Applications of the Crossing Number

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Abstract

Let $G$ be a graph of $n$ vertices that can be drawn in the plane by straight-line segments so that no $k + 1$ of them are pairwise crossing. We show that $G$ has at most $c_k n \log^{2k-2} n$ edges. This gives a partial answer to a dual version of a well-known problem of Avital–Hanani, Erdős, Kupitz, Perles, and others. We also construct two point sets $\{p_1, \ldots, p_n\}, \{q_1, \ldots, q_n\}$ in the plane such that any piecewise linear one-to-one mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(p_i) = q_i$ ($1 \leq i \leq n$) is composed of at least $\Omega(n^2)$ linear pieces. It follows from a recent result of Souvaine and Wenger that this bound is asymptotically tight. Both proofs are based on a relation between the crossing number and the bisection width of a graph.

Keywords: Crossing number, geometric graph, bisection width, triangulation.

*Supported by NSF grant CCR-91-22103, PSC-CUNY Research Award 663472 and OTKA-4269. The extended abstract of this paper was presented at the 10th Annual ACM Symposium on Computational Geometry, Stony Brook, NY, 1994.
1 Introduction

A geometric graph is a graph drawn in the plane by (possibly crossing) straight-line segments i.e., it is defined as a pair of \((V(G), E(G))\), where \(V(G)\) is a set of points in the plane in general position and \(E(G)\) is a set of closed segments whose endpoints belong to \(V(G)\).

The following question was raised by Avital and Hanani [AH], Erdős, Kupitz [K] and Perles: What is the maximum number of edges that a geometric graph of \(n\) vertices can have without containing \(k+1\) pairwise disjoint edges? It was proved in [PT] that for any fixed \(k\) the answer is linear in \(n\). (The cases when \(k \leq 3\) had been settled earlier by Hopf and Pannwitz [HF], Erdős [E], Alon and Erdős [AE], O’Donnel and Perles [OP], and Goddard, Katchalski and Kleitman [GKK].)

In this paper we shall discuss the dual counterpart of the above problem. We say that two edges of \(G\) cross each other if they have an interior point in common. Let \(e_k(n)\) denote the maximum number of edges that a geometric graph of \(n\) vertices can have without containing \(k+1\) pairwise crossing edges. If \(G\) has no two crossing edges, then it is a planar graph. Thus, it follows from Euler’s polyhedral formula that

\[
e_1(n) = 3n - 6 \quad \text{for all } n \geq 3.
\]

It was shown in [P] that \(e_2(n) < 13n^{3/2}\) and that, for any fixed \(k\),

\[
e_k(n) = O(n^{2-1/25(k+1)^2}).
\]

However, we suspect that \(e_k(n) = O(n)\) holds for every fixed \(k\) as \(n\) tends to infinity. We know that the corresponding statement is true if we restrict our attention to convex geometric graphs, i.e., to geometric graphs whose vertices are in convex position [CP]. Our next theorem brings us fairly close to this bound for arbitrary geometric graphs.

**Theorem 1.1** Let \(G\) be a geometric graph of \(n\) vertices, containing no \(k+1\) pairwise crossing edges. Then the number of edges of \(G\) satisfies

\[
|E(G)| \leq c_k n \log^{2k-2} n,
\]

with a suitable constant \(c_k\) depending only on \(k\).
The proof is based on a general result relating the crossing number of a graph to its bisection width (see Theorem 2.1). A nice feature of our approach is that we do not use the assumption that the edges of $G$ are line segments. Theorem 1.1 remains valid for graphs whose edges are represented by arbitrary Jordan arcs such that any two arcs meet at most once (or at most a bounded number of times).

The same ideas can be used to settle the following problem. Let $T_1$ and $T_2$ be triangles in the plane, and let $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_n\}$ be two $n$-element point sets lying in the interior of $T_1$ and $T_2$, respectively. A homeomorphism $f$ from $T_1$ onto $T_2$ is a continuous one-to-one mapping with continuous inverse. $f$ is called piecewise linear if there exists a triangulation of $T_1$ such that $f$ is linear on each of its triangles. The size of $f$ is defined as the minimum number of triangles in such a triangulation. Recently, Souvaine and Wenger [SW] have shown that one can always find a piecewise linear homeomorphism $f : T_1 \to T_2$ with $f(p_i) = q_i$ ($1 \leq i \leq n$) such that the size of $f$ is $O(n^2)$. Our next result shows that this bound cannot be improved.

**Theorem 1.2** There exist a triangle $T$ and two point sets $\{p_1, \ldots, p_n\}$, $\{q_1, \ldots, q_n\} \subseteq \text{int } T$ such that the size of any piecewise linear homeomorphism $f : T \to T$ which maps $p_i$ to $q_i$ ($1 \leq i \leq n$) is at least $cn^2$ (for a suitable constant $c > 0$).

For some closely related problems consult [S] and [ASS].

## 2 Crossing number and bisection width

Let $G$ be a graph of $n$ vertices with no loops and no multiple edges. For any partition of the vertex set $V(G)$ into two disjoint parts $V_1$ and $V_2$, let $E(V_1, V_2)$ denote the set of edges with one endpoint in $V_1$ and the other endpoint in $V_2$. Define the bisection width of $G$ as

$$b(G) = \min_{|V_1| \geq n/3} |E(V_1, V_2)|,$$

where the minimum is taken over all partitions $V(G) = V_1 \cup V_2$ with $|V_1|$, $|V_2| \geq n/3$.

Consider now a drawing of $G$ in the plane, where the vertices are represented by distinct points and the edges are represented by Jordan arcs.
connecting them such that (1) no arc passes through a vertex different from its endpoints and (2) no three arcs have an interior point in common. The crossing number \( c(G) \) of \( G \) is defined as the minimum number of crossings in a drawing of \( G \) satisfying the above conditions, where a crossing is a common interior point of two arcs. It is easy to show that the minimum number of crossings can always be realized by a drawing such that

(3) no two arcs meet in more than one point (including their endpoints).

We need the following result which is an easy consequence of a weighted version of the Lipton-Tarjan separator theorem for planar graphs [LT].

**Theorem 2.1** Let \( G \) be a graph with \( n \) vertices of degree \( d_1, \ldots, d_n \). Then

\[
b^2(G) \leq (1.58)^2 \left( 16c(G) + \sum_{i=1}^{n} d_i^2 \right),
\]

where \( b(G) \) and \( c(G) \) denote the bisection width and the crossing number of \( G \), respectively.

**Proof:** Let \( H \) be a plane graph on the vertex set \( V(H) = \{v_1, \ldots, v_N\} \) such that each vertex has a non-negative weight \( w(v_i) \) and \( \sum_{i=1}^{N} w(v_i) = 1 \). Let \( d(v_i) \) denote the degree of \( v_i \) in \( H \). It was shown by Gazit and Miller [GM] that, by the removal of at most

\[
1.58 \left( \sum_{i=1}^{N} d^2(v_i) \right)^{1/2}
\]
edges, \( H \) can be separated into two disjoint subgraphs \( H_1 \) and \( H_2 \) such that

\[
\sum_{v_i \in V(H_1)} w(v_i) \geq \frac{1}{3}, \quad \sum_{v_i \in V(H_2)} w(v_i) \geq \frac{1}{3}.
\]

(See also [M] and [DDS].)

Consider now a drawing of \( G \) with \( c(G) \) crossing pairs of arcs satisfying conditions (1)–(3). Introducing a new vertex at each crossing, we obtain a plane graph \( H \) with \( N = n + c(G) \) vertices. Assign weight 0 to each new vertex and weights of \( 1/n \) to all other vertices. The above result implies that, by the deletion of at most

\[
1.58 \left( 16c(G) + \sum_{i=1}^{n} d_i^2 \right)^{1/2}
\]
edges, $H$ can be separated into two parts $H_1$ and $H_2$ such that both of the sets $V_1 = V(H_1) \cap V(G)$ and $V_2 = V(H_2) \cap V(G)$ have at least $n/3$ elements. Hence,

$$b(G) \leq |E(V_1, V_2)| \leq 1.58 \left( 16c(G) + \sum_{i=1}^{n} d_i^2 \right)^{1/2},$$

and the result follows. $\square$

In the special case when every vertex of $G$ is of degree at most 4, Theorem 2.1 was established by Leighton [L] and it proved to be an important tool in VLSI design (see [U]).

## 3 Geometric graphs

The aim of this section is to prove the following generalization of Theorem 1.1 for curvilinear graphs.

**Theorem 3.1** Let $G$ be a graph with $n$ vertices that has a drawing with Jordan arcs such that no arc passes through any vertex other than its endpoints, no two arcs meet in more than one point, and there are no $k+1$ pairwise crossing arcs ($k \geq 1$). Then

$$|E(G)| \leq 3n(10\log_2 n)^{2k-2}.$$ 

**Proof:** By double induction on $k$ and $n$. The assertion is true for $k = 1$ and for all $n$. It is also true for any $k > 1$ and $n \leq 6 \cdot 10^{2k-2}$, because for these values the above upper bound exceeds $\binom{n}{2}$.

Assume now that we have already proved the theorem for some $k$ and all $n$, and we want to prove it for $k + 1$. Let $n \geq 6 \cdot 10^{2k}$, and suppose that the theorem holds for $k + 1$ and for all graphs having fewer than $n$ vertices.

Let $G$ be a graph of $n$ vertices that can be drawn in the plane so that no two edges meet more than once and there are no $k+2$ pairwise crossing edges. For the sake of simplicity, this drawing will also be denoted by $G = (V(G), E(G))$. For any arc $e \in E(G)$, let $G_e$ denote the graph consisting of all arcs that cross $e$. Clearly, $G_e$ has no $k + 1$ pairwise crossing arcs. Thus, by the induction hypothesis,
\[ c(G) \leq \frac{1}{2} \sum_{e \in E(G)} |E(G_e)| \]
\[ \leq \frac{1}{2} \sum_{e \in E(G)} 3n(10 \log_2 n)^{2k-2} \]
\[ \leq \frac{3}{2} |E(G)| n(10 \log_2 n)^{2k-2}. \]

Since \( \sum_{i=1}^{n} d_i^2 \leq 2|E(G)|n \) holds for every graph \( G \) with degrees \( d_1, \ldots, d_n \), Theorem 2.1 implies that
\[ b(G) \leq 1.58 \left(16c(G) + \sum_{i=1}^{n} d_i^2\right)^{1/2} \]
\[ \leq 9\sqrt{n|E(G)|}(10 \log_2 n)^{k-1}. \]

Consider a partition of \( V(G) \) into two parts \( V_1 \) and \( V_2 \), each containing at least \( n/3 \) vertices, such that the number of edges connecting them is \( b(G) \). Let \( G_1 \) and \( G_2 \) denote the subgraphs of \( G \) induced by \( V_1 \) and \( V_2 \), respectively. Since neither of \( G_1 \) or \( G_2 \) contains \( k + 2 \) pairwise crossing edges and each of them has fewer than \( n \) vertices, we can apply the induction hypothesis to obtain
\[ |E(G)| = |E(G_1)| + |E(G_2)| + b(G) \leq 3n_1(10 \log_2 n_1)^{2k} + 3n_2(10 \log_2 n_2)^{2k} + b(G), \]
where \( n_i = |V_i| \ (i = 1, 2) \). Combining the last two inequalities we get
\[ |E(G)| - 9\sqrt{n}(10 \log_2 n)^{k-1} \sqrt{|E(G)|} \leq 3n_1(10 \log_2 n_1)^{2k} + 3n_2(10 \log_2 n_2)^{2k} \]
\[ \leq 3n(10 \log_2 n)^{2k}(1 - \frac{k}{\log_2 n}). \]

If the left hand side of this inequality is negative, then \( |E(G)| \leq 3n(10 \log_2 n)^{2k} \) and we are done. Otherwise,
\[ f(x) = x - 9\sqrt{n}(10 \log_2 n)^{k-1} \sqrt{x} \]
is a monotone increasing function of $x$ when $x \geq |E(G)|$. An easy calculation shows that

\[ f(3n(10 \log_2 n)^{2k}) > 3n(10 \log_2 n)^{2k}(1 - \frac{k}{\log_2 n}). \]

Hence,

\[ f(|E(G)|) < f(3n(10 \log_2 n)^{2k}), \]

which in turn implies that

\[ |E(G)| < 3n(10 \log_2 n)^{2k}, \]

as required. □

4 Avoiding snakes

In [ASS], Aronov, Seidel and Souvaine constructed two polygonal regions $P$ and $Q$ with vertices $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_n\}$ in clockwise order such that the size of any piecewise linear homeomorphism $f : P \rightarrow Q$ with $f(p_i) = q_i$ ($1 \leq i \leq n$) is at least $cn^2$ (for an absolute constant $c > 0$). Their ingenious construction heavily relies on some special geometric features of “snakelike” polygons.

Our Theorem 1.2 (stated in the introduction) provides the same lower bound for a modified version of this problem due to J.E. Goodman. The proof given below is purely combinatorial, and avoids the use of “snakes.”

**Proof of Theorem 1.2:** Let $T_1$ and $T_2$ be two triangles containing two convex $n$-gons $P$ and $Q$ in their interiors, respectively. Let $p_{\pi(1)}, \ldots, p_{\pi(n)}$ denote the vertices of $P$ in clockwise order, where $\pi$ is a permutation of $\{1, \ldots, n\}$ to be specified later. Furthermore, let $q_1, \ldots, q_n$ denote the vertices of $Q$ in clockwise order. Let $f : T_1 \rightarrow T_2$ be a piecewise linear homeomorphism with $f(p_i) = q_i$ ($1 \leq i \leq n$), and fix a triangulation $T_1'$ of $T_1$ with $|T_1'| = \text{size}(f)$ triangles such that $f$ is linear on each of them. By subdividing some members of $T_1'$ if necessary, we obtain a new triangulation $T_1''$ of $T_1$ such that each $p_i$ is a vertex of $T_1''$ and $|T_1''| \leq |T_1'| + 3n$.

Obviously, $f$ will map $T_1''$ into an isomorphic triangulation $T_2''$ of $T_2$. The image of each segment $p_{\pi(i)}p_{\pi(i+1)}$ is a polygonal path connecting $q_{\pi(i)}$ and $q_{\pi(i+1)}$ ($1 \leq i \leq n$). The collection of these paths together with the segments $q_1q_2$ is a drawing of the graph $G = G_\pi$ defined by:
\[ V(G) = \{ q_1, \ldots, q_n \}; \]

\[(*) \quad E(G) = \{ q_i q_{i+1} \mid 1 \leq i \leq n \} \cup \{ q_{\pi(i)} q_{\pi(i+1)} \mid 1 \leq i \leq n \}. \]

Suppose that this drawing has \( c \) crossing pairs of arcs. Notice that each crossing must occur between a path \( q_{\pi(i)} q_{\pi(i+1)} \) and a segment \( q_j q_{j+1} \). By the convexity of \( Q \), any line can intersect at most two segments \( q_j q_{j+1} \). Hence the total number of subsegments of the concatenation of the polygons \( f(p_{\pi(i)} p_{\pi(i+1)}) \), \( 1 \leq i \leq n \), is at least \( c/2 \). On the other hand, by the convexity of \( P \), each triangle belonging to \( T'_1 \) intersects at most two sides of the form \( p_{\pi(i)} p_{\pi(i+1)} \). Thus, \( |T'_1| \geq c/4 \), which yields that

\[ \text{size}(f) = |T_1| \geq |T'_1| - 3n \geq \frac{c(G)}{4} - 3n, \]

where \( c(G) \) stands for the crossing number of \( G \). Applying Theorem 2.1, we obtain that

\[ c(G) > \frac{b^2(G)}{40} - 1. \]

Therefore,

\[ \text{size}(f) \geq \frac{b^2(G)}{160} - 3n - \frac{1}{4}. \]

To complete the proof of Theorem 1.2, it is sufficient to show that for a suitable permutation \( \pi \) the bisection width of the graph \( G = G_\pi \) defined by \( (*) \) is at least constant times \( n \). We use a counting argument (cf. \([AS]\)). The family of graphs \( G_\pi \) has size \( n! \). We bound from above the number of those members of this family whose bisection width is at most \( k \). We will see that for \( k \leq n/20 \) this number is less than \( n! \).

Let \( b(G_\pi) \leq k \). Let \( (V_1, V_2) \) be a partition of \( V(G_\pi) \) with \( |V_1|, |V_2| \geq n/3 \) and \( E(V_1, V_2) \leq k \). Define

\[ E_1(V_1, V_2) = \{ q_i q_{i+1} \mid 1 \leq i < n \} \cap E(V_1, V_2), \]
\[ E_2(V_1, V_2) = \{ q_{\pi(i)} q_{\pi(i+1)} \mid 1 \leq i < n \} \cap E(V_1, V_2). \]

Since \( |E_1(V_1, V_2)| \leq k \), the partition \( (V_1, V_2) \) should be of a special form. If we delete all elements of \( E_1(V_1, V_2) \) from the path \( q_1 \ldots q_n \), it splits into at
most $k + 1$ paths (or points) lying alternately in $V_1$ and in $V_2$. This yields a $2(k + 1)\binom{n}{k}$ upper bound on the number of partitions in question.

The order in which the elements of $V_i$ ($i = 1, 2$) occur in the sequence $q_{\pi(1)} \ldots q_{\pi(n)}$ can be represented by a function $\sigma_i : \{1, \ldots, |V_i|\} \to V_i$ ($i = 1, 2$). For a fixed partition $(V_1, V_2)$, there are at most $|V_1|!$ choices for $\sigma_1$ and $|V_2|!$ choices for $\sigma_2$. If $\sigma_1$ and $\sigma_2$ are also fixed, then the number of possible permutations is bounded again by $2(k + 1)\binom{n}{k}$. Thus the total number of permutations $\pi$ for which $b(G_\pi) \leq k$ cannot exceed

$$\sum_{(V_1, V_2)} |V_1|!|V_2|!2(k + 1)\binom{n}{k} \leq \sum_{(V_1, V_2)} n!\left(\frac{n}{n/3}\right)^{-1} 2(k + 1)\binom{n}{k} \leq 4(k + 1)^2 \binom{n}{k}^2 \left(\frac{n}{n/3}\right)^{-1} n!,$$

which is less than $n!$ if $k \leq n/20$, and $n$ is sufficiently large. □

References


