A generalization of the Erdős-Szekeres theorem
to disjoint convex sets

János Pach* and Géza Tóth†
Courant Institute, NYU and Hungarian Academy of Sciences

Abstract

Let $\mathcal{F}$ denote a family of pairwise disjoint convex sets in the plane. $\mathcal{F}$ is said to be in convex position, if none of its members is contained in the convex hull of the union of the others. For any fixed $k \geq 3$, we estimate $P_k(n)$, the maximum size of a family $\mathcal{F}$ with the property that any $k$ members of $\mathcal{F}$ are in convex position, but no $n$ are. In particular, for $k = 3$, we improve the triply exponential upper bound of T. Bisztriczky and G. Fejes Tóth by showing that $P_3(n) < 16^n$.

1 Introduction

In their classical paper [ES1], Erdős and Szekeres proved that any set of more than $\binom{2n-4}{n-2}$ points in general position in the plane contains $n$ points which are in convex position, i.e., they form the vertex set of a convex $n$-gon. T. Bisztriczky and G. Fejes Tóth [BF1], [F] extended this result to families of convex sets.

Throughout this paper, by a family $\mathcal{F} = \{A_1, \ldots, A_t\}$ we always mean a family of pairwise disjoint compact convex sets in the plane in general position, i.e., no three of them have a common supporting line. $\mathcal{F}$ is said to be in convex position if none of its members is contained in the convex hull of the union of the others, i.e., if $\text{bd conv}(\bigcup \mathcal{F})$, the boundary of the convex hull of the union of all members of $\mathcal{F}$, contains a piece of the boundary of each $A_i$. Evidently, any two members of $\mathcal{F}$ are in convex position.

T. Bisztriczky and G. Fejes Tóth proved that there exists a function $P(n)$ such that if $|\mathcal{F}| > P(n)$ and any three members of $\mathcal{F}$ are in convex position, then $\mathcal{F}$ has $n$ members in convex position. Improving their initial result, in [BF2] they showed that this statement is true with a function $P(n)$, triply exponential in $n$. They also remarked that "it seems that none of the" previous proofs of the Erdős-Szekeres theorem "can be modified so as to obtain

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a proof of our theorem.” One of the aims of the present note is to show that the idea of
the original proof of Erdős and Szekeres can be applied to deduce the Bisztriczky-Fejes Tóth
theorem with a much better function \( P(n) < 16^n \).

**Theorem 1.** Let \( \mathcal{F} \) be a family of \( n \) pairwise disjoint compact convex sets in the plane, any
three of which are in convex position. If

\[
|\mathcal{F}| > \left( \frac{2n - 4}{n - 2} \right)^2,
\]

then \( \mathcal{F} \) has \( n \) members in convex position.

If any \( k \) members of \( \mathcal{F} \) are in convex position, then we say that \( \mathcal{F} \) satisfies property \( P_k \). If
no \( n \) members of \( \mathcal{F} \) are in convex position, then we say that \( \mathcal{F} \) satisfies property \( P^n \). Property
\( P^n_k \) means that both \( P_k \) and \( P^n \) are satisfied. Using these notions, Theorem 1 states that if
a family \( \mathcal{F} \) satisfies property \( P^n_3 \), then \( |\mathcal{F}| \leq \left( \frac{2n-4}{n-2} \right)^2 \).

T. Bisztriczky and G. Fejes Tóth [BF2] raised the following more general question. What
is the maximum size \( P_k(n) \) of a family \( \mathcal{F} \) satisfying property \( P^n_k \)? They gave an exponential
upper bound on \( P_4(n) \), and quadratic upper bounds on \( P_k(n) \) for any fixed \( k \geq 5 \), as \( n \) tends
to infinity. Some of these estimates can be improved as follows.

**Theorem 2.** \( 2 \left[ \frac{n+1}{4} \right]^2 \leq P_4(n) < n^3 \)

**Theorem 3.** \( P_1(n) \leq cn \log n \)

Obviously, \( P_1(n) \leq P_k(n) \) holds for every \( l \geq k \).

2 Proof of Theorem 1

The combinatorial seed of the original proof of the Erdős-Szekeres theorem was isolated and
generalized by Chvátal and Komlós. A complete graph, whose edges are arbitrarily oriented,
is called a tournament. An acyclic tournament is said to be *transitive*.

**Lemma 2.1** [CK] Let \( T \) be a transitive tournament with more than \( n \) vertices, and let \( f \) be any real-valued function defined on its edge set.

Then there is an oriented path \( v_1 v_2 \cdots v_n \) with \( n \) vertices such that the sequence \( f(\bar{v}_1 v_2), f(\bar{v}_2 v_3), \ldots, f(\bar{v}_{n-1} v_n) \) is either monoton increasing or strictly decreasing.

We use this statement to establish the following result, whose part (ii) was proved in
[BF2].

**Lemma 2.2.** Let \( \mathcal{F} \) be a family of compact convex sets in the plane, satisfying property \( P^n_3 \)
and at least one of the following two conditions:
(i) any two members of $\mathcal{F}$ can be separated by a vertical line;
(ii) there is a line intersecting all members of $\mathcal{F}$.

Then $\mathcal{F}$ has at most $t = \binom{2^n - 4}{n-2}$ members.

**Proof.** In case (ii), we can assume without loss of generality that the common transversal of the elements of $\mathcal{F}$ is horizontal.

Let $A_1, A_2, \ldots, A_t$ be the members of $\mathcal{F}$ listed from left to right (with respect to their projections onto the $x$-axis in case (i), and with respect to their intersections with the common transversal in case (ii)). For any $1 \leq i < j \leq t$, there are four uniquely determined points $p_1, q_1 \in \text{bd} A_i$; $p_2, q_2 \in \text{bd} A_j$ such that the segments $p_1p_2, q_1q_2$ belong to the boundary of $\text{conv}(A_i \cup A_j)$, and along this boundary the counter-clockwise order of these points is $p_2, p_1, q_1, q_2$. Let $f(i, j)$ and $g(i, j)$ denote the counter-clockwise angles from the direction of the positive $x$-axis to $\overrightarrow{p_2p_1}$ and $\overrightarrow{q_2q_1}$, respectively (see Fig. 1).

Since $\mathcal{F}$ satisfies property $P_3$, for any $i < j < k$ with $f(i, j) \leq f(j, k)$, we have $g(i, j) < g(j, k)$.

Define a transitive tournament with vertices $v_1, v_2, \ldots, v_t$, such that every edge is oriented toward its endpoint of larger index. For any $i < j$, assign to the edge $\overrightarrow{v_iv_j}$ the value $f(i, j)$. By Lemma 2.1, if $t > \binom{2^n - 4}{n-2}$, then there is a directed path $v_{i_1}, v_{i_2}, \ldots, v_{i_n}$ such that either

$$f(i_1, i_2) \leq f(i_2, i_3) \leq \ldots \leq f(i_{n-1}, i_n)$$

or

$$f(i_1, i_2) > f(i_2, i_3) > \ldots > f(i_{n-1}, i_n).$$
In both cases, it is easy to verify that \((A_{i_1}, A_{i_2}, \ldots, A_{i_m})\) are in convex position (see Fig. 2). □

\[ \text{Fig. 2. } f(1, 2) > f(2, 3) > f(3, 4) > f(4, 5) \]

Now we are ready to prove Theorem 1. Let \( \mathcal{F} \) be a family of more than \((2^n - 4)^2\) convex sets in the plane satisfying property \( P_3 \). Projecting these sets onto the \( z \)-axis, we obtain a system of intervals \( \mathcal{I} \). A well-known result of Gallai (see [B], p.373) implies that \( \mathcal{I} \) has more than \((2^n - 4)\) elements that are either pairwise disjoint or all of them have a point in common. In the first case, the corresponding elements of \( \mathcal{F} \) can be separated by vertical lines, in the second case all of them can be intersected by one line. In either case, we can apply Lemma 2.1 to finish the proof. □

3 Proof of Theorem 2

Let \( \mathcal{F} = \{A_1, A_2, \ldots, A_t\} \) be a family of pairwise disjoint convex sets in general position in the plane. Denote the convex hull of \( \cup \mathcal{F} = \bigcup_{i=1}^{t} A_i \) by \( \text{conv} \mathcal{F} \). The boundary of \( \text{conv} \mathcal{F} \), \( \text{bd} \text{conv} \mathcal{F} \), consists of finitely many boundary pieces of the \( A_i \)'s, called \textit{vertex-arcs}, connected by straight-line segments, called \textit{edge-arcs}. (This terminology reflects the picture in the special case when every set \( A_i \) is a single point.)

The elements \( A_i \in \mathcal{F} \) contributing at least one vertex-arc to the boundary of \( \text{conv} \mathcal{F} \) will be called \textit{vertices of conv} \( \mathcal{F} \) or, simply, \textit{vertices of} \( \mathcal{F} \). If \( A \) is not a vertex, then it is said to be an \textit{internal member of} \( \mathcal{F} \).

**Lemma 3.1** [BF2] Let \( k \geq 4 \) and let \( \mathcal{F} \) be a family of pairwise disjoint convex sets in the plane satisfying property \( P_k \). If \( \mathcal{F} \) has \( m \) vertices then there are \( \left\lfloor \frac{2m}{k-3} \right\rfloor \) lines such that any internal member of \( \mathcal{F} \) is intersected by at least one of them.
Lemma 3.2 Let $\mathcal{F}$ be a family of disjoint convex sets satisfying property $P^n_4$, and assume that there is a line $\ell$ intersecting all members of $\mathcal{F}$. Then $\mathcal{F}$ has at most $(n-2)^2 + 1$ members.

**Proof.** Let $A_1, A_2, \ldots, A_t$ be the members of $\mathcal{F}$ listed in the order of their intersections with $\ell$. For any $A_i, A_j$, $1 \leq i < j \leq t$, define $f(i, j)$ and $g(i, j)$ exactly as in the proof of Lemma 2.2.

If $f(i_1, i_2) > f(i_2, i_3) > \ldots > f(i_{k-1}, i_k)$ for some $i_1 < i_2 < \ldots < i_k$, then $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ are said to form an upper chain of length $k$. They form a lower chain of length $k$, if $g(i_1, i_2) < g(i_2, i_3) < \ldots < g(i_{k-1}, i_k)$. It is easy to see that, in both cases $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ are in convex position.

For any $2 \leq i < j \leq t$, let $u_i$ (resp. $l_i$) be the length of the longest upper (resp. lower) chain that ends with $A_i$. Clearly, $u_i, l_i \geq 2$.

**Claim.** If $i \neq k$, then $(u_i, l_i) \neq (u_k, l_k)$.

Indeed, if $u_i = u_k = u$, $l_i = l_k = l$ for some $i < k$, then neither the longest upper chain $A_{i_1}, \ldots, A_{i_u} = A_i$ nor the longest lower chain $A_{j_1}, \ldots, A_{j_l} = A_i$ ending with $A_i$ could be extended by $A_k$ to a longer (upper resp. lower) chain. Therefore, $f(i_{u-1}, i) < f(i, k)$ and $g(j_{l-1}, i) > g(i, k)$, which would imply $\text{conv}(A_{i_{u-1}} \cup A_{j_{l-1}} \cup A_k) \ni A_i$, contradicting property $P_4$. (See Fig. 3.)

It follows from the Claim and from the fact that $u_i, l_i \geq 2$ for every $2 \leq i \leq t$ that, if $t > (n-2)^2 + 1$, then there is an $i$ such that either $u_i \geq n$ or $l_i \geq n$. So, there is an upper (resp. lower) chain of length $n$, and its elements are in convex position. □

![Fig. 3. $f(i_{u-1}, i) < f(i, k)$, $g(j_{l-1}, i) > g(i, k)$](image-url)
Proof of Theorem 2. First we prove the upper bound. Let $\mathcal{F}$ be a family satisfying property $P^n_4$ and suppose for contradiction that $|\mathcal{F}| \geq (n - 4)((n - 2)^2 + 1) + n$.

By Lemma 3.1, one can select at most $n - 4$ lines such that every internal member of $\mathcal{F}$ intersects at least one of them. Since $\mathcal{F}$ has at least $(n - 4)((n - 2)^2 + 1) + 1$ internal members, one of the lines intersects at least $(n - 2)^2 + 2$ members of $\mathcal{F}$. By Lemma 3.2, $\mathcal{F}$ has $n$ members in convex position, contradicting property $P^n$.

Fig. 4

The lower bound is shown by the following construction. Suppose for simplicity that $n = 4k + 3$ for some $k$, and let $\mathcal{F}$ denote the family of vertical segments

$$S_{ij} = \{(x, y) \mid x = x_{ij}, y_{ij} \leq y \leq y'_{ij}\},$$

$1 \leq i \leq 2k + 2, 1 \leq j \leq 2\min(i, 2k - i + 3) - 1$, where

$$x_{ij} = i + \varepsilon j, \quad y_{ij} = (2k - i + 2)^2 + (\varepsilon j)^2, \quad y'_{ij} = (2k + 3)^2 - i^2 - (\varepsilon (k - j))^2,$$

and $\varepsilon$ is an extremely small positive number (see Fig. 4). Clearly, $|\mathcal{F}| = 2(k + 1)^2 > \frac{n^2}{8}$.

For any $S = S_{i,j} \in \mathcal{F}$, let $i(S) = i, j(S) = j$.  

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Let $\mathcal{F}'$ be a subfamily of $\mathcal{F}$, $S_{ij} \in \mathcal{F}'$. Observe that if $(x_{ij}, y_{ij}')$ is not a vertex of $\text{conv}\mathcal{F}'$, then there are $S_1, S_2 \in \mathcal{F}'$ such that $i(S_1) > i$, $i(S_2) = i$, and $j(S_2) < j$. Similarly, if $(x_{ij}, y_{ij})$ is not a vertex of $\text{conv}\mathcal{F}'$, then there are $S_3, S_4 \in \mathcal{F}'$ such that $i(S_3) < i$, $i(S_4) = i$, and $j(S_4) > j$. Therefore, if $S_{ij}$ is not a vertex of $\mathcal{F}'$, then $\mathcal{F}'$ has at least four other members. This shows that $\mathcal{F}'$ satisfies property $P_4$.

It remains to show that $\mathcal{F}$ satisfies property $P^n$. To see this, consider a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \geq n > 4k + 2$. It is easy to see that there are $S_1, S_2, S_3, S_4, S_5 \in \mathcal{F}'$ such that $i(S_1) < i(S_2) = i(S_3) = i(S_4) < i(S_5)$, and $j(S_2) < j(S_3) < j(S_4)$. Then, by the above observation, $S_3$ is not a vertex of $\mathcal{F}'$, so the members of $\mathcal{F}'$ are not in convex position. This completes the proof of Theorem 2. □

4 Proof of Theorem 3

Lemma 4.1. Let $\mathcal{F}$ be a family of disjoint convex sets in the plane, satisfying property $P_5$ and at least one of the following two conditions:

(i) any two members of $\mathcal{F}$ can be separated by a vertical line;

(ii) there is a line intersecting all members of $\mathcal{F}$.

Then $\mathcal{F}$ is in convex position.

Proof. Case (ii) was settled by Bisztriczky and Fejes Tóth [BF2]. So we have to prove the assertion only in case (i).

Let $A_1, A_2, \ldots, A_t$ denote the members of $\mathcal{F}$ listed from left to the right. Clearly, $A_1$ and $A_t$ are vertices of $\mathcal{F}$, so we can choose two points, $x \in A_1$, $y \in A_t$, that belong to the boundary of $\text{conv}\mathcal{F}$. Let $a(xy)$ (and $a(yx)$) denote the counter-clockwise oriented arcs from $x$ to $y$ (from $y$ to $x$, respectively).

Suppose that $A_j$ is not a vertex of $\text{conv}\mathcal{F}$ for some $1 < j < t$. Let

$$\alpha = \max \{i \mid i < j, A_i \text{ meets } a(xy)\},$$

$$\beta = \min \{i \mid i > j, A_i \text{ meets } a(xy)\},$$

$$\gamma = \max \{i \mid i < j, A_i \text{ meets } a(yx)\},$$

$$\delta = \min \{i \mid i > j, A_i \text{ meets } a(yx)\}.$$

(Since $A_1$ and $A_t$ meet both $a(xy)$ and $a(yx)$, these numbers are well defined.) Notice that $\text{conv}(A_\alpha \cup A_\beta \cup A_\gamma \cup A_\delta) \supset A_j$, contradicting property $P_5$. □

Lemma 4.2. Let $\mathcal{F}$ be a family of disjoint convex sets in the plane, satisfying property $P^n$. Suppose that there are $m$ vertical lines such that every member of $\mathcal{F}$ intersects at least one of them.
Then one can choose at most \( \lceil m/2 \rceil \) vertical lines so that every internal member of \( \mathcal{F} \) intersects at least one of them.

**Proof.** Suppose that every member of \( \mathcal{F} \) intersects at least one of the vertical lines \( \ell_1, \ell_2, \ldots, \ell_m \), ordered from left to right. For any \( i \), let \( \mathcal{F}_i, \mathcal{F}_{<i}, \) and \( \mathcal{F}_{>i} \) denote the families of all members of \( \mathcal{F} \) intersecting \( \ell_i \), lying in the open half-plane to the left of \( \ell_i \), and in the open half-plane to the right of \( \ell_i \), respectively.

It is sufficient to show that every internal member of \( \mathcal{F} \) intersects at least two distinct lines \( \ell_i \), and then it follows that \( \ell_2, \ell_4, \ldots, \ell_{2\lceil m/2 \rceil} \) meet the requirements of the lemma.

Suppose, for contradiction, that there is an internal member \( A \in \mathcal{F} \) which intersects only one line \( \ell_i \), and assume that \( 1 < i < m \). (The cases when \( i = 1 \) or \( m \) are similar, but somewhat simpler.)

Let \( X \) and \( Y \) be two vertex-arcs on the boundary of \( \text{conv} \mathcal{F} \) such that there is a point \( x \in X \) in the closed half-plane to the left of \( \ell_1 \), and there is a point \( y \in Y \) in the closed half-plane to the right of \( \ell_m \). Let \( a(xy) \) and \( a(yx) \) denote the counter-clockwise oriented arcs of the boundary of \( \text{conv} \mathcal{F} \) from \( x \) to \( y \), and from \( y \) to \( x \), respectively.

Let \( V_1 \) (and \( V_4 \)) denote the last (resp. first) vertex-arc along \( a(xy) \), which belongs to a member of \( \mathcal{F}_{<i} \) (of \( \mathcal{F}_{>i} \), respectively). If there is no such vertex-arc, let \( V_1 = X \) (resp. \( V_4 = Y \)). Clearly, if there is any vertex-arc on \( a(x,y) \) between \( V_1 \) and \( V_4 \), it must belong to an element of \( \mathcal{F}_i \). Let \( V_2 \) (resp. \( V_3 \)) denote the vertex-arc succeeding \( V_1 \) (resp. preceding \( V_4 \)) along \( a(xy) \). Similarly, define the vertex-arcs \( U_1, U_2, U_3, U_4 \) along the oriented arc \( a(yx) \).

Let \( A_1, A_2, \ldots, A_s \) denote the members of \( \mathcal{F}_i \) listed from top to bottom, in order of their intersections with \( \ell_i \). (\( A \) appears in this list, i.e., \( A = A_r \), for some \( 1 \leq r \leq s \).) By Lemma 5.1(ii), \( \mathcal{F}_i \) is in convex position. Let \( x' \in \text{bd} A_1 \) and \( y' \in \text{bd} A_s \) be two boundary points of \( \text{conv} \mathcal{F}_i \). Let \( a(x'y') \) (and \( a(y'x') \)) denote the oriented arcs connecting \( x' \) to \( y' \) (resp. \( y' \) to \( x' \)) along \( \text{bd} \text{conv} \mathcal{F}_i \). Assume without loss of generality that \( A \) has a boundary point on \( a(y'x') \).

We distinguish two cases.

If \( A \) has a boundary point on \( a(x'y') \), then let us define \( \mathcal{G} \) as the collection of those members (vertices) of \( \mathcal{F} \) which correspond to the vertex-arcs \( V_1, \ldots, V_s, U_1, \ldots, U_4 \).

If \( A \) does not have a boundary point on \( a(x'y') \), then let

\[
\alpha = \max \{ i \mid i < r, A_i \text{ has a point on } a(x'y') \}, \\
\beta = \min \{ i \mid i > r, A_i \text{ has a point on } a(x'y') \}.
\]

Since both \( x' \in A_1 \) and \( y' \in A_s \) belong to \( a(x'y') \), \( \alpha \) and \( \beta \) are well defined. Now let \( \mathcal{G} \) consist of \( A_\alpha, A_\beta \), and the members of \( \mathcal{F} \), corresponding to \( V_1, \ldots, V_s, U_1, \ldots, U_4 \).

In both cases, \( \mathcal{G} \) has at most 10 members. It is easy to check that none of the edge-arcs of \( \text{conv} \mathcal{G} \) can be met by \( A \). Since \( A \cap \ell_i \not\subseteq \text{conv} \mathcal{G} \), we obtain that \( A \) must be contained in the convex hull of \( \mathcal{G} \), contradicting property \( P_{11} \) (see Fig. 5). \( \square \)
Now we can prove Theorem 3. Let $\mathcal{F}$ be a family of disjoint convex sets in the plane satisfying property $P_n^m$. In view of Lemma 4.1(i), no $n$ members of $\mathcal{F}$ can be separated from each other by vertical lines. Thus, according to a well-known result of T. Gallai (cited before), one can find $n - 1$ vertical lines such that every member of $\mathcal{F}$ intersects at least one of them.

Let $\mathcal{F}_1$ denote the family of all internal members of $\mathcal{F}$. Clearly, $|\mathcal{F}_1| > |\mathcal{F}| - n$. By Lemma 4.2, all members of $\mathcal{F}_1$ can be pierced by $\lceil \frac{n-1}{2} \rceil < n/2$ vertical lines. Similarly, the family $\mathcal{F}_2$ of all internal members of $\mathcal{F}_1$ has more than $|\mathcal{F}| - 2n$ members, and all of them can be intersected by fewer than $n/4$ vertical lines. Applying Lemma 4.2 repeatedly, after at most $\lceil \log_2 n \rceil$ steps, we end up with a subfamily of $\mathcal{F}$, which has more than $|\mathcal{F}| - n \log_2 n$ members, and they all intersect the same line. By Lemma 4.1(ii), this implies that

$$|\mathcal{F}| - n \log_2 n < n,$$

concluding the proof of Theorem 3.

![Fig. 5]
References


