Comment on Fox News

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Abstract

Does there exist a constant $c > 0$ such that any family of $n$ continuous arcs in the plane, any pair of which intersect at most once, has two disjoint subfamilies $A$ and $B$ with $|A|, |B| \geq cn$ with the property that either every element of $A$ intersects all elements of $B$ or no element of $A$ intersects any element of $B$? Based on a recent result of Fox, we show that the answer is no if we drop the condition that two arcs can cross at most once.

1 Introduction

It was shown in [4] that any family of $n$ segments in the plane has two disjoint subfamilies $A$ and $B$, each of size at least constant times $n$, such that either every element of $A$ intersects all elements of $B$ or no element of $A$ intersects any element of $B$. In [1], this result was extended to families of algebraic curves with bounded degree at most $D$, where the corresponding constant depends on $D$.

More generally, let $G$ be the intersection graph of $n$ $d$-dimensional semialgebraic sets of degree at most $D$. Then there exist two disjoint subsets $A, B \subset V(G)$ such that $|A|, |B| \geq c(d, D)n$ and one of the following two conditions is satisfied:

1. $ab \in E(G)$ for all $a \in A, b \in B$,

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2. \( ab \notin E(G) \) for all \( a \in A, b \in B \).

Here \( c(d, D) \) is a positive constant depending only on \( d \) and \( D \).

It is not completely clear whether the assumption that the sets
are semialgebraic can be weakened. For example, a similar result
may hold for intersection graphs of plane convex sets. Clearly, the
same theorem is false for intersection graphs of three-dimensional
convex bodies, because any finite graph can be represented in such
a way, and a random graph \( G \) with \( n \) vertices almost surely does not
have \( A, B \subset V(G) \) satisfying conditions 1 or 2 with \( |A|, |B| \geq c \log n \),
if \( c \) is large enough.

It would be interesting to analyze intersection graphs of continuous
arcs in the plane. (These are often called “string graphs” in the
literature [2].) We have been unable to answer the following question
even for \( k = 1 \), that is, for pseudo-segments.

**Problem 1.1.** Is it true that any family of \( n \) continuous arcs in
the plane, any pair of which intersect at most \( k \) times, has two
disjoint subfamilies \( A \) and \( B \) with \( |A|, |B| \geq c_k n \) such that either
every element of \( A \) intersects all elements of \( B \) or no element of \( A \)
intersects any element of \( B \)? (Here \( c_k > 0 \) is a suitable constant.)

It follows from a beautiful recent result of Jacob Fox [3] (see
Theorem 2.2 below) that the answer to the above question is negative
if we drop the condition on pairwise intersections.

**Proposition 1.2.** Fix \( \varepsilon \in (0, 1) \). For every \( n \), there is a family of
\( n \) continuous real functions defined on \([0, 1]\) such that their inter-
section graph \( G \) has no complete bipartite subgraph with at least
\( c(\varepsilon) \frac{n}{\log n} \) vertices in each of its vertex classes, and every vertex of \( G \)
is connected to all but at most \( n^\varepsilon \) other vertices.

Obviously, the last condition implies that \( G \) has no two disjoint
nonempty sets of vertices \( A \) and \( B \) with \( |A \cup B| > n^\varepsilon \) such that no
vertex in \( A \) is connected to any element of \( B \) by an edge.

## 2 Proof of Proposition 1.2

We need a simple representation lemma.
Lemma 2.1. The elements of every finite partially ordered set 
\( \{p_1, p_2, \ldots, p_r\}, < \) can be represented by continuous real functions \( f_1, f_2, \ldots \)
defined on the interval \([0, 1]\) such that \( f_i(x) < f_j(x) \) for every \( x \) if and only if \( p_i < p_j \) (\( i \neq j \)).

Moreover, we can assume that the graphs of any pair of functions \( f_i \) and \( f_j \) are either disjoint or have finitely many points in common, at which they properly cross.

Proof. Let \( P = \{p_1, p_2, \ldots, p_r\} \). We describe a recursive construction with the additional property that for any extension of \((P, <)\) to a total order \( p_{k(1)} < p_{k(2)} < \cdots < p_{k(\ell)} \), there exists \( x \in [0, 1] \) such that \( f_{k(1)}(x) < f_{k(2)}(x) < \cdots < f_{k(\ell)}(x) \).

The proof is by induction on the number of elements of \( P \). For \( \ell = 1 \), there is nothing to prove. For \( \ell = 2 \), there are two possibilities. If \( p_1 < p_2 \) then the functions \( f_1 \equiv 1 \), \( f_2 \equiv 2 \) meet the requirements. If \( p_1 \) and \( p_2 \) are incomparable, then let \( f_1(x) = x \), \( f_2(x) = 1 - x \). Now \((P, <)\) can be extended to a total order in two different ways. Accordingly, \( f_1(x) < f_2(x) \) at \( x = 0 \) and \( f_2(x) < f_1(x) \) at \( x = 1 \).

Let \( \ell \geq 3 \), and suppose without loss of generality that \( p_r \) is a minimal element of \( P \). Assume recursively that we have already constructed continuous real functions \( f_1, f_2, \ldots, f_{\ell - 1} \) with the required properties representing the elements of the partially ordered set \((P \setminus \{p_r\}, <)\). Consider now an extension of \((P, <)\) to a total order \( p_{k(1)} < p_{k(2)} < \cdots < p_{k(\ell)} \). Clearly, \( p_r \) appears in this sequence, i.e., \( \ell = k(m) \) for some \( 1 \leq m \leq \ell \). By our assumption, there exists \( x \in [0, 1] \) such that

\[
  f_{k(1)}(x) < \cdots < f_{k(m - 1)}(x) < f_{k(m + 1)}(x) < \cdots < f_{k(\ell)}.
\]

In fact, there exists a whole interval \( I \subset [0, 1] \) such that the above inequalities hold for all \( x \in I \). Now pick a point \( x^* \in I \) and a number \( y^* \) such that \( f_{k(m - 1)}(x^*) < y^* < f_{k(m + 1)}(x^*) \), and define

\[
  f_\ell(x^*) := y^*.
\]

Repeating this procedure for every permutation \((k(1), k(2), \ldots, k(\ell))\) for which \( p_{k(1)} < p_{k(2)} < \cdots < p_{k(\ell)} \) is an extension of \((P, <)\) to a total order, we define the function \( f_\ell \) at finitely many points. (To avoid inconsistencies, we can make sure that we pick a different point \( x^* \) for each permutation.)
It remains to verify that this partially defined function can be extended to a continuous function $f_{\ell} : [0, 1] \to \mathbb{R}$ meeting the requirements. The following two conditions must be satisfied:

1. if $p_\ell < p_j$ in $(P, <)$ for some $j \neq \ell$, then $f_\ell(x) < f_j(x)$ for all $x \in [0, 1]$;

2. if $p_\ell$ and $p_j$ are incomparable in $(P, <)$ for some $j \neq \ell$, then the graphs of $f_\ell$ and $f_j$ cross each other.

Notice that each point $(x^*, y^*)$ constructed during the above procedure lies below the lower envelope (pointwise minimum) of the functions $f_j(x)$ over all $j$ for which $p_j > p_\ell$ in $(P, <)$. Pick a point $x_0 \in [0, 1]$ distinct from all previously selected points $x^* \in [0, 1]$, and let $f_\ell(x_0) := y_0$ for some

$$y_0 < \min_{1 \leq j < \ell} f_j(x_0).$$

Extend $f_\ell$ to a continuous function on $[0, 1]$ whose graph lies strictly below

$$\min\{f_j(x) : \text{for all } j \text{ such that } p_j > p_\ell\}.$$ 

Obviously, $f_\ell$ satisfies condition 1. To see that condition 2 is also satisfied, fix an index $j$ such that $p_\ell$ and $p_j$ are incomparable in $(P, <)$. Consider an extension of $(P, <)$ to a total order in which $p_j < p_\ell$. It follows from our construction that there exists a point $x \in [0, 1]$ at which the values $f_\ell(x)$ are in the same total order as the elements $p_i$ ($1 \leq i \leq \ell$). In particular, we have $f_j(x) < f_\ell(x)$. On the other hand, by definition, $f_\ell(x_0) = y_0 < f_j(x_0)$. Therefore, the graphs of $f_\ell$ and $f_j$ must cross each other, completing the proof. □

**Theorem 2.2.** (Fox) Fix $\varepsilon \in (0, 1)$. For every $n$, there is a partially ordered set $(P, <)$ of size $n$ with the following two properties. (i) There are no two disjoint subsets $A, B \subset P$ such that $|A|, |B| \geq c(\varepsilon) \frac{n}{\log n}$ and no element of $A$ is comparable to any element of $B$. (ii) Every element of $P$ is comparable to at most $n^\varepsilon$ other elements. □

To deduce Proposition 1.2, apply Lemma 2.1 to the partially ordered set whose existence is guaranteed by Theorem 2.2. To see that the intersection graph $G$ of the resulting functions meets the requirements, it is enough to notice that two vertices of $G$ are connected by an edge if and only if the corresponding elements of $P$ are incomparable.
References


