MIDPOINTS OF SEGMENTS
INDUCED BY A POINT SET

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Abstract
Applying some well known results in additive number theory, we partially answer two geometric questions due to V. Bálint et al. and F. Hurtado. (1) Let $m(n)$ be the largest integer $m$ with the property that from every set of $n$ points in the plane one can select $m$ elements so that none of them is the midpoint of two others. It is shown that $n^{1-c/\sqrt{\log n}} \leq m(n) \leq n/\log'^\alpha n$. (2) Let $\mu(n)$ be the smallest number of distinct midpoints of all segments induced by $n$ points in the plane, no 3 of which are collinear. It is proved that $\lim_{n \to \infty} \mu(n)/n = \infty$ and that $\mu(n) \leq nc'' \sqrt{\log n}$. Here $c, c', c''$ denote suitable positive constants.

1 Introduction

Many extremal problems in discrete geometry lead to questions in additive number theory [12]. This is partly due to the fact that their solutions are known or conjectured to be lattice-like, i.e., affinely equivalent to the integer lattice. Here we present two planar examples.

Bálint et al. [1] (see also [10], p. 27.) investigated the following question. A set of points in the plane is said to be midpoint-free if it has no pair of elements whose midpoint also belongs to the set. Let $m(n)$

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denote the largest number \( m \) such that every set of \( n \) points in the plane has a midpoint-free subset of size \( m \). It was proved in \([1]\) that
\[
\left\lfloor \frac{-1 + \sqrt{8n+1}}{2} \right\rfloor \leq m(n),
\]
and it was conjectured that the order of magnitude of this bound cannot be improved, i.e., we have \( m(n) = O(\sqrt{n}) \). However, it follows from the existence of relatively dense sets of integers containing no 3-term arithmetic progression that this conjecture is wrong.

**Theorem 1.** There are positive constants \( c, c' \) such that
\[
n^{1-c/\sqrt{\log n}} \leq m(n) \leq n/\log^c n.
\]

F. Hurtado raised the following problem. For any point set \( P \), let \( M(P) \) denote the set of midpoints of all the \( \binom{n}{2} \) segments spanned by point pairs in \( P \). Determine \( \mu(n) = \min_{|P|=n} |M(P)| \), where the minimum is taken over all sets of \( n \) points in the plane, no 3 of which are collinear.

Hurtado and Urrutia showed that \( \mu(n) = O(n^{\log_2 3}) \approx O(n^{1.585}) \), but no superlinear lower bound was known. Using an idea of Behrend and Freiman’s theory of set addition, we prove

**Theorem 2.** There is a positive constant \( c \) such that
\[
\mu(n) \leq ne^{c\sqrt{\log n}}.
\]
Furthermore, we have \( \lim_{n \to \infty} \mu(n)/n = \infty \).

In the next two sections, we establish Theorems 1 and 2, resp., while in the last section some related questions are discussed.

## 2 Proof of Theorem 1

Consider a set \( P \) of \( n \) points in the plane with no midpoint-free subset of size larger than \( m(n) \). First, choose (e.g., randomly) a straight line \( \ell \) so that the orthogonal projection \( \phi : P \to \ell \) takes \( P \) into an \( n \)-element set \( P' \) satisfying the following condition: for any \( p_i, p_j, p_k \in P' \), the midpoint of the segment \( p_i p_k \) is \( p_j \) if and only if \( \phi(p_i), \phi(p_j), \) and \( \phi(p_k) \) (in this
order) form an arithmetic progression of length 3. Using simultaneous
approximation [8], for any positive integer \( q \), we can replace each point
\( \phi(p_i) \) by a rational number \( r_i/q \), such that \( r_i = r_i(q) \) is an integer and

\[
|\phi(p_i) - \frac{r_i}{q}| \leq \frac{1}{q^{1+1/n}}
\]

holds for all \( 1 \leq i \leq n \).

There exists a sufficiently large \( q \) satisfying the following condition:
each triple \( (\phi(p_i), \phi(p_j), \phi(p_k)) \) forms an arithmetic progression (in this
order) if and only if \( (r_i, r_j, r_k) \) does. Indeed, we have

\[
|\phi(p_i) + \phi(p_k) - 2\phi(p_j) - (r_i + r_k - 2r_j)| \leq
\]

\[
|q\phi(p_i) - r_i| + |q\phi(p_k) - r_k| + 2|q\phi(p_j) - r_j| \leq \frac{4}{q^{1/n}}.
\]

Assuming that \( q > 4^n \), if \( \phi(p_i) + \phi(p_k) - 2\phi(p_j) = 0 \) holds for some triple,
we obtain that \( |r_i + r_k - 2r_j| < 1 \) so that \( r_i + r_k - 2r_j = 0 \) must also be true.
In the reverse direction, assume indirectly that \( \phi(p_i) + \phi(p_k) - 2\phi(p_j) \) is
not equal to zero, but \( r_i(q) + r_k(q) - 2r_j(q) = 0 \) holds for infinitely many
values of \( q \). For these values, we have

\[
|\phi(p_i) + \phi(p_k) - 2\phi(p_j)| \leq \frac{4}{q^{1+1/n}},
\]

which leads to a contradiction, as \( q \) tends to infinity.

Thus, we have reduced the problem to the following: determine the
largest positive integer \( m_3'(n) \) such that every set of \( n \) integers has a
subset of size \( m_3'(n) \) which contains no arithmetic progression of length
3.

Let \( m_3(n) \) denote the largest number of elements that can be chosen
from the first \( n \) positive integers without containing a 3-term arithmetic
progression. Clearly, we have \( m_3'(n) \leq m_3(n) \) for every \( n \). It was proved
by Komlós, Sulyok, and Szemerédi [11] in a more general setting that
there exists a constant \( c > 0 \) such that \( m_3'(n) \geq cn_3(n) \). Thus, Theo-
rem 2 immediately follows from well known estimates on \( m_3(n) \), due to
Behrend [2], Heath-Brown [9], and Szemerédi [14].

Note that the same argument can be applied in higher dimensions.
3 Proof of Theorem 2

First we establish the upper bound, by adapting the arguments in [5]. Assume, for the sake of simplicity, that \( n = \lfloor \frac{2^{d(d-2)}}{d} \rfloor \) for some natural number \( d \geq 4 \). Consider the set \( L \) of all lattice points \( (x_1, \ldots, x_d) \in \mathbb{R}^d \) with integer coordinates \( 0 \leq x_i < 2^d \). The number of distinct distances determined by \( L \) is at most \( d(2^d)^2 \), because there are at most that many numbers of the form \( (\sum_{i=1}^d (x_i - x_i')^2)^{1/2} \), where \( 0 \leq x_i, x'_i < 2^d \). In particular, there is a sphere around the origin which contains at least

\[
\frac{|L|}{d(2^d)^2} = \frac{(2^d)^d}{d(2^d)^2} \geq \left[ \frac{2^{d(d-2)}}{d} \right] = n
\]

elements of \( L \). Let \( P \) denote the set of these points.

Let \( M(P) \) denote the set of midpoints of all segments determined by \( P \). Clearly, we have \( |M(P)| = |P + P| \), where \( P + P = \{ p_1 + p_2 \mid p_1, p_2 \in P \} \). Observe that every element of \( P + P \) is a vector \( (x_1, \ldots, x_d) \in \mathbb{R}^d \) with integer coordinates \( 0 \leq x_i < 2^{d+1} \), hence

\[
|P + P| \leq (2^{d+1})^d < n2^8\sqrt{\log n}.
\]

Fix a 2-dimensional plane \( \Pi \) in \( \mathbb{R}^d \), and for any \( p \in P \) let \( p' \) denote the orthogonal projection of \( p \) into \( \Pi \). Evidently, we can choose \( \Pi \) so as to meet the following two conditions: (i) the projections of no two elements of \( P \) coincide, (ii) no 3 elements of \( P' \) are collinear. In view of the fact that \( p_1 + p_2 = p_3 + p_4 \) implies \( |p_1' + p_2'| = |p_3' + p_4'| \), we have that the number of distinct midpoints of all segments determined by \( P' \) satisfies

\[
|M(P')| = |P' + P'| \leq |P + P| < n2^8\sqrt{\log n},
\]

as required. This argument easily extends to the general case when \( n \) can take any positive integer value.

We prove the second part of Theorem 2 by contradiction. Assume that for infinitely many values of \( n \) there are \( n \)-element point sets \( P_n \) with no 3 collinear points in the plane such that the the number of midpoints of all segments spanned by \( P_n \) satisfies \( |M(P_n)| = |P_n + P_n| < Cn \), for an absolute constant \( C \).

We need the following well known result of Freiman [6]: For any integer \( C \), there exists \( C' \) with the property that any \( n \)-element set \( P_n \)
in the plane with \( P_n + P_n \leq Cn \) can be covered by the projection of a lattice of dimension \( C \) and size \( C' n \). That is,

\[ P_n \subseteq \{ v_0 + m_1 v_1 + \cdots + m_C v_C \mid 1 \leq m_i \leq n_i \} , \]

for suitable vectors \( v_i \in \mathbb{R}^2 \) and natural numbers \( n_i \) satisfying \( \prod_{i=1}^{C} n_i \leq C' n \). (See Ruzsa [13] for a simple proof.)

Without loss of generality assume that \( n_1 \geq n^{1/C} \). Obviously, we can fix some values \( \tilde{m}_2, \ldots, \tilde{m}_C \) so that

\[ v_0 + m_1 v_1 + \tilde{m}_2 v_2 + \cdots + \tilde{m}_C v_C \in P_n \]

for at least

\[ \frac{n}{n_2 n_3 \cdots n_C} \geq \frac{n_1}{C'} \geq \frac{n^{1/C}}{C'} \]

different integers \( m_1 \). However, the corresponding points of \( P_n \) are all on a line, contradicting our assumption.

## 4 Related problems

### 4.1. It was noticed by Cockayne and Hedetniemi [3] that the problem of placing queens on the diagonal of an \( n \times n \) chessboard so as to cover all squares is equivalent to the problem of finding a midpoint-free set of integers up to \( n/2 \), i.e., one containing no 3-term arithmetic progression.

### 4.2. Erdős raised the following problem related to Theorem 1. Determine the largest integer \( a(n) \) such that every set of \( n \) points in the plane, no four on a line, has an \( a(n) \)-element subset with no collinear triples. The best known bounds, due to Füredi [7], leave plenty of room for improvement:

\[ \Omega(\sqrt{n \log n}) \leq a(n) \leq o(n) . \]

### 4.3. Erdős, Fishburn, and Füredi [4] studied the following question, strongly related to Theorem 2. Given a set \( P \) of \( n \) points in convex position in the plane, let \( M(P) \) denote the set of midpoints of its \( \binom{n}{2} \) sides and diagonals. How small can the cardinality \( \mu_e(n) \) of \( M \) be for fixed \( n \)? One might guess that the answer is \( (0.5 - o(1)) n^2 \). However, it
was shown in [4] that this minimum is somewhere between $0.40n^2$ and $0.45n^2$. In fact, we have
\[
\left(\frac{n}{2}\right) - \left\lfloor \frac{n(n + 1)(1 - e^{-1/2})}{4} \right\rfloor \leq \mu_e(n) \leq \left(\frac{n}{2}\right) - \left\lfloor \frac{n^2 - 2n + 12}{20} \right\rfloor,
\]
for all $n \geq 3$. The upper bound follows from the fact that the number of multiple midpoints can be as large as $\lfloor (n^2 - 2n + 12)/20 \rfloor$.

Woodall [15] solved a similar problem of R. Hall, concerning the minimum number of midpoints induced by an $n$-element subset of the vertex set of a $d$-dimensional cube ($n \leq 2^d$).

References


