Geometric Graph Theory

Midterm Test

1. Let G be a planar graph with n vertices, whose every face has at least 6 sides. In terms of n, give an upper bound on
(i) the number of edges of G,
(ii) the number of faces of G.

2. Let $n \geq 8$ be divisible by 4. Construct a topological graph with n vertices and $4n - 8$ edges, in which every edge crosses at most one other edge.

3. (i) Prove that C_4 cannot be drawn as a thrackle.
(ii) Is it possible to draw C_4 as a "generalized thrackle," i.e., in such a way that any two adjacent edges cross an *even* number of times (not counting their common endpoint), and any two non-adjacent edges cross an *odd* number of times? (Two edges are called *adjacent* if they share an endpoint. Warning: no two edges are allowed to be tangent to each other!)
(iii) Let G be a graph that can be obtained from a cycle with 8 vertices $v_1, v_2, \ldots, v_8$ (in this order) by adding the edges v_2v_5 and v_1v_6. Can G be drawn as a thrackle?

4. Prove that for every n the crossing number of the complete bipartite graph $K_{n,n}$ with n vertices in each of its classes is at least $\binom{n}{2}^2/9$.

5. Is it true that every planar graph with n vertices has a straight-line drawing with non-crossing edges satisfying the following two conditions: (a) every vertex has nonnegative integer coordinates smaller than n^5 and (b) no two vertices have the same x-coordinate?

Please explain all of your answers! Good luck! - J.P.