Math 233 Note

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1 Lecture Video and Textbook

1. Normal approximation to the binomial distribution
Firstly, we start by introducing DeMoivre-Laplace Limit theroem, which states that if we standardize the binomial by first substracting its mean np and then dividing the result by its standard deviation $\sqrt{np(1-p)}$, then the distribution function of this standardized random variable (which has mean 0 and variance 1) will converge to the standardized normal distribution function as $n \longrightarrow \infty$.

Formally, we can define as following:
If S_n denotes the number of successes that occur when n independent trials, each resulting in a success with probability p, are performed, then, for any $a < b$, $P[a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b] \longrightarrow \Phi(b) - \Phi(a)$ as $n \longrightarrow \infty$.

Let X be distributed as binomial with (n,p). Then, we get, where k is a particular integer.

$$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

We see that when n is large, this $P[X = k]$ is complicated to compute. Thus, the application of normal distribution to binomial distribution helps to reduce the amount of computational tasks.

Now, we see that since the random binomial variable is discrete, $P[x = k]$ is also equal to $P[k - \frac{1}{2} \leq x \leq k + \frac{1}{2}]$, which is the form that we can apply normal approximation.

Therefore, $P[k - \frac{1}{2} \leq x \leq k + \frac{1}{2}] = P[\frac{k - \frac{1}{2} - np}{\sqrt{np(1-p)}} \leq \frac{X - np}{\sqrt{np(1-p)}} \leq \frac{k + \frac{1}{2} - np}{\sqrt{np(1-p)}}]$. By the DeMovie-Laplace approximation, the random variable, $\frac{X - np}{\sqrt{np(1-p)}}$ in the middle is approximately equal to standard normal random variable $N(0, 1)$ as $n \longrightarrow \infty$. Thus, we get $P[\frac{k - \frac{1}{2} - np}{\sqrt{np(1-p)}} \leq \frac{X - np}{\sqrt{np(1-p)}} \leq \frac{k + \frac{1}{2} - np}{\sqrt{np(1-p)}}] \approx P[\frac{k - \frac{1}{2} - np}{\sqrt{np(1-p)}} \leq Z \leq \frac{k + \frac{1}{2} - np}{\sqrt{np(1-p)}}] = \Phi(\frac{k + \frac{1}{2} - np}{\sqrt{np(1-p)}}) - \Phi(\frac{k - \frac{1}{2} - np}{\sqrt{np(1-p)}})$.

Now, we complete the process of normal approximation to the binomial distribution and we define the step setting $P[x = k] = P[k - \frac{1}{2} \leq x \leq k + \frac{1}{2}]$ as Continuity Correction, which helps to relate a continuous random variable to a discrete random variable. To employ the normal approximation, note that because the binomial is a discrete integer-valued random variable, whereas the normal is a continuous random variable,it is best to write $P[X = i]$ as $P[i - \frac{1}{2} < X < i + \frac{1}{2}]$ before applying the normal approximation. Also, the larger the n is, the closer the approximation of $\frac{X-np}{\sqrt{np(1-p)}}$ to Z and therefore we can get a closer to approximation of $P[\frac{X-np}{\sqrt{np(1-p)}} \leq \frac{X-np}{\sqrt{np(1-p)}} \leq \frac{X+np}{\sqrt{np(1-p)}}]$ by $\Phi(\frac{X-np}{\sqrt{np(1-p)}}) - \Phi(\frac{X-np}{\sqrt{np(1-p)}})$.

2. Exponential random variable A continuous random variable whose probability density function is given, for some $\lambda > 0$, by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{x} \geq 0 \\ 0 & \text{x} < 0 \end{cases}$$

is said to be an exponential random variable with parameter λ. The cumulative distribution function $F(a)$ of an exponential random variable is given by

$$F(a) = P\{X \leq a\} = \int_a^b \lambda e^{-\lambda x} \,\mathrm{d}x = 1 - e^{-\lambda a}, a \geq 0$$

Note that $F(\infty) = \int_0^\infty \lambda e^{-\lambda x} \,\mathrm{d}x = 1$. In practice, In practice, the exponential distribution often arises as the distribution of the amount of time until some specific event occurs. For instance, the amount of time (starting from now) until an earthquake occurs, or until a new war breaks out, or until a telephone call you receive turns out to be a wrong number are all random variables that tend in practice to have exponential distributions.

Example: Let X be an exponential random variable with parameter λ. Calculate $(a)E[X]$ and $(b)Var(X)$.

Solution:(a) Since the density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{x} \geq 0 \\ 0 & \text{x} < 0 \end{cases}$$

We obtain, for $n > 0$,

$$E[X^n] = \int_0^\infty x^n \lambda e^{-\lambda x} \,\mathrm{d}x = 1$$

Integration by parts yields

$$E[X^n] = 0 + \frac{n}{\lambda} \int_0^\infty x^{n-1} \lambda e^{-\lambda x} \,\mathrm{d}x = \frac{n}{\lambda} E[X^{n-1}]$$

Setting $n = 1$ and then $n = 2$ gives

$$E[X] = \frac{1}{\lambda}, E[X^2] = \frac{2}{\lambda}E[X] = \frac{2}{\lambda^2}$$

(b)Hence,

$$Var(X) = \frac{2}{\lambda^2} - (\frac{1}{\lambda})^2 = \frac{1}{\lambda^2}$$

Thus, we can easily see that Thus, the mean of the exponential is the reciprocal of its parameter λ, and the variance is the mean squared.
Also, there is also one variation of the exponential distribution, which is the distribution of a random variable that is equally likely to be either positive or negative and whose absolute value is exponentially distributed with parameter λ, $\lambda \geq 0$. Such a random variable is said to have a Laplace distribution, and its density is given by

$$f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}, -\infty < x < \infty$$

and its distribution function is given by

$$f(x) = \begin{cases} \frac{1}{2}e^{\lambda x} & \text{x} < 0 \\ 1 - \frac{1}{2}e^{-\lambda x} & \text{x} > 0 \end{cases}$$

3. Gamma distribution A random variable is said to have a gamma distribution with parameters (α, λ), $\lambda > 0$, $\alpha > 0$, if its density function is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & \text{x} \geq 0 \\ 0 & \text{x} < 0 \end{cases}$$

where

$$\Gamma(\alpha) = \int_a^b f(x)\, \mathrm{d}x. \tag{1}$$

If we integrate $\Gamma(\alpha)$ by parts repeatedly, it follows that for integral value of n, $\Gamma(n) = (n-1)!$.

4. Cauchy distribution

A random variable is said to have a Cauchy distribution with parameter θ , $-q < \theta < q$, if its density is given by

$$f(x) = \frac{1}{\pi}\frac{1}{1 + (x-\theta)^2}, -\infty < x < \infty$$

2 Graphs and Exercise

1. Recap and Graphs

(a) Normal approximation to the binomial distribution in graph $P[\frac{k-\frac{1}{2}-np}{\sqrt{np(1-p)}} \leq \frac{X-np}{\sqrt{np(1-p)}} \leq \frac{k+\frac{1}{2}-np}{\sqrt{np(1-p)}}] \approx P[\frac{k-\frac{1}{2}-np}{\sqrt{np(1-p)}} \leq Z \leq \frac{k+\frac{1}{2}-np}{\sqrt{np(1-p)}}] = \Phi(\frac{k+\frac{1}{2}-np}{\sqrt{np(1-p)}}) - \Phi(\frac{k-\frac{1}{2}-np}{\sqrt{np(1-p)}})$

Figure 3 below shows how the graph of Gamma distribution goes.

(2) Cauchy distribution density function:

$$f(x) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2}, -\infty < x < \infty$$

Figure 2 below shows how the graph of Gamma distribution goes.
Remark: Tail of Cauchy distribution is much thicker than Normal distribution.

(3) Gamma distribution

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & \text{x} \geq 0 \\ 0 & \text{x} < 0 \end{cases}$$

where

$$\Gamma(\alpha) = \int_a^b f(x)\,\mathrm{d}x. \tag{2}$$

Figure 1 below shows how the graph of Gamma distribution goes.

(4). Related Exercise

Suppose that the length of a phone call in minutes is an exponential random variable with parameter =110. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait
1. more than 10 minutes;
2. between 10 and 20 minutes.

Solution: Let X denote the length of the call made by the person in the booth. Then the desired probabilities are
1. $P[X > 10]$=1$F(10) = e^{-1} \approx 0.368$
2. $P[10 < X < 20] = F(20)$–$F(10) = e^{-1} - e^{-2} \approx 0.233$

We say that a nonnegative random variable X is memoryless if $P[X > s+t|X > t] = P[X > s]$ for all $s, t \geq 0$ It states that the probability that the waiting time is at least $s + t$ minutes, given that you have waited for t minutes, is the same as the initial probability that you wait for at least s minutes. $P[X > s + t|X > t] = P[X > s]$ for all $s, t \geq 0$ is equivalent to $P[X > s+t, X > t]P[X > t] = P[X > s]$ or $P[X > s+t] = P[X > s]P[X > t]$.

This is satisfied when X is exponentially distributed (for $e^{-\lambda(s+t)} = e^{-\lambda s} e^{-\lambda t}$), it follows that exponentially distributed random variables are memoryless.

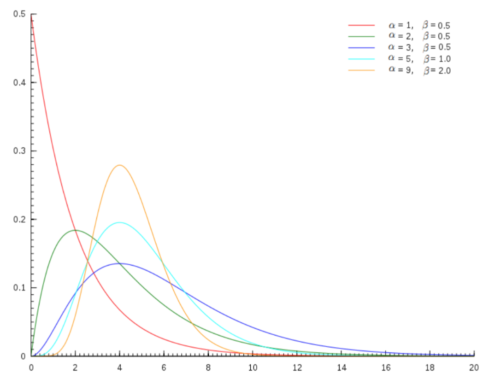

Figure 1: Gamma distribution

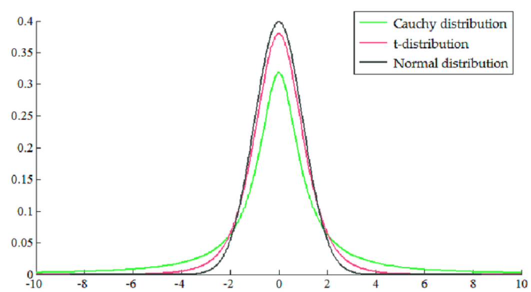

Figure 2: Cauchy distribution

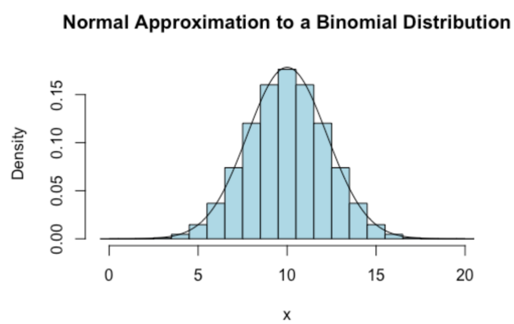

Figure 3: Normal distribution