4. More on straight-line embeddings

Theorem 4.1. Every outerplanar graph G on n vertices admits a noncrossing straight-line drawing on any set P of n points in general position in the plane as its vertex set.

Proof. We can assume without loss of generality that all interior faces of G are triangles, i.e., $|E(G)| = 2n-3$.

We prove a stronger statement by induction on n: Given any two consecutive vertices v_1, v_2 on the outer face of G and any two consecutive vertices p_1, p_2 of the convex hull of P, there is a drawing in which v_1 is represented by p_1 and v_2 is represented by p_2. For $n=3$, the statement is true. Suppose now that $n>3$ and that the assertion is true for all outerplanar graphs with fewer than n vertices.

Let $v_1, v_2, \dots, v_n$ denote the vertices of G along the outer face, in counterclockwise order. Let v_i be the third vertex of the unique triangle

sitting on the edge $v_1v_2 \in E(G)$. It is not hard to see that there is a point $p \in P \setminus \{p_1, p_2\}$ such that (i) $\Delta p_1 p_2 p$ has no element of P in its interior; (ii) there is a line ℓ through p and an interior point of the segment p_1p_2 that has precisely $i-2$ elements of P strictly on its "right-hand" side (i.e., in the open half-plane bounded by ℓ that contains p_2).

To find such a point, first select $p' \in P$ such that p_2p' has precisely $i-3$ elements of P strictly on its right-hand side. If $\Delta p_1 p_2 p'$ is empty, let $p := p'$; otherwise choose p to be the point in $\Delta p_1 p_2 p'$ for which the angle $p_2 p_1 p$ is minimum.

Now let H_1 and H_2 denote the half-planes on the right-hand side resp. left-hand side of ℓ, and let G_1 and G_2 be the subgraphs of G induced by $\{v_2, v_3, \dots, v_i\}$ resp. $\{v_i, v_{i+1}, \dots, v_n, v_1\}$. By the induction hypothesis G_j has a noncrossing straight-line drawing on the (vertex) set $P \cap H_j$. $(j = 1, 2)$. □

Corollary 4.2. Every tree with n vertices admits a noncrossing straight-line drawing on any set of n points in general position in the plane.

Actually, Ikebe, Perles, Tamura, and Shinnichi proved a stronger statement:

Theorem 4.3. Every rooted tree with n vertices admits a noncrossing straight-line drawing on any set P of n points in the plane in general position such that the root is embedded in an arbitrarily specified element of P.

Problem 4.4. Given two edge disjoint trees on the same vertex set V, do they always admit simultaneous noncrossing straight-line embeddings on some set of $|V|$ points in general position in the plane? (Two edges belonging to different trees may cross.)

Homework: Prove that the answer is no if
(i) we are not allowed to choose the point set; or
(ii) the edges belonging to different trees cannot cross either.

A far-reaching generalization of Fáry's theorem was proved much earlier (!) by Koebe:

Theorem 4.5. The vertices of any planar graph can be represented by disjoint open disks in the plane s.t. two disks touch each other if and only if the corresponding vertices are adjacent.

Homework: Deduce from Koebe's theorem that the vertices of any planar graph can be colored by ≤ 6 colors so that no 2 vertices of the same color are connected by an edge.

Putting a graph on a small grid guarantees that the picture does not get too "crowded". Another parameter is the angular resolution of the graph G, which is the smallest angle between two adjacent edges in the best drawing of G. Clearly, if the maximum degree of the vertices in G is D, the angular resolution cannot exceed $2\pi/D$. As far as I know, this parameter is NP-hard to compute (even for $D=4$), but it is not known to be in NP. (?)

Theorem 4.6. (Formann et al.) Every graph with maximum degree D has a straight-line drawing with resolution $\geq$ constant $\cdot \frac{1}{D^2}$.

Proof. Let G^2 denote the graph obtained from G by adding an edge between any two vertices of G that can be connected by a path of length (at most) two. Clearly, the maximum

degree in G^2 is $\leq D^2$, therefore the chromatic number of G^2 is $\leq D^2+1$. Take a coloring of G^2 with the smallest number of colors, and embed every vertex of G that received color i near the i-th vertex of a regular (D^2+1)-gon. This embedding will meet the requirements, because the smallest angle determined by three distinct vertices of a regular (D^2+1)-gon is $\frac{1}{D^2+1}$. □

With a little extra care one can make sure that all vertices are embedded into distinct points of a polynomial size grid and the resolution remains roughly the same. For planar graphs, the situation is better: a resolution $\geq$ constant. $\frac{1}{D}$ can be attained. However, in such a representation the edges may cross. For noncrossing drawings, based on Koebe's theorem, Malitz and Papakostas proved

Theorem 4.7. Every planar graph with maximum degree D admits a noncrossing straight-line drawing with resolution $\geq \alpha^D$, where $\alpha > 0$ is an absolute constant.

It turned out that in this setting there is no lower bound decreasing less than exponentially fast for this parameter.

The improved resolution for planar graphs is an immediate consequence of the following

Lemma 4.8. Let G be a planar graph with maximum degree D. Then $\chi(G^2) \leq \text{constant} \cdot D$.

Homework: Prove Lemma 4.8.