GLOBAL SOLUTIONS FOR 3D QUADRATIC SCHRÖDINGER EQUATIONS

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Abstract. This is the first of three papers where we present a new method based on the concept of space-time resonance to prove global existence of small solutions to nonlinear dispersive equations. The idea is that time resonances (dynamical systems resonances) correspond to interactions between plane waves; but since for dispersive equations we deal with localized solutions, it is crucial to take also into account the traveling speeds of the different wave packets. Here we show how this idea, and the analytical method that this naturally suggests, leads to a simple proof of global existence and scattering for quadratic nonlinear Schrödinger equations in three dimensions.

1. Introduction

There is a vast body of literature dealing with the global existence of small solutions to nonlinear Schrödinger (NLS) equations. A prototype of these equations is

$$\partial_t u = -i\Delta u + Q(u)$$

where Q is a quadratic function of u and $\bar{u}$. Most of the proofs given rely on weighted L^2 estimates and L^∞ decay of solutions. Although it is natural to consider these types of estimates for dispersive equations, the construction of some of the proofs seems to be ad hoc and tailored to a given equation rather than relying on or developing a general method for dealing with nonlinear equations.

Our aim in this paper is to present a transparent proof of global existence based upon the concept of space-time resonances and normal forms, which we will explain below. Specifically we will provide the details of the proof for the initial value problem[1]

$$\text{(NLS)} \qquad \begin{cases} \partial_t u + i\Delta u = \alpha u^2 \\ u_{|t=2} = u_2 \overset{def}{=} e^{-2i\Delta} u_*, \end{cases}$$

where α is a complex number and where u is a complex-valued function of $(t,x) \in \mathbb{R} \times \mathbb{R}^3$. We also remark afterwards how we can trivially generalize the work to the case of $Q(u) = \alpha u^2 + \beta \bar{u}^2$, or more generally to systems of the form

$$\text{(SNLS)} \qquad \begin{cases} \partial_t u_\ell = -ic_\ell \Delta u_\ell + \sum_{m,n} A_{\ell mn} u_m u_n \\ u_{\ell|t=2} = u_{\ell 2}, \end{cases}$$

where the c_ℓ are nonzero real numbers[2]. Moreover the same method we employ here works for problems in two space dimensions, which is the subject of a second paper in preparation.

Space-time resonance. The concept of space-time resonance is a natural generalization of resonance for ODEs. If one considers a linear dispersive equation on $\mathbb{R}^n$

$$\partial_t u = iL\Big(\frac{1}{i}\nabla\Big)u$$

[1] The choice of initial data to be given at $t = 2$ instead of the standard $t = 0$ is related to the choice of norms and will be explained later.

[2] It is not at all obvious how to generalize existing proofs to deal with systems such as (SNLS).

then the quadratic time resonance can be found by considering plane wave solutions $u = e^{i(L(\xi)t+\xi\cdot x)}$. In this case time resonance for u^2 corresponds to

$$\mathscr{T} = \{(\xi_1, \xi_2); L(\xi_1) + L(\xi_2) = L(\xi_1 + \xi_2)\}.$$

However, time resonances tell only part of the story for dispersive equations when one considers spatially localized solutions. Specifically, if one considers two solutions u_1 and u_2 with data localized in space around the origin and in frequency around ξ_1 and ξ_2, respectively, then the solutions u_1 and u_2 at large time t will be spatially localized around $(-\nabla L(\xi_1)t)$ and $(-\nabla L(\xi_2)t)$. Thus quadratic spatial resonance is defined as the set $(\xi_1, \xi_2) \in \mathscr{S}$ where

$$\mathscr{S} = \{(\xi_1, \xi_2); \nabla L(\xi_1) = \nabla L(\xi_2)\}.$$

Thus we define quadratic space-time resonance as

$$\mathscr{R} = \mathscr{T} \cap \mathscr{S}.$$

Spatial localization can be measured by computing weighted norms. Thus if the initial data is spatially localized around the origin, i.e., $\|x^k u(0,\cdot)\|_{L^2}$ is bounded, then the localization of the solution u can be measured by computing $\|\partial_\xi^k e^{-iL(\cdot)t}\hat{u}(t,\cdot)\|_{L^2}$, where $\hat{}$ is the Fourier transform in space. This leads to the observation that the absence of space-time resonance can be captured using Duhamel's formula for the profile $\hat{f}(t,\xi) = e^{-iL(\xi)t}\hat{u}(t,\xi)$ and integrating by parts either in time or in frequency. These ideas which we have been developing for the past two years will be illustrated in detail for various dispersive equations in a forthcoming review article.

Known results. Global existence of small solutions to nonlinear Schrödinger equations have been studied extensively. This question was first studied using energy estimates, and dispersion properties of the Schrödinger semi-group, e.g., see Strauss [14], Klainerman and Ponce [8], Shatah [12]. This approach works for quadratic nonlinearities in dimension greater or equal to 4, or in dimension 3 for nonlinearities of order strictly greater than 2. The problem which is considered in the present paper, namely quadratic nonlinearities in dimension 3, is far more subtle. If a polynomial nonlinearity, in three space dimension, has a quadratic term it is necessary to take resonances into account. In this case global existence is known only if the dimension of $\mathscr{R}$ is zero, in particular we mention the work of Hayashi and Naumkin [11], Hayashi Mizumachi and Naumkin [6], Kawahara [7]. These authors employ vector fields methods to obtain global existence. The vector field approach which we understand as a method that addresses space resonances, can be cumbersome or unintuitive to apply (as in the (SNLS) case), or even limited in applicability since it does not deal with time resonances directly.

In the case of the nonlinearity $|u|^2$, the space time resonant set $\mathscr{R}$ is three-dimensional. In this case almost global existence has been proved by Ginibre and Hayashi [3], but it is not clear at all if global existence is true. For the gauge invariant nonlinearity $|u|u$, the Fourier side approach should of course be avoided; but using the pseudo-conformal transform, one can obtain global existence: see Cazenave and Weissler [1], Nakanishi and Ozawa [10].

Finally, let us remark that the quadratic Klein-Gordon equation in dimension 3 is easier to deal with: global existence is known for all type of quadratic nonlinearities since the papers of Klainerman [9] and Shatah [13]. This might seem surprising since the dispersive properties of the (linear) Klein-Gordon and Schrödinger operators are very close, but the main difference is of course that for Klein-Gordon $\mathscr{T} = \emptyset$.

Main results. For equation (NLS) let f denote the profile of u, i.e.,

$$\hat{f}(t,\xi) = e^{-i|\xi|^2 t}\hat{u}(t,\xi)\ ,$$

then by Duhamel's formula

$$\hat{f}(t,\xi) = \hat{u}_*(\xi) + \frac{\alpha}{(2\pi)^{3/2}} \int_2^t \int e^{is\varphi(\xi,\eta)} \hat{f}(s,\eta)\hat{f}(s,\xi-\eta) d\eta ds, \tag{1}$$

where $\varphi = -|\xi|^2 + |\eta|^2 + |\xi-\eta|^2$ and $\hat{f}(\xi)$ denotes the Fourier transform of f given by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{3/2}} \int e^{-ix.\xi} f(x)\ dx.$$

With $\xi_1 = \eta$ and $\xi_2 = \xi - \eta$, space time resonances correspond to the sets where $\varphi = 0$ and $\partial_\eta \varphi = 0$ respectively. Therefore space-time resonance corresponds to a point $\mathscr{R} = \{(0,0)\}$. Consequently we can prove the following

Theorem 1. *Define the Banach space X by its norm*

$$\|u\|_X = \|f\|_{L^\infty L^2} + \left\| \frac{x}{\log t} f \right\|_{L^\infty L^2} + \left\| \frac{x^2}{\sqrt{t}} f \right\|_{L^\infty L^2} + \|\hat{f}\|_{L^\infty L^\infty} + \|t^{3/2} u\|_{L^\infty L^\infty}.$$

where f is the profile of u, namely $\hat{f}(t,\xi) = e^{-i|\xi|^2 t}\hat{u}(t,\xi)$. For data u_* such that $\|e^{-it\Delta}u_*\|_X$ is small enough, there exists a solution of (NLS) in X. Furthermore, this solution scatters; i.e., $f(t)$ has a limit in L^2 as $t \to \infty$.

Here we used the short hand $L^p L^q = L^p_t([2,\infty), L^q_x(\mathbb{R}^3))$. The choice of initial data at $t = 2$ is made to avoid having singularities at $t = 0$ and $t = 1$ in the norm on X. If we choose the data to be given at $t = 0$, then in the definition of the norm on X t should be replaced by $\langle t \rangle = \sqrt{2 + t^2}$. The norm on X can be explained as follows. The first and the last term in the norm are just the L^2 control and the L^∞ decay of the linear solution. For data which is strongly localized in space we expect the profile $f(t)$ to be in $L^\infty L^1$. However we can not prove that and instead we show that the profile is such that $\hat{f} \in L^\infty L^\infty$ which is the fourth term in the definition of the norm. Similarly one expects $x^{\frac{3}{2}} f$ to be in $L^\infty L^2$, however to prove this requires the use of fractional derivatives in η which is cumbersome to say the least. Thus we opt to use stronger integer weights and allow for weak time growth of the stronger weighted norms. Finally one expect xf to be in $L^\infty L^2$, which is true but is stronger than what is needed for the existence proof. The easier estimate where we allow logarithmic growth is sufficient for our proof.

The method of proof of theorem 1, which is based on dealing with space-time resonances, enables us to easily treat a system of Schrödinger equations with different dispersion coefficients (SNLS). In this case the profile of u_ℓ satisfies

$$\hat{f}_\ell(t,\xi) = \hat{u}_{\ell *}(\xi) + \frac{1}{(2\pi)^{3/2}} \sum_{m,n} A_{\ell m n} \int_2^t \int e^{is\varphi_{\ell m n}(\xi,\eta)} \hat{f}_m(s,\eta)\hat{f}_n(s,\xi-\eta) d\eta ds,$$

where $\varphi_{\ell mn} = -c_\ell|\xi|^2 + c_m|\eta|^2 + c_n|\xi-\eta|^2$. The space-time resonances are $\mathscr{R} = \{(0,0)\}$ unless $c_m + c_n = 0$ or $\frac{1}{c_\ell} = \frac{1}{c_m} + \frac{1}{c_n}$, in which case $\mathscr{R}$ is three dimensional. Consequently we have

Theorem 2. *Assume that $A_{\ell mn} = 0$ if $c_m + c_n = 0$ or $\frac{1}{c_\ell} = \frac{1}{c_m} + \frac{1}{c_n}$ (no quadratic resonances). Then for small enough initial data the (SNLS) has a unique solution in X.*

The idea of the proof. For simplicity in notation we give the idea of the proof for theorem 1. Consider the phase $\varphi = -\xi^2 + \eta^2 + (\xi - \eta)^2$ which appears in the Duhamel's formula for the profile f (1). Since the space-time resonance are a point $\mathscr{R} = \{(0,0)\}$, we can take advantage of the oscillation of the phase in the Duhamel's formula (1) by integrating by parts in either s or η. In order to implement this strategy, we notice that $\partial_s e^{is\varphi} = i\varphi e^{is\varphi}$ and $\partial_\eta e^{is\varphi} = is\partial_\eta\varphi e^{is\varphi}$, thus for any P,

$$\frac{1}{iZ}\left(\partial_s + \frac{P}{s}\partial_\eta\right) e^{is\varphi} = e^{is\varphi} \tag{2}$$

where $Z \stackrel{def}{=} \varphi + P \cdot \partial_\eta \varphi$. We now pick a P such that Z vanishes only at $(0,0)$. There are many functions P that will do the trick and

$$P = -\eta + \frac{1}{2}\xi,$$

is just one of them. For this specific P we have

$$Z = \varphi + P \cdot \partial_\eta \varphi = -2\eta^2 - \xi^2 + 2\xi\eta < -\frac{1}{10}(\eta^2 + \xi^2), \tag{3}$$

which vanishes only at the point where φ and $\partial_\eta \varphi$ are zero, which is $(\xi,\eta) = (0,0)$. To deal with the singularity of $\frac{1}{Z}$ we also consider the smoothened version of (2)

$$\frac{1}{\frac{1}{s} + iZ}\left(\frac{1}{s} + \partial_s + \frac{P}{s}\partial_\eta\right) e^{is\varphi} = e^{is\varphi}. \tag{4}$$

The integration by parts using (2) or (4) takes full advantage of the oscillations of $e^{is\varphi}$ and thus introduces additional decay in s. However after the integration by parts, the resulting expressions are not simple products in space anymore; instead they belong to a well-known class of bilinear operators, the Coifman-Meyer operators, which behave as products under Hölder's inequality. This fact and the added decay in s make the resulting equation for the profile similar to a Schrödinger equation with a cubic nonlinearity which has global small solutions.

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2. Background Material

The following are standard inequalities that we include for the convenience of the reader.

Coifman-Meyer operators. Throughout the proof of theorem 1 we will be dealing with Coifman-Meyer operators. These operators are defined via a Fourier multiplier $m(\xi,\eta)$ by

$$T_m(f,g) = \mathcal{F}^{-1} \int m(\xi,\eta)\hat{f}(\eta)\hat{g}(\xi-\eta)d\eta.$$

The fundamental theorem of Coifman and Meyer states that these operators have the same boundedness properties as the ones given by Hölder's inequality for the standard product.

Theorem C-M (Coifman-Meyer)**.** *Suppose that m satisfies*

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi,\eta)| \leq \frac{C}{(|\xi| + |\eta|)^{|\alpha|+|\beta|}} \tag{5}$$

for sufficiently many multi-indices (α,β). Then the operator

$$T_m : L^p \times L^q \to L^r$$

is bounded for

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad 1 < p,q \leq \infty \quad \textit{and} \quad 0 < r < \infty. \tag{6}$$

Remark. *1) For condition (5) to hold, it suffices for m to be homogeneous of degree 0, and of class $\mathcal{C}^\infty$ on a (ξ,η) sphere. 2) If $m(\xi,\eta)$ is a Coifman-Meyer multiplier, so is $m_t(\xi,\eta) = m(t\xi,t\eta)$, for t a real number. Furthermore, the bounds (5) are independent of t, and consequently so are the norms of T_{m_t} as an operator from $L^p \times L^q$ to L^r, for (p,q,r) satisfying (6).*

A stationary phase lemma. The $L^1 \to L^\infty$ decay of solutions to the Schrödinger equation is just the stationary phase lemma. The following is a more detailed version of this fact.

Lemma 2.1. *The Schrödinger semigroup satisfies*

$$(e^{-it\Delta}g)(x) = \frac{1}{(-2it)^{3/2}} e^{-i\frac{x^2}{4t}} \hat{g}\left(-\frac{x}{2t}\right) + \frac{1}{t^{7/4}} O(\|x^2 g\|_2)$$

with the convention that $\frac{1}{(-i)^{3/2}} = e^{i\frac{3\pi}{4}}$. *In particular*

$$\|e^{-it\Delta}g\|_{L^\infty} \lesssim \frac{1}{t^{\frac{3}{2}}}\|\hat{g}\|_{L^\infty} + \frac{1}{t^{7/4}}\|x^2 g\|_{L^2}.$$

Proof. It is well known that

$$\begin{aligned}(e^{-it\Delta}g)(x) &= \frac{1}{(-4i\pi t)^{3/2}} \int e^{-i\frac{|x-y|^2}{4t}} g(y)dy \\ &= \frac{1}{(-2it)^{3/2}} e^{-i\frac{x^2}{4t}} \hat{g}\left(-\frac{x}{2t}\right) + \frac{1}{(-4i\pi t)^{3/2}} e^{-i\frac{x^2}{4t}} \int e^{i\frac{xy}{2t}} \left(e^{-i\frac{y^2}{4t}} - 1\right) g(y)dy.\end{aligned}$$

In order to prove the lemma, it suffices to bound the second term in the last line. This is easily done as follows

$$\begin{aligned}\left| e^{-i\frac{x^2}{4t}} \int e^{2\frac{xy}{2s}} \left(e^{i\frac{y^2}{4t}} - 1\right) g(y)dy \right| &\leq \int_{|y|\leq\sqrt{t}} \frac{y^2}{4t}|g(y)|dy + \int_{|y|\geq\sqrt{t}} |g(y)|dy \\ &\leq \frac{C}{t^{1/4}}\|y^2 g\|_2.\end{aligned}$$

□

A Gagliardo-Nirenberg type inequality. For Schrödinger equation the generator of the pseudo conformal transformation $J \stackrel{def}{=} x - 2it\nabla$ plays the role of partial differentiation. Thus we have

Lemma 2.2. *The following inequality holds*

$$\left\|e^{-it\Delta}(xf)\right\|_4^2 \leq \left\|e^{-it\Delta}f\right\|_\infty \left\|e^{-it\Delta}(x^2 f)\right\|_2$$

Proof. The proof relies on the observation that $e^{-it\Delta}x = Je^{-it\Delta}$, and that $J = 2ite^{-i\frac{x^2}{4t}}\nabla e^{i\frac{x^2}{4t}}$. Thus we get

$$\begin{aligned}\|e^{-it\Delta}xf\|_4 = \|Je^{-it\Delta}f\|_4^2 &= 4t^2\|e^{-i\frac{x^2}{4t}}\nabla e^{i\frac{x^2}{4t}} e^{-it\Delta}f\|_4^2 \lesssim t^2\|e^{-it\Delta}f\|_\infty \|\Delta e^{i\frac{x^2}{4t}} e^{-it\Delta}f\|_2 \\ &\lesssim \|e^{-it\Delta}f\|_\infty \|J^2 e^{-it\Delta}f\|_2 \lesssim \|e^{-it\Delta}f\|_\infty \|e^{-it\Delta}x^2 f\|_2,\end{aligned}$$

where we used the standard Gagliardo-Nirenberg inequality for the first inequality. □

Fractional integration. By integrating by parts in Duhamel's formula we will introduce Z and $\frac{1}{t} + Z$ in the denominators. These terms behave like $-\Delta$ and $\frac{1}{t} - \Delta$ in the denominators, and thus we need the following inequalities which are standard.

Lemma 2.3 (Fractional integration). *Let* $\Lambda^\beta \stackrel{def}{=} (-\Delta)^{\beta/2}$ *and* $\Lambda_t^\beta \stackrel{def}{=} \left(\frac{1}{t} - \Delta\right)^{\beta/2}$, *then*

- *If* $\alpha \geq 0$, $1 < p, q < \infty$, *and* $\frac{1}{q} - \frac{1}{p} = \frac{\alpha}{3}$,
$$\left\|\Lambda^{-\alpha} f\right\|_p \lesssim \|f\|_q.$$
- *If* $\alpha \geq 0$, *and denoting* $L^{p,q}$ *for the usual Lorentz space,*
$$\|\Lambda^{-\alpha} f\|_\infty \lesssim \|f\|_{L^{\frac{3}{\alpha},1}}.$$

- *If* $\alpha \geq 0$, $1 \leq p, q \leq \infty$, *and* $0 \leq \frac{1}{q} - \frac{1}{p} < \frac{\alpha}{3}$

$$\left\|\Lambda_t^{-\alpha} f\right\|_p \lesssim t^{\frac{\alpha}{2}+\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_q.$$

We can bound Z and $\frac{1}{t} + Z$ in the denominator using Coifman-Meyer Theorem and Λ or Λ_t in the following manner.

Lemma 2.4. *Let* $Z = \varphi + P \cdot \partial_\eta \varphi$ *as in* (3) *and let* P_ℓ *denote a homogeneous polynomial in* (ξ, η) *of degree* ℓ*. Then*

(1) *The multiplier*

$$m(\xi,\eta) = \frac{P_{2k-1}(\xi,\eta)}{Z^k}$$

satisfies

$$\|T_m(f,g)\|_r \lesssim \left(\left\|\Lambda^{-1} f\right\|_p \|g\|_q + \|f\|_p \left\|\Lambda^{-1} g\right\|_q\right).$$

(2) *The multiplier*

$$m_t(\xi,\eta) = \frac{P_\ell(\xi,\eta)}{(\frac{1}{t}+Z)^k},$$

satisfies

$$\|T_{m_t}(f,g)\|_r \lesssim \left(\left\|\Lambda_t^{\ell-2k} f\right\|_p \|g\|_q + \|f\|_q \left\|\Lambda_t^{\ell-2k} g\right\|_p\right)$$

with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ *as in the C-M Theorem.*

Proof. Let ψ_1 and ψ_2 be two functions of ξ and η, homogeneous of degree 0 and $\mathcal{C}^\infty$ outside $(0,0)$, such that

$$\begin{aligned} &\psi_1(\xi,\eta) + \psi_2(\xi,\eta) = 1 \quad \text{for any } (\xi,\eta) \\ &\psi_1(\xi,\eta) = 0 \quad \text{if } |\eta| \geq \tfrac{1}{4}|\xi-\eta| \\ &\psi_2(\xi,\eta) = 0 \quad \text{if } |\xi-\eta| \geq \tfrac{1}{4}|\eta|. \end{aligned} \tag{7}$$

To prove (1) we decompose the Fourier multiplier m into two multipliers

$$m(\xi,\eta) = \psi_1(\xi,\eta) m(\xi,\eta) + \psi_2(\xi,\eta) m(\xi,\eta) \overset{def}{=} m_1(\xi,\eta) + m_2(\xi,\eta) \tag{8}$$

We can rewrite m_1 as

$$m_1(\xi,\eta) = \psi_1(\xi,\eta) \frac{P_{2k-1}(\xi,\eta)|\eta|}{Z^k} \frac{1}{|\eta|}.$$

Observe that $\psi_1(\xi,\eta)\frac{P_{2k-1}(\xi,\eta)|\eta|}{Z^k}$ satisfies the hypothesis of the C-M Theorem. Thus we have

$$\|T_{m_1}(f,g)\|_r = \left\|T_{\frac{P_{2k-1}(\xi,\eta)|\eta|}{Z^k}}\left(\Lambda^{-1} f, g\right)\right\|_r \lesssim \left\|\Lambda^{-1} f\right\|_p \|g\|_q.$$

The case of m_2 being entirely similar permuting the roles of f and g, we finish the proof of (1).

To prove (2) we similarly decompose m_t into $m_{t1} + m_{t2}$ and rewrite m_{t1} as

$$m_{t1}(\xi,\eta) = \psi_1(\xi,\eta) \frac{P_\ell(\xi,\eta)\left(\frac{1}{t}+\eta^2\right)^{k-\frac{l}{2}}}{(\frac{1}{t}+Z)^k} \frac{1}{\left(\frac{1}{t}+\eta^2\right)^{k-\frac{l}{2}}} = \mu(t\xi, t\eta) \frac{1}{\left(\frac{1}{t}+\eta^2\right)^{k-\frac{l}{2}}}$$

where

$$\mu(\xi,\eta) \overset{def}{=} \psi_1(\xi,\eta) \frac{P_\ell(\xi,\eta)\left(1+\eta^2\right)^{k-\frac{l}{2}}}{(1+Z)^k}$$

satisfies (5). By the C-M Theorem and the remark following it we have

$$\|T_{m_{t1}}(f,g)\|_r = \left\|T_{\mu(t\xi,t\eta)}\left(\Lambda_t^{\ell-2k}f,g\right)\right\|_r \lesssim \left\|\Lambda_t^{\ell-2k}f\right\|_p \|g\|_q.$$

The case of m_{t2} being entirely similar, we finish the proof of the lemma. □

3. Proof of Theorem 1

The solution u will be constructed using Picard's fixed point theorem. Thus we will simply show that

$$\hat{f}(t,\xi) \to \hat{u}_*(\xi) + \frac{\alpha}{(2\pi)^{3/2}}\int_2^t \int e^{is\varphi(\xi,\eta}\hat{f}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds \stackrel{def}{=} \hat{u}_*(\xi) + \alpha\hat{B}(f,f)(t,\xi),$$

is a contraction on a neighborhood of the origin in X. In other words, we will prove the estimate

$$\|B(f,f)\|_X \lesssim \|f\|_X^2 \ .$$

Notice that scattering follows immediately from the belonging of f to X. Indeed the L^∞ decay of u, for f in X, is integrable in time.

Control of $\|\hat{B}(f,f)\|_{L^\infty L^\infty}$. It suffices to estimate the bilinear term $\hat{B}$ which can be written as

$$\hat{B}(f,f)(t,\xi) = \int_2^t e^{-is\frac{|\xi|^2}{2}} \int e^{2is|\zeta|^2}\hat{f}(s,\frac{\xi}{2}+\zeta)\hat{f}(s,\frac{\xi}{2}-\zeta)d\zeta ds.$$

Bounding $e^{-is\frac{|\xi|^2}{2}}$ by 1, and the inner integral by Lemma 2.1, we get

$$\begin{aligned}\left|\hat{B}(f,f)(t,\xi)\right| &\lesssim \int_2^t \left(\frac{1}{s^{3/2}}\left|\hat{f}\left(s,\frac{\xi}{2}\right)\right|^2 + \frac{1}{s^{7/4}}\left\|\partial_\zeta^2\left(\hat{f}\left(s,\frac{\xi}{2}+\zeta\right)\hat{f}\left(s,\frac{\xi}{2}-\zeta\right)\right)\right\|_{L^2_\zeta}\right)ds \\ &\lesssim \int_2^t \frac{1}{s^{3/2}}\|f\|_X^2 ds + \int_2^t \frac{1}{s^{7/4}}\left(\|\partial_\zeta^2\hat{f}(s)\|_{L^2_\zeta}\|\hat{f}(s)\|_{L^\infty_\zeta} + \|\partial_\zeta\hat{f}(s)\|^2_{L^4_\zeta}\right)ds \\ &\lesssim \|f\|_X^2 + \int_2^t \frac{1}{s^{7/4}}\|\partial_\zeta^2\hat{f}(s)\|_{L^2_\zeta}\|\hat{f}(s)\|_{L^\infty_\zeta}ds \quad \text{by Gagliardo-Nirenberg's} \\ &\lesssim \|f\|_X^2 + \|f\|_X^2\int_2^t \frac{1}{s^{7/4}}s^{1/2}ds \lesssim \|f\|_X^2.\end{aligned}$$

Control of $\|B(f,f)\|_{L^\infty L^2}$. This is given by a simple energy estimate

$$|B(f,f)|_{L^\infty L^2} \lesssim \int_2^\infty \frac{1}{s^{\frac{3}{2}}}\|t^{3/2}u\|_{L^\infty L^\infty}\|f\|_{L^\infty L^2}ds \lesssim \|f\|_X^2.$$

Control of $\left\|\frac{x}{\log t}B(f,f)\right\|_{L^\infty L^2}$. Applying ∂_ξ, one gets

$$\partial_\xi\hat{B}(f,f) = \int_2^t\int is\varphi_\xi e^{is\varphi}\hat{f}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds + \int_2^t\int e^{is\varphi}\hat{f}(s,\eta)\partial_\xi\hat{f}(s,\xi-\eta)d\eta ds \stackrel{def}{=} I + II.$$

The standard Hölder inequality suffices to estimate II:

$$\begin{aligned}\|II\|_2 &= \left\|\int_2^t e^{is\Delta}\left(e^{-is\Delta}fe^{-is\Delta}(xf)\right)ds\right\|_2 \\ &\lesssim \int_2^t \|u\|_\infty\|xf\|_2 ds \\ &\lesssim \int_2^t \frac{ds}{s^{3/2}}\|f\|_X^2 \lesssim \|f\|_X^2.\end{aligned} \tag{9}$$

In order to evaluate I, we integrate by parts using $\frac{1}{iZ}\left(\partial_s + \frac{P}{s}\partial_\eta\right) e^{is\varphi} = e^{is\varphi}$

$$
\begin{aligned}
I &= \int_2^t \int s\varphi_\xi \frac{1}{Z}\left(\partial_s + \frac{P}{s}\partial_\eta\right) e^{is\varphi} \hat{f}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds \\
&= + \int \frac{\varphi_\xi}{Z} t e^{it\varphi} \hat{f}(t,\eta)\hat{f}(t,\xi-\eta)d\eta \qquad \text{(10a)} \\
&\quad - 2\int \frac{\varphi_\xi}{Z} e^{i2\varphi} \hat{u}_*(\eta)\hat{u}_*(\xi-\eta)\,d\eta \qquad \text{(10b)} \\
&\quad - \int_2^t \int \frac{\varphi_\xi}{Z} e^{is\varphi} \hat{f}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds \qquad \text{(10c)} \\
&\quad - \int_2^t \int s\frac{\varphi_\xi}{Z} e^{is\varphi} \partial_s\hat{f}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds \quad + \text{similar term} \qquad \text{(10d)} \\
&\quad - \int_2^t \int \partial_\eta\left(\frac{P\varphi_\xi}{Z}\right) e^{is\varphi} \hat{f}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds \qquad \text{(10e)} \\
&\quad - \int_2^t \int \frac{P\varphi_\xi}{Z} e^{is\varphi} \partial_\eta\hat{f}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds \quad + \text{similar term} \qquad \text{(10f)}
\end{aligned}
$$

The "similar terms" appearing in the above sum correspond to the cases where the s or η derivative hits $\hat{f}(s,\xi-\eta)$ instead of $\hat{f}(s,\eta)$.

Now observe that, interpolating between the different components of the X norm, one gets easily that

$$\|u\|_{L^{3,1}} \lesssim \frac{1}{\sqrt{t}} \quad \text{and} \quad \|f\|_{6/5} \lesssim t^\epsilon \ .$$

where $\epsilon > 0$ can be chosen as small as one wants.

Using the above bounds, and applying lemma 2.4 and lemma 2.3 we get:

$$\|(10a)\|_2 = t\left\|e^{it\Delta}T_{\frac{\varphi_\xi}{Z}}(e^{-it\Delta}f, e^{-it\Delta}f)\right\|_2 \lesssim t\|\Lambda^{-1}e^{it\Delta}f\|_2\|e^{it\Delta}f\|_\infty \lesssim t\|f\|_{6/5}\|u\|_\infty \lesssim \|f\|_X^2.$$

The treatment of (10b) (10c) and (10e) is very similar, so we skip it. Coming to (10d), we write

$$
\begin{aligned}
\|(10c)\|_2 &\lesssim \int_2^t s\left\|e^{is\Delta}T_{\frac{\varphi_\xi}{Z}}\left(u^2,u\right)\right\|_2 ds \lesssim \int_2^t s\left(\left\|\Lambda^{-1}u\right\|_\infty \|u^2\|_2 + \|u\|_2\|\Lambda^{-1}u^2\|_\infty\right) ds \\
&\lesssim \int_2^t s\left(\|u\|_{3,1}\|u\|_\infty\|u\|_2\right) ds \lesssim \|f\|_X^3 \int_2^t \frac{ds}{s} \lesssim \|f\|_X^3 \log t.
\end{aligned}
$$

Finally, it suffices to apply the Coifman-Meyer theorem to deal with (10f), as follows

$$\|(10e)\|_2 \lesssim \int_2^t \left\|e^{is\Delta}T_{\frac{P\varphi_\xi}{Z}}\left(e^{-is\Delta}f, e^{-is\Delta}xf\right)\right\|_2 ds \lesssim \int_2^t \|xf\|_2\|u\|_\infty ds \lesssim \|f\|_X^2 \int_2^t s^\epsilon \frac{ds}{s^{3/2}} \lesssim \|f\|_X^2.$$

Control of $\|t^{3/2}e^{-it\Delta}B(f,f)\|_{L^\infty L^\infty}$. The bound on $\|e^{-it\Delta}B(f,f)\|_\infty$ is usually achieved by using normal forms, i.e., integrating by parts in time. Using $\frac{1}{\frac{1}{s}+iZ}\left(\frac{1}{s} + \partial_s + \frac{P}{s}\partial_\eta\right) e^{is\varphi} = e^{is\varphi}$, we have

$$\hat{B}(f,f) = \int_2^t \int \frac{1}{\frac{1}{s}+iZ}\left(\frac{1}{s} + \partial_s + \frac{P}{s}\partial_\eta\right) e^{is\varphi}\hat{f}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds = \hat{g}(\xi) + \hat{h}(\xi)$$

where

$$\hat{g}(\xi) = \hat{g}_1(\xi) + \hat{g}_2(\xi) \overset{def}{=} \int \frac{1}{\frac{1}{t} + iZ} e^{it\varphi} \hat{f}(t,\eta)\hat{f}(t,\xi-\eta) - \frac{1}{\frac{1}{2} + iZ} e^{i2\varphi} \hat{u}_*(\eta)\hat{u}_*(\xi-\eta) d\eta$$

$$\hat{h}(\xi) \overset{def}{=} \int_2^t \int \frac{\frac{1}{s} + P\partial_\eta \varphi}{\frac{1}{s} + iZ} e^{is\varphi} \hat{f}(s,\eta)\hat{f}(s,\xi-\eta) d\eta ds - i \int_2^t \int \frac{1}{s^2} \frac{1}{(\frac{1}{s} + iZ)^2} e^{is\varphi} \hat{f}(s,\eta)\hat{f}(s,\xi-\eta) d\eta ds$$
$$- 2 \int_2^t \int \frac{1}{\frac{1}{s} + iZ} e^{is\varphi} \partial_s \hat{f}(s,\eta)\hat{f}(s,\xi-\eta) d\eta ds.$$

We want to estimate the L^∞ norm of $e^{-it\Delta} B(f,f) = e^{-it\Delta} g + e^{-it\Delta} h$, which will be achieved in different manners for each term.

To estimate $e^{-it\Delta} g$, let us first notice that the second summand, g_2, is very easily dealt with since it is constant in time. Focusing on g_1, we can proceed directly using that

$$\mathcal{F}(e^{-it\Delta} g_1)(\xi) = \int \frac{1}{\frac{1}{t} + iZ} \hat{u}(t,\eta)\hat{u}(t,\xi-\eta) d\eta$$

so that by the C-M Theorem and fractional integration (lemma 2.3) we have

$$\|e^{-it\Delta} g_1\|_\infty = \left\| \mathcal{F}^{-1} \frac{1}{\frac{1}{t} + \xi^2} \int \frac{\frac{1}{t} + \xi^2}{\frac{1}{t} + Z} \hat{u}(\eta)\hat{u}(s,\xi-\eta) d\eta \right\|_\infty = \left\| \Lambda_t^{-2} T_{\frac{\frac{1}{t}+\xi^2}{\frac{1}{t}+Z}} (u,u) \right\|_\infty$$
$$\lesssim t^{3/4} \left\| T_{\frac{\frac{1}{t}+\xi^2}{\frac{1}{t}+Z}} (u,u) \right\|_6 \lesssim t^{3/4} \|u\|_6 \|u\|_\infty \lesssim \|f\|_X^2 t^{-7/4}.$$

Also notice that the norm of $\hat{g}_1$ in L^∞_ξ is easily bounded:

$$|\hat{g}_1|_{L^\infty_\xi} \leq \int \frac{1}{|\eta|^2} \left| \hat{f}(s,\eta) \right| \left| \hat{f}(s,\xi-\eta) \right| d\eta \lesssim \|\hat{f}\|^2_{L^\infty_\eta \cap L^2_\eta} \lesssim \|f\|_X^2. \tag{11}$$

The L^∞ norm of the term $e^{-it\Delta} h$ is harder to estimate directly. However we will prove in the next subsection that

$$\|x^2 h(t)\|_2 \lesssim t^\epsilon, \tag{12}$$

with ϵ a constant arbitrarily small. This estimate on h is to be contrasted with the L^2 norms of $x^2 f$ or $x^2 g$, for which we only expect a bound of $C\sqrt{t}$. In other words, having removed g from f, the remainder $h = B(f,f) - g$ has a better decay. This enables us to use Lemma 2.1 as follows

$$\|e^{-it\Delta} h\|_\infty \lesssim \frac{1}{t^{3/2}} \|\hat{h}\|_\infty + \frac{1}{t^{7/4}} \|x^2 h\|_2 \lesssim \frac{1}{t^{3/2}} \left(\|\hat{B}(f,f)\|_\infty + \|\hat{g}\|_\infty \right) + \frac{1}{t^{7/4}} \|x^2 h\|_2 \lesssim \frac{1}{t^{3/2}} \|f\|_X^2,$$

where the last inequality follows from (11) and (12).

So to conclude the proof of Theorem 1, the only thing which is left is the proof of (12); this is the aim of the next subsection.

Control of $\|\frac{x^2}{t^{1/2}} g\|_{L^\infty L^2}$ and $\|\frac{x^2}{t^\epsilon} h\|_{L^\infty L^2}$. The idea is always to look at these quantities expressed in the Fourier variable, apply ∂_ξ^2, and estimate each of the terms which appear. In the case of $\|\frac{x^2}{t^{1/2}} g\|_{L^\infty L^2}$ this is straightforward, so we skip this and focus on estimating $\|\frac{x^2}{t^\epsilon} h\|_{L^\infty L^2}$, which requires much more work.

Recall that we denote P_ℓ any homogeneous polynomial in (ξ,η) of degree ℓ. Applying ∂_ξ^2 to $\hat{h}(\xi)$ produces terms of the following types:

$$\int_2^t \int \frac{1}{s^j} \frac{P_{2k-4-2j}}{\left(\frac{1}{s}+iZ\right)^k} e^{is\varphi} \partial_s \hat{f}(s,\eta) \hat{f}(s,\xi-\eta) d\eta ds \quad \text{with } k \geq 0 \text{ and } k-2 \geq j \geq -2 \tag{13a}$$

$$\int_2^t \int \frac{1}{s^j} \frac{P_{2k-3-2j}}{\left(\frac{1}{s}+iZ\right)^k} e^{is\varphi} \partial_s \hat{f}(s,\eta) \partial_\xi \hat{f}(s,\xi-\eta) d\eta ds \quad \text{with } k \geq 0 \text{ and } k-\tfrac{3}{2} \geq j \geq -1 \tag{13b}$$

$$\int_2^t \int \frac{1}{s^j} \frac{P_{2k-2-2j}}{\left(\frac{1}{s}+iZ\right)^k} e^{is\varphi} \partial_s \hat{f}(s,\eta) \partial_\xi^2 \hat{f}(s,\xi-\eta) d\eta ds \quad \text{with } k \geq 0 \text{ and } k-1 \geq j \geq 0 \tag{13c}$$

$$\int_2^t \int \frac{1}{s^j} \frac{P_{2k-2-2j}}{\left(\frac{1}{s}+iZ\right)^k} e^{is\varphi} \hat{f}(s,\eta) \hat{f}(s,\xi-\eta) d\eta ds \quad \text{with } k \geq 0 \text{ and } k-1 \geq j \geq 0 \tag{13d}$$

$$\int_2^t \int \frac{1}{s^j} \frac{P_{2k-1-2j}}{\left(\frac{1}{s}+iZ\right)^k} e^{is\varphi} \hat{f}(s,\eta) \partial_\xi \hat{f}(s,\xi-\eta) d\eta ds \quad \text{with } k \geq 0 \text{ and } k-\tfrac{1}{2} \geq j \geq 0 \tag{13e}$$

$$\int_2^t \int \frac{1}{s^j} \frac{P_{2k-2j}}{\left(\frac{1}{s}+iZ\right)^k} e^{is\varphi} \hat{f}(s,\eta) \partial_\xi^2 \hat{f}(s,\xi-\eta) d\eta ds \quad \text{with } k \geq 0 \text{ and } k-1 \geq j \geq 0 \tag{13f}$$

$$\int_2^t \int s \frac{P_{2k}}{\left(\frac{1}{s}+iZ\right)^k} e^{is\varphi} \hat{f}(s,\eta) \hat{f}(s,\xi-\eta) d\eta ds \tag{13g}$$

$$\int_2^t \int s \frac{P_{2k+1}}{\left(\frac{1}{s}+iZ\right)^k} e^{is\varphi} \hat{f}(s,\eta) \partial_\xi \hat{f}(s,\xi-\eta) d\eta ds \tag{13h}$$

$$\int_2^t \int s^2 \varphi_\xi^2 \frac{P\varphi_\eta}{\left(\frac{1}{s}+iZ\right)} e^{is\varphi} \hat{f}(s,\eta) \hat{f}(s,\xi-\eta) d\eta ds. \tag{13i}$$

• *Control of (13a), (13b), (13c).* These three terms are easy to deal with, and can be handled in a similar fashion. We will only illustrate the estimate on (13a). By lemmata (2.3) and 2.4,

$$\begin{aligned}
\|(13\mathrm{a})\|_2 &\leq \int_2^t \frac{1}{s^j} \left\| e^{is\Delta} T_{\frac{P_{2k-4-2j}}{\left(\frac{1}{s}+iZ\right)^k}} \left(e^{-is\Delta} f, u^2\right) \right\|_2 ds \\
&\lesssim \int_2^t \frac{1}{s^j} \left(\left\|\Lambda_s^{-2j-4} e^{-is\Delta} f\right\|_2 \left\|u^2\right\|_\infty + \left\|e^{-is\Delta} f\right\|_2 \left\|\Lambda_s^{-2j-4} u^2\right\|_\infty \right) ds \\
&\lesssim \int_2^t \frac{1}{s^j} s^{j+2} \|u^2\|_\infty \|f\|_2 ds \lesssim \|f\|_X^2 \int_2^t s^2 \frac{1}{s^3} ds \lesssim \|f\|_X^2 \log t.
\end{aligned}$$

• *Control of (13d).* By lemmata 2.3 and 2.4,

$$\begin{aligned}
\|(13\mathrm{d})\|_2 &\leq \int_2^t \frac{1}{s^j} \left\| e^{is\Delta} T_{\frac{P_{2k-2-2j}}{\left(\frac{1}{s}+iZ\right)^k}} \left(e^{-is\Delta} f, e^{-is\Delta} f\right) \right\|_2 ds \\
&\lesssim \int_2^t \frac{1}{s^j} \left\|\Lambda_s^{-2-2j} e^{-is\Delta} f\right\|_2 \left\|e^{-is\Delta} f\right\|_\infty ds \\
&\lesssim \int_2^t \frac{1}{s^j} s^{j+\frac{1}{2}} \|f\|_{6/5} \|u\|_\infty ds \lesssim \|f\|_X^2 \int_2^t \sqrt{s} s^\epsilon \frac{1}{s^{3/2}} ds \lesssim \|f\|_X^2 t^\epsilon.
\end{aligned}$$

• *Control of (13e).* By lemmata (2.3) and 2.4,

$$\|(13\mathrm{e})\|_2 \le \int_2^t \frac{1}{s^j}\left\| e^{is\Delta} T_{\frac{P_{2k-1-2j}}{\left(\frac{1}{s}+Z\right)^k}}\left(e^{-is\Delta}f, e^{-is\Delta}xf\right)\right\|_2 ds$$

$$\lesssim \int_2^t \frac{1}{s^j}\left(\left\|\Lambda_s^{-2j-1}e^{-is\Delta}xf\right\|_2\left\|e^{-is\Delta}f\right\|_\infty + \left\|e^{-is\Delta}xf\right\|_2\left\|\Lambda_s^{-2j-1}e^{-is\Delta}f\right\|_\infty\right)ds$$

$$\lesssim \int_2^t \frac{1}{s^j}s^{j+\frac12}\|xf\|_2\|u\|_\infty ds \lesssim \|f\|_X^2\int_2^t\sqrt{s}\log s\frac{1}{s^{3/2}}ds \lesssim \|f\|_X^2 t^\epsilon.$$

• *Control of (13f).* By lemmata (2.3) and 2.4,

$$\|(13\mathrm{f})\|_2 \le \int_2^t \frac{1}{s^j}\left\| e^{is\Delta} T_{\frac{P_{2k-2j}}{\left(\frac{1}{s}+iZ\right)^k}}\left(e^{-is\Delta}f, e^{-is\Delta}x^2f\right)\right\|_2 ds$$

$$\lesssim \int_2^t \frac{1}{s^j}\left(\left\|\Lambda_s^{-2j}e^{-is\Delta}x^2f\right\|_2\left\|e^{-is\Delta}f\right\|_\infty + \left\|e^{-is\Delta}x^2f\right\|_2\left\|\Lambda_s^{-2j}e^{-is\Delta}f\right\|_\infty\right)ds$$

$$\lesssim \int_2^t \frac{1}{s^j}s^{j}\|x^2f\|_2\|u\|_\infty ds \lesssim \|f\|_X^2\int_2^t\sqrt{s}\frac{1}{s^{3/2}}ds \lesssim \|f\|_X^2 \log t.$$

• *Control of (13g).* In order to deal with this term, we need to integrate by parts using the identity $\frac{1}{\frac{1}{s}+iZ}\left(\frac{1}{s}+\partial_s+\frac{P}{s}\partial_\eta\right)e^{is\varphi} = e^{is\varphi}$ which means writing

$$(13\mathrm{g}) = \int_2^t\int s\frac{P_{2k}}{\left(\frac{1}{s}+iZ\right)^k}\frac{1}{\frac{1}{s}+iZ}\left(\frac{1}{s}+\partial_s+\frac{P}{s}\partial_\eta\right)e^{is\varphi}\hat f(s,\eta)\hat f(s,\xi-\eta)d\eta ds$$

and performing the integrations by parts in s and η. Upon doing so, all terms that appear are of the form (13a) - (13f), except the time boundary integral term, which is easily estimated. This gives the desired estimate

$$\|(13\mathrm{g})\|_2 \lesssim \|f\|_X^2 t^\epsilon.$$

• *Control of (13h).* In this case we proceed as above that is we write

$$(13\mathrm{h}) = \int_2^t\int s\frac{P_{2k+1}}{\left(\frac{1}{s}+iZ\right)^k}\frac{1}{\frac{1}{s}+iZ}\left(\frac{1}{s}+\partial_s+\frac{P}{s}\partial_\eta\right)e^{is\varphi}\hat f(s,\eta)\partial_\xi\hat f(s,\xi-\eta)d\eta ds$$

and perform the integration by parts in s and η. All terms which appear then are of the form (13a) - (13f), except for two terms. The first

$$\int_2^t\int\frac{P_{2k+1}P}{\left(\frac{1}{s}+iZ\right)^{k+1}}e^{is\varphi}\partial_\eta\hat f(s,\eta)\partial_\xi\hat f(s,\xi-\eta)d\eta ds$$

which by C-M Theorem and Lemma 2.2, can be bounded by

$$\int_2^t\left\|e^{-it\Delta}f\right\|_\infty\left\|e^{-it\Delta}(x^2f)\right\|_2 ds.$$

The second

$$\int_2^t\int s\frac{P_{2k+1}}{\left(\frac{1}{s}+iZ\right)^{k+1}}e^{is\varphi}\hat f(s,\eta)\partial_s\partial_\xi\hat f(s,\xi-\eta)d\eta ds,$$

for which we remove the ∂_ξ integral from the last term, using the fact that $\partial_\xi \hat{f}(s,\xi-\eta) = -\partial_\eta \hat{f}(s,\xi-\eta)$, and then integrating by parts in η. The terms resulting from this integration by parts are all of the form (13a) - (13f), thus controlled. This gives the desired estimate

$$\|\mathcal{F}^{-1}(13g)\|_2 \lesssim \|f\|_X^2 t^\epsilon.$$

• *Control of (13i).* We just need to rewrite this term as

$$(13i) = \int_2^t \int s\varphi_\xi^2 \frac{P}{\left(\frac{1}{s}+iZ\right)} \partial_\eta e^{is\varphi} \hat{f}(s,\eta)\hat{f}(s,\xi-\eta)d\eta ds.$$

Performing the integration by parts in η, one obtains terms of the form (13g) and (13h). Since these terms are controlled, this concludes the bound on $\|\frac{x^2}{t^\epsilon}h\|_{L^\infty L^2}$. Thus if the initial data $\|e^{-it\Delta}u_*\|_X$ is small enough then the map $f \to u_* + B(f,f)$ is a contraction, and this completes the proof of Theorem 1.

Sketch of the proof of theorem 2. From Duhamel's formula the bilinear terms for the (SNLS) are

$$\hat{B}_{\ell mn}(f_m,f_n)(t,\xi) = A_{\ell mn} \int_2^t \int e^{is\varphi_{\ell mn}(\xi,\eta}\hat{f}_m(s,\eta)\hat{f}_n(s,\xi-\eta)d\eta ds$$

where $\varphi_{\ell mn} = -c_\ell|\xi|^2 + c_m|\eta|^2 + c_n|\xi-\eta|^2$. Since all $\varphi_{\ell mn}$ have space-time resonances $\mathscr{R} = \{(0,0)\}$ then we can choose a function

$$P_{\ell mn} = (c_\ell c_m + c_\ell c_n - c_m c_n + 1)(\eta - \frac{c_n}{c_m+c_n}\xi)$$

which is well defines since $c_n + c_m \neq 0$ by the non resonance hypothesis. It is easy to check that if we define $Z_{\ell mn} = \varphi_{\ell mn} + P_{\ell mn}\cdot\partial_\eta\varphi_{\ell mn}$ then

$$|Z_{\ell mn}| \gtrsim |c_\ell c_m + c_\ell c_n - c_m c_n|(|\xi|^2+|\eta|^2) \gtrsim (|\xi|^2+|\eta|^2),$$

since $c_\ell c_m + c_\ell c_n - c_m c_n \neq 0$ by the non resonance hypothesis. From here on the proof of theorem 2 proceeds verbatim as in the proof of theorem 1.

□

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