Homework 6

Due: 12:30pm May 3, 2018

Notes on the assignment:

Submission: Homework assignments must be submitted in hardcopy at the beginning of class on the due date. Late homework will not be accepted. You are encouraged to use the original homework LaTeX document as a template to write-up your homework. If you are required to hand in code, this will explicitly be stated on the homework assignment.

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1. (10 pts) A quadrature formula on the interval $[-1,1]$ uses the quadrature nodes $x_1 = -\alpha$ and $x_2 = \alpha$, where $\alpha \in (0,1]$:

$$\int_{-1}^{1} f(x)\,dx \approx w_1\,f(-\alpha) + w_2\,f(\alpha).$$

The formula is required to be exact whenever f is a polynomial of degree 1. Show that $w_1 = w_2 = 1$, independent of the value of α. Show also that there is one particular value of α for which the formula is exact also for all polynomials of degree 2. Find this α, and show that, for this value, the formula is also exact for all polynomials of degree 3.

2. (10 pts) Denote by $T_n f$ the composite trapezoidal rule applied to f on the interval $[a,b]$,

$$T_n f = h\left(\frac{1}{2}f(x_0 = a) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n = b)\right),$$

with $h = x_j - x_{j-1}$, and by $S_{2n} f$ the composite Simpson rule applied to f on the same interval,

$$S_{2n} f = \frac{h}{3}\left(f(x_0 = a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n} = b)\right).$$

Show that

$$S_{2n} f = \frac{4T_{2n} f - T_n f}{3}.$$

3. Construct the 10-point Gauss-Legendre quadrature rule on the interval $[-1,1]$ in the following steps:

   (a) (4 pts) Write a code to find the zeros of P_{10}, the Legendre polynomial of degree 10.

   (b) (3 pts) Write a code which computes the Gauss-Legendre weights $w_1, \ldots, w_{10}$ by enforcing that the following integrals are *exactly* computed by the quadrature rule:

$$I_m = \int_{-1}^{1} x^m\,dx = \sum_{i=1}^{10} w_i\,x_i^m, \qquad \text{for } m = 0, \ldots, 9.$$

   (c) (3 pts) Show that the quadrature rule constructed gives the exact (up to machine precision) answer for the integrals

$$I_m = \int_{-1}^{1} x^m\,dx, \qquad \text{for } m = 0, \ldots, 19.$$

You should turn in all your work and a printout of your code, as well as a list of all the quadrature nodes and weights, shown to 12 significant digits.

Hint: The Legendre polynomials satisfy the following recurrence relationship:

$$P_{n+1}(x) = \frac{2n+1}{n+1}\, x\, P_n(x) - \frac{n}{n+1}\, P_{n-1}(x).$$

The derivative $P'_{n+1}(x)$ can be computed by differentiating the above expression.

4. (10 pts) Using Taylor's Theorem, derive the error term for the approximation:

$$f'(x) \approx \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}.$$

What is the round-off error in the above finite difference? (You can ignore the error in computing h, x, $x+h$, and $x+2h$.)

5. (a) (5 pts) Show that Euler's method fails to approximate the solution

$$y(x) = \left(\frac{4x}{5}\right)^{5/4}$$

of the initial value problem

$$\begin{aligned} y' &= y^{1/5} \\ y(0) &= 0. \end{aligned}$$

Justify your answer.

(b) (5 pts) Now consider approximating the solution to the same initial value problem with the implicit Euler method. Show that there is a solution of the form $y_n = (c_n\, h)^{5/4}$, for $n \geq 0$, with

$$\begin{aligned} c_0 &= 0, \\ c_1 &= 1, \\ c_n &> 1, \qquad \text{for all } n \geq 2. \end{aligned}$$