Preface to an Earlier Version of RPDG

My friend Leo Moser (1921–1970) was an avid creator, collector, and solver of problems in number theory and combinatorics. At the 1963 Number Theory Conference in Boulder, Colorado, he distributed mimeographed copies of his list of fifty problems, which he called "Poorly formulated unsolved problems in combinatorial geometry." Although some parts of this collection have been reproduced several times, the entire list in its original form appeared in print only recently (*Discrete Applied Math.* **31** (1991), 201–225).

After Leo Moser's death, his brother Willy put together his *Research Problems in Discrete Geometry (RPDG),* which was based on some questions proposed by Leo and was first distributed among the participants of the Discrete Geometry week in Oberwolfach, July 1977. This collection has been revised and largely extended by W. Moser and J. Pach. It has become an excellent resource book of fascinating open problems in combinatorial and discrete geometry which had nine different editions circulating in more than a thousand copies. In the last fifteen years it has reached virtually everybody interested in the field, and has generated a lot of research. In addition to the many new questions, a number of important but badly forgotten problems have also been publicized in these collections. They include Heilbronn's (now famous) triangle problem and my old questions about the distribution of distances among n points in the plane, just to mention two areas where much progress has been made recently. The present book is an updated "final" version of a large subset of the problems that appeared in the previous informal editions of *Research Problems in Discrete Geometry.* The authors have adopted a very pleasant style that allows the reader to get not only a feel for the problems but also an overview of the field.

And now let me say a few words about discrete geometry. As a matter of fact, I cannot even give a reasonable definition of the subject. Perhaps it is not inappropriate to recall the following old anecdote. Some years ago, when pornography was still illegal in America, a judge was asked to define pornography. He answered: "I cannot do this, but I sure can recognize it when I see it."

Perhaps discrete geometry started with the feud between Newton and Gregory about the largest number of solid unit ball spheres that can be placed to touch a "central" unit ball sphere. Newton believed this number to be twelve, while Gregory believed it was thirteen. This controversy was settled in Newton's favor only late in the last century. Even today little is known about similar problems in higher dimensions, although these questions were kept alive by the nineteenth century crystallographers and have created a lot of interest among physicists and biologists.

Minkowski's book *Geometrie der Zahlen* (1896) opened a new and important chapter in mathematics. It revealed some surprising connections between number theory and convex geometry, particularly between diophantine approximation and packing problems. This branch of discrete geometry was developed in books by Cassels (*An Introduction to the Geometry of Numbers*), Lekkerkerker (*Geometry of Numbers*), Coxeter (*Regular Polytopes*), and L. Fejes Tóth (*Lagerungen in der Ebene, auf der Kugel und im Raum*). "Alles Konvexe interessiert mich," said Minkowski, and I share his feeling.

Another early source is Sylvester's famous "orchard problem." In 1893 he also raised the following question: Given n points in the plane, not all on a line, can one always find a line passing through exactly two points? This problem remained unsolved and was completely forgotten before I rediscovered it in 1933. I was reading the Hilbert and Cohn-Vossen book (*Anschauliche Geometrie*) when the question occurred to me, and I thought it was new. It looked innocent, but to my surprise and annoyance I was unable to resolve it. However, I immediately realized that an affirmative answer would imply that any set of n noncollinear points in the plane determines at least n connecting lines. A couple of days later, Tibor Gallai came up with an ingenious short proof which turned out to be the first solution of Sylvester's problem. This was the starting point of many fruitful investigations about the incidence structure of sets of points and lines, circles, etc. Recently, these results have attracted a lot of attention, because they proved to be relevant in computational geometry.

In 1931, E. Klein observed that from any five points in the plane in general position one can choose four that determine a convex quadrilateral, and she asked whether the following generalization was true: For any $k \geq 4$ there exists an integer n_k such that any n_k-element set of points in general position in the plane contains the vertex set of a convex k-gon. Szekeres and I managed to establish this result; for the first proof we needed, and Szekeres rediscovered, Ramsey's theorem! Our paper raised many fascinating new questions which, I think, gave a boost to the development of combinatorial geometry and extremal combinatorics. A large variety of problems of this kind is discussed in the books of Hadwiger and Debrunner (*Combinatorial Geometry in the Plane*, translated and extended by Klee), Grünbaum (*Convex Polytopes*), Croft, Falconer, and Guy (*Unsolved Problems in Geometry*), and in the collection of my papers (*The Art of Counting*). I hope that the reader will forgive me that the above sketch of the recent history of combinatorial and discrete geometry is very subjective and, of course, overemphasizes my own contribution to the field.

There are certain areas of mathematics where individual problems are less important. However, I feel that problems play a very important role in elementary number theory and geometry. Hilbert and Hermann Weyl had the same opinion, but many eminent mathematicians disagree. I cannot decide who is right, but I am certainly on the side of Grünbaum in his old controversy with Dieudonné, who claimed that geometry is "dead." We are convinced that if a subject is rich in simple and fascinating unsolved problems, then it has a great future! The present collection of research problems by Moser and Pach proves beyond doubt the richness of discrete geometry.

I wish the reader good luck with the solutions!

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