or dedicated compute cards such as Graphics Processing Units (GPUs) or specialized co-processors, instead of relying on an increase in clock speed or density of transistors. Many of the following fast algorithms, since they are often built on hierarchical data structures, are compatible with such compute architecture.

The question that remains is: what happens to computational science when the exponential growth in compute power dies off? Or rather, what types of algorithms will be compatible with limited computing resources?

While not complete by any means, there are several characteristics that all the fast analysis-based algorithms shared that will be covered in these notes. These algorithms differ from many of the classical algorithms rooted in theoretical computer science (sorting, etc.).

Asymptotic scaling: In part because Moore’s Law has held true, asymptotic scaling of fast computational algorithms is perhaps their most important characteristic. For example, a standard matrix-vector multiplication requires $O(n^2)$ operations. If for twice the money, a computer can be purchased which is twice as fast and has twice as much memory (this is not exactly true), then only a matrix of dimension $\sqrt{2} \approx 1.4$ times larger can be applied to vectors of length $N$ in the amount of time. This is not a sustainable scaling of cost vs. computational ability in the long-term. Going forward, by fast algorithm we mean one whose computational complexity scales as $O(N \log^p N)$, for some reasonably small $p$, as the size $N$ of the inputs grows.
Controlled precision: Analysis-based algorithms strongly take advantage of the fact that modern day computers operate natively as finite precision machines. For example, the Fast Fourier Transform relies on particular algebraic observations concerning complex exponentials, as well as relying on the data to be sampled at equispaced intervals. No numerical approximation is made in the design of the algorithm; it is exact in infinite precision. This being said, often times that degree of precision is not needed. The algorithms detailed in this course often scale as $O(N \log^p N \log^q 1/\epsilon)$, where $\epsilon$ is the precision to which the output of the algorithm is desired. Obtaining more correct digits in the output only affects the constant implicit in the $O(\cdot)$ notation, not the asymptotic scaling with $N$. Because of only the need for finite precision solutions, many ideas in approximation theory can be used to accelerate and control intermediate calculations.

Technical implementations: Most, if not all of the algorithms described in these notes rely on hierarchical data structures, efficient linear algebra routines, and/or efficient and accurate evaluation of many of the special functions of classical mathematical physics (Bessel functions, orthogonal polynomials, etc.). Each of these tasks can be an entire project in and of themselves, but once implemented, become black box subroutines that can be relied on. In fact, the efficient and accurate evaluation of certain classes of special functions is an ongoing research topic in several groups.

The above characteristics of many fast analysis-based algorithms have some interesting consequences. For example, as in the FFT, evaluating $N$-body interactions (for very large $N$) using the fast multipole method is often more accurate that doing the direct calculation via matrix-vector multiplication. The reason for this is the inherent round-off error from computing the dot-product of two vectors of length $N$. In fact, the round-off error in computing $x \cdot y$ can be shown to be:

$$\epsilon \leq N \epsilon_{mach} \sum_j |x_j||y_j|.$$  \hspace{1cm} (1.9)

Since fast multipole methods never add up $N$ numbers for each data point, the round-off error is often much smaller.

1.3 Notes on the notes

In the following manuscript, we have explicitly avoided any such discuss of function spaces. When necessary, and of interest, references are provided to direct the reader to sources that may clarify the requirements on functions, kernels, operators, etc. For the most part, unless otherwise noted, boundaries are assumed to be smooth and kernels and functions are assumed to be square integrable. This assumption naturally corresponds to the classic $L^2$ theory of integral operators, as is commonly seen in works by Fredholm, Hilbert, and Riesz.
2 The Laplace equation

In the introduction, we saw that Poisson’s equation arises naturally from physical considerations in electrostatics merely by using Coulomb’s law for the force exerted on charged particles. Similarly, Laplace’s equation with boundary conditions (and Poisson’s equation) also arise naturally in fluid flow and elasticity. These problems will be treated in later chapters. We first discuss the simplest case, that of Laplace’s equation in two dimensions.

2.1 Two dimensions

In two dimensions, the Green’s function $g$ for Laplace’s equation satisfies

$$\Delta g(x, x') = \delta(x - x').$$  \hspace{1cm} (2.1)

Since the right hand side is only a function of $r$, we can assume that so is the Green’s function $g = g(r)$. With this in mind, and switching to polar coordinates, we see that the Green’s function, away from $r = 0$ must satisfy:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}\right) g(r) = 0.$$  \hspace{1cm} (2.2)

The solution to this ODE is $g(r) = c_1 \log r + c_2$. Taking $c_2 = 0$, the constant $c_1$ can be determined by integrating both sides of (2.1) over some ball of radius $R$ and applying the Divergence Theorem. We have that

$$\iiint_{B(0,R)} \Delta g(r) \, dV(r) = \iiint_{B(0,R)} \nabla \cdot \nabla g(r) \, dV(r)$$

$$= \int_{\partial B(0,R)} \mathbf{n} \cdot \nabla g(r) \, dA(r)$$

$$= \int_0^{2\pi} \left( \frac{\partial}{\partial r} \log r \right) |_{r=R} R \, d\phi$$

$$= 2\pi.$$  \hspace{1cm} (2.3)

This result is independent of $R$, and since $\iint \delta = 1$, by definition the definition of the $\delta$ function, we have that

$$g(x, x') = g(x - x') = \frac{1}{2\pi} \log |x - x'|.$$  \hspace{1cm} (2.4)

Using this Green’s function the solution to the Poisson problem in all of $\mathbb{R}$,

$$\Delta u = f,$$  \hspace{1cm} (2.5)
can explicitly be written as:
\[ u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - x'| f(x') \, dV(x'). \] (2.6)

The log kernel is absolutely integrable, and therefore \( u \) is well-defined everywhere.

For functions \( \sigma \) and \( \mu \) supported on the boundary \( \Gamma \) of some open region \( \Omega \subset \mathbb{R}^2 \), we define the single and double layer potential operator:
\[
S\sigma(x) = \int_{\Gamma} g(x, x') \sigma(x') \, ds(x') \\
D\mu(x) = \int_{\Gamma} \left( \frac{\partial}{\partial n'} g(x, x') \right) \mu(x') \, ds(x'),
\] (2.7)
where \( \partial / \partial n = n \cdot \nabla \) and it is assumed that \( x \notin \Gamma \). It is worth noting that no matter what \( \sigma \) and \( \mu \) are, the functions \( S\sigma \) and \( D\mu \) are harmonic everywhere by construction.

In the limit as \( x \) approaches the boundary \( \Gamma \), special case must be made to in order to obtain the correct limit in these integrals. Since the Green’s function \( g \) exhibits only a logarithmic singularity, it is absolutely integrable and we have that
\[
\lim_{h \to 0} S\sigma(x + hn) = S\sigma(x). \] (2.8)

That is to say, the function \( S\sigma \) is continuous across the boundary \( \Gamma \). On the other hand, it can be shown that the double layer potential \( D\mu \) exhibits a jump of size \( \mu \) across \( \Gamma \):
\[
\lim_{h \to 0} D\mu(x + hn) = \pm \frac{1}{2} \mu(x) + D\mu(x), \] (2.9)
where for \( x \in \Gamma \), in somewhat of an abuse of notation, \( D\mu(x) \) is interpreted in its principal value sense:
\[
D\mu(x) = \lim_{\epsilon \to 0} \int_{\Gamma \setminus B(x, \epsilon)} \frac{-n' \cdot (x - x')}{{2\pi|x - x'|}^2} \mu(x') \, ds(x'), \] (2.10)
where \( B(x, \epsilon) \) is the ball of radius \( \epsilon \) centered at \( x \). A careful calculation along the smooth curve \( \Gamma \) can show that the integral over \( \Gamma \cap B(x, \epsilon) \) is merely \( \pm \mu / 2 \):
\[
\lim_{\epsilon \to 0} \lim_{h \to 0} \int_{\Gamma \cap B(x, \epsilon)} \frac{-n' \cdot (x + hn - x')}{{2\pi|x + hn - x'|}^2} \mu(x') \, ds(x') = \pm \frac{1}{2} \mu(x). \] (2.11)

Furthermore, the double layer kernel is actually continuous at \( x = x' \), with
\[
\lim_{x \to x'} \frac{-n' \cdot (x - x')}{{2\pi|x - x'|}^2} = \kappa(x), \] (2.12)
where \( \kappa \) denotes the signed curvature of \( \Gamma \). See [?] for a full derivation. Related formulas in the complex analytic case, known as Sokhotski-Plemelj formulae, can easily be derived using the residue theorem.
Lemma 1 (Plemelji formula). Let $f$ be a complex analytic function in a neighborhood of the curve $\Gamma$ which bounds a simply connected region $\Omega \subset \mathbb{C}$. Then for $w \in \Gamma$,

$$
\lim_{\epsilon \to 0^{+}} \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - w} \, dz = \pm \frac{1}{2} f(w) + \text{PV} \int_{\Gamma} \frac{f(z)}{z - w} \, dz,
$$

(2.13)

where $+$ denotes the limit from the exterior $(\mathbb{C} \setminus \Omega)$, $-$ the limit from the interior, and PV the principal value.

Proof. We prove the case for the interior limit only, as the exterior limit is virtually identical. To this end, let $w \in \Gamma$, and let $\Gamma_{\epsilon} = \Gamma \cap B(w, \epsilon)$. Denote by $\partial B^{*}(w, \epsilon)$ the part of the boundary of this ball that lies exterior to $\Omega$. The interior limit corresponds to the limit as $\epsilon \to 0$ in Figure 2.1. The direction (counter-clockwise) of integration is also shown in the figure. The integral can therefore be split as follows:

$$
\int_{\Gamma} \frac{f(z)}{w - z} \, dz = \lim_{\epsilon \to 0} \int_{\partial B^{*}(w, \epsilon)} \frac{f(z)}{w - z} \, dz + \text{PV} \int_{\Gamma} \frac{f(z)}{w - z} \, dz.
$$

(2.14)

Under the assumption that $f$ is analytic in a neighborhood of $\Gamma$, $\epsilon$ can be chosen small enough so that $f$ can be written in $B(w, \epsilon)$ as

$$
f(z) = f(w) + f'(w)(z - w) + \frac{f''(w)}{2}(z - w)^2 + \ldots.
$$

(2.15)

Inserting this expansion into the first integral on the right of (2.14), we have (as in a residue calculation):

$$
\int_{\partial B^{*}(w, \epsilon)} \frac{f(z)}{w - z} \, dz = \int_{\partial B^{*}(w, \epsilon)} \frac{f(w)}{w - z} \, dz + \int_{\partial B^{*}(w, \epsilon)} \frac{f'(w)(z - w)}{w - z} \, dz + \ldots
$$

$$
= f(w) \int_{\theta_{1}}^{\theta_{2}} \frac{-ie^{i\theta}}{\epsilon e^{i\theta}} \, d\theta + f'(w) \int_{\partial B^{*}(w, \epsilon)} \frac{ie^{2i\theta}}{\epsilon e^{i\theta}} \, d\theta + \ldots
$$

(2.16)

$$
= -i f(w) \int_{\theta_{1}}^{\theta_{2}} \, d\theta + i e f'(w) \int_{\theta_{1}}^{\theta_{2}} \, e^{i\theta} \, d\theta + \ldots
$$

where the curve has been parameterized according to $w - z = \epsilon e^{i\theta}$. Taking the limit as $\epsilon \to 0$, we see that $\theta_{2} - \theta_{1} \to \pi$, and we have that

$$
\lim_{\epsilon \to 0} \int_{\partial B^{*}(w, \epsilon)} \frac{f(z)}{w - z} \, dz = -i\pi f(w).
$$

(2.17)

□

A similar calculation can be done for the real-valued double layer kernel, as it can be shown that

$$
\Re \left( \frac{1}{2\pi i} \frac{dz'}{z - z'} \right) = \frac{-n' \cdot (x - x')}{|x - x'|^{2}} \, ds(x'),
$$

(2.18)

where we have associated the complex values $z, z'$ with the real vectors $x$ and $x'$, respectively.
To summarize, we have the following limited behaviors for the single and double layer potentials:

\[
\lim_{h \to 0^\pm} S \sigma(x + hn) = S \sigma(x), \\
\lim_{h \to 0^\pm} D \mu(x + hn) = \pm \frac{1}{2} \mu(x) + D \mu(x). \tag{2.19}
\]

The kernel of \( S \) is singular, but integrable, and the kernel of \( D \) is continuous (and therefore integrable). Analogous results can be shown for their normal derivatives along \( \Gamma \), denoted by \( S' \) and \( D' \):

\[
\lim_{h \to 0^\pm} S' \sigma(x + hn) = \mp \frac{1}{2} \sigma(x) + S' \sigma(x), \\
\lim_{h \to 0^\pm} D' \mu(x + hn) = D' \mu(x). \tag{2.20}
\]

The operator \( D' \) is known as a hypersingular operator, and has to be interpreted as a finite part integral. That is to say, for \( x \in \Gamma \),

\[
\text{FP} D' \mu(x) = \lim_{\epsilon \to 0} \left( \int_{\Gamma \setminus B(x, \epsilon)} \frac{\partial^2 g(x, x')}{\partial n \partial n'} \mu(x') ds(x') - \frac{2 \mu(x)}{\epsilon} \right). \tag{2.21}
\]

As before, principal value and finite part integrals are implied when the argument is located on the boundary of integration.

### 2.1.1 The Interior Dirichlet Problem

The canonical boundary value problem for the Laplace equation is that of the Interior Dirichlet problem. In this section we derive a corresponding integral equation for this boundary value problem, prove that it has a unique solution, and then discuss discretization and quadrature techniques in the next section. Other boundary value problems are treated in subsequent sections, but the techniques are very similar.

For some connected bounded region \( \Omega \) with smooth boundary \( \Gamma \), and \( f \) defined on \( \Gamma \), the interior Dirichlet boundary value problem is given as:

\[
\Delta u = 0, \quad \text{in } \Omega, \\
u = f, \quad \text{on } \Gamma. \tag{2.22}
\]
This boundary value problem arises naturally in electrostatic scattering problems from perfect electric conductors. The uniqueness of \( u \) is established by Green’s First Identity (see Appendix). If \( u_1 \) and \( u_2 \) were two different solutions to (2.22), then the difference \( u = u_1 - u_2 \) would satisfy the interior Dirichlet problem with \( f = 0 \). Setting \( v = u \) in Green’s First Identity \( \text{(A.2)} \), we have that

\[
\int_{\Omega} |\nabla u|^2 \, dV = 0. 
\]  
(2.23)

Therefore, \( u \) must be a constant. But since \( u = 0 \) on \( \Gamma \), \( u = 0 \) everywhere.

Instead of numerically solving (2.22) in differential form, we now change variables and represent the solution \( u \) in \( \Omega \) in terms of a double layer potential due to an unknown distribution \( \sigma \) (motivation to be discussed later on). For \( x \) in the interior of \( \Omega \), we write:

\[
u(x) = D\sigma(x) = \int_{\Gamma} \left( \frac{\partial}{\partial n'} \frac{1}{2\pi} \log |x - x'| \right) \sigma(x') \, ds(x').
\]  
(2.24)

Note that since the Green’s function was used in the representation of \( u \), we automatically have that \( \Delta u = 0 \) in \( \Omega \). It merely remains to satisfy the boundary condition on \( \Gamma \) that \( u = f \). Using the limiting values of the layer potential operators discussed in the previous section, enforcing the boundary condition \( u = f \) on \( \Gamma \) yields the following integral equation for \( \sigma \):

\[
-\frac{1}{2} \sigma + D\sigma = f,
\]  
(2.25)

where the limit to the boundary has been taken from the domain of interest (i.e., the interior, inside \( \Omega \)). Since \( D\sigma \) defines a harmonic function everywhere inside \( \Omega \), any such solution to the above integral equation will yield a solution \( u \) to the interior Dirichlet problem by setting \( u = D\sigma \). It remains to be shown that there is a unique solution to this integral equation.

By the Fredholm Alternative (see Appendix \[B\]), since the operator \( D : \Gamma \rightarrow \Gamma \) is compact, to show the uniqueness of \( \sigma \) it suffices to show that the homogeneous integral equation

\[
\left( -\frac{1}{2} I + D \right) \sigma = 0
\]  
(2.26)

only has the solution \( \sigma = 0 \).

### 2.1.2 Discretization and quadrature
3 The 3D Laplace Fast Multipole Method
A Green’s Identities

All three Green’s identities are a direct consequence of the Divergence Theorem (valid in any dimension):
\[ \int_{\Omega} \nabla \cdot F \, dV = \int_{\Gamma} n \cdot F \, dA, \quad (A.1) \]
where \( \Omega \) is a bounded (assumed to be open) region with boundary \( \Gamma \), and \( F \) is a smooth vectorfield. Green’s First Identity can be derived by setting \( F = u \nabla v \), where \( u \) and \( v \) are smooth functions defined on the closure of \( \Omega \):
\[ \int_{\Omega} \left( \nabla u \cdot \nabla v + u \Delta v \right) \, dV = \int_{\Gamma} u \frac{\partial v}{\partial n} \, dA, \quad (A.2) \]
where as always, \( \frac{\partial}{\partial n} \) denoted differentiation in the direction outward from \( \Omega \).
Green’s Second Identity is derived by setting \( F = v \nabla u \), and subtracting the resulting equation from that in (A.2):
\[ \int_{\Omega} (u \Delta v - v \Delta u) \, dV = \int_{\Gamma} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, dA. \quad (A.3) \]
Finally, Green’s Third Identity is obtained by setting \( v(x') = g(x,x') \), where \( g \) is the Green’s function for the Laplace operator, and integrating in the \( x' \) variable:
\[ u = \int_{\Omega} g(\cdot, x') \Delta u(x') \, dV(x') + \int_{\Gamma} \left( u(x') \frac{\partial g(\cdot, x')}{\partial n'} - g(\cdot, x') \frac{\partial u(x')}{\partial n'} \right) \, dA(x') \]
\[ = \nabla \Delta u + D u - S \frac{\partial u}{\partial n}. \quad (A.4) \]
If \( u \) happens to be a harmonic function, this identity reduces to what is known as Green’s Reproducing Identity:
\[ u = D u - S \frac{\partial u}{\partial n}. \quad (A.5) \]
For values of \( x \) on the boundary of \( \Omega \), the limiting form of the above reproducing formula is:
\[ u = \pm \frac{1}{2} u + D u - S \frac{\partial u}{\partial n}. \quad (A.6) \]
where the sign depends on which side the limit is taken from, and the integral operator \( D \) is implicitly interpreted in its principal value sense (even though it has a continuous kernel).
B Fredholm theory