Multipliers, Paramultipliers, and weak-strong uniqueness for the Navier-Stokes equations

Pierre Germain
Centre de Mathématiques Laurent Schwartz, Ecole Polytechnique
U.M.R. 7640 du C.N.R.S.
91128 Palaiseau Cedex

Abstract
In this article, we describe spaces $\mathcal{P}$ such that : if u is a weak (in the sense of Leray [26]) solution of the Navier-Stokes system for some initial data u_0, and if u belongs to $\mathcal{P}$, then u is unique in the class of weak solutions. We say then that weak-strong uniqueness holds. It turns out that the proof of such results relies on the boundedness of a trilinear functional $F : L^{2/\alpha}\dot{H}^\alpha \times L^{2/\beta}\dot{H}^\beta \times \mathcal{P} \to \mathbb{R}$, where α, β belong to $[0,1]$. In order to find optimal conditions for the boundedness of F, we are led to describing spaces of multipliers and of paramultipliers (that is, functions which map, by classical pointwise product or by paraproduct, a given Sobolev spaces in another given Sobolev space). The study of these spaces enables us to give conditions for weak-strong uniqueness which generalise all previously known results, from the famous Serrin criterion [41], to the recent conditions formulated by Lemarié [25].

Keywords
Navier-Stokes equations, Leray solutions, weak-strong uniqueness, multipliers, Sobolev spaces, paraproduct

1 Introduction

1.1 The Cauchy problem for the Navier-Stokes equations

We shall in this paper study uniqueness criteria for the solutions of the Cauchy problem associated to the Navier-Stokes equations. We shall consider these equations in the whole space $\mathbb{R}^d$, where $d \geq 2$. The Cauchy problem reads then

$$(NS) \left\{ \begin{array}{l} \partial_t u - \Delta u + u \cdot \nabla u = \nabla p \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0 \ . \end{array} \right.$$

It describes the evolution of a viscous fluid filling the whole space : $u(x,t)$ and $p(x,t)$ are, respectively, the velocity and the pressure of the fluid. The initial condition $u_{|t=0} = u_0$ gives its velocity at $t = 0$. The fluid is furthermore supposed to be incompressible (hence the condition $\operatorname{div} u = 0$) and of viscosity $\nu = 1$.

The modern theory of the Navier-Stokes equations goes back to Leray [26] who first constructed weak solutions of finite energy of (NS), for an initial data u_0 in L^2 ; a solution is said to be of finite energy if it belongs to the space

$$\mathcal{L} \overset{\text{def}}{=} L^\infty([0,\infty),L^2)\cap L^2([0,\infty),\dot H^1)\ , \tag{1}$$

where $\dot H^1$ is the homogeneous Sobolev space, ie the space of functions whose gradient belongs to L^2. These weak solutions are global in time ; they are known to be unique for $d=2$ but for $d\geq 3$, their uniqueness is still an open problem today.
Another approach is the one of strong solutions, also already considered by Leray. One of the major steps in that domain was accomplished by Fujita and Kato [17], who built, for an initial data u_0 in $\dot H^{d/2-1}$, solutions in the space

$$\mathcal{C}([0,T],\dot H^{d/2-1})\ \text{ for some }\ T>0\ .$$

Strong solutions are in general unique ; for small initial data, they are defined for any time, but for large initial data it is not known whether they might blow up in finite time or not.

1.2 The weak-strong uniqueness problem

Let us now come to the weak-strong uniqueness problem ; we shall remain here rather sketchy and formal, but explain everything more thoroughly in Parts 2 and 3.
Weak-strong uniqueness is an attempt to reconcile the two points of view which have been described : weak and strong solutions. More precisely, the problem is to find conditions on a strong solution u of (NS) such that all weak solutions which share the same initial condition u_0 equal u. Leray already considered this problem ; however the articles of Prodi [38] and Serrin [41] were a very important improvement of the theory.

How does one proceed to prove weak-strong uniqueness ? The almost universal method is to establish the boundedness of the trilinear functional

$$F \overset{\text{def}}{:} \begin{array}{c}(u,v,h)\mapsto \displaystyle\int_0^T\int_{\mathbb{R}^d}(u\cdot\nabla v)h\,dxds\\ \mathcal{L}\times\mathcal{L}\times\mathcal{P}\longrightarrow\mathbb{R}\ ,\end{array}$$

where $T>0$ is given and $\mathcal{P}$ is to be determined. Notice that this functional is classical in the framework of the Navier-Stokes equations. If it is bounded for a certain $\mathcal{P}$, then, using energy estimates and a Gronwall type argument, one can in general conclude that weak-strong uniqueness holds.

So the problem reduces to finding $\mathcal{P}$ so that F be continuous. Prodi and Serrin suggested Lebesgue spaces in time and space, and, in a series of articles ([2] [13] [19] [23] [25] [39] [45]), spaces always more refined were considered. The common point of all these articles is that the authors actually do not prove the boundedness of F on $\mathcal{L}^2\times P$ but on $L^\alpha\dot H^{2/\alpha}\times L^\beta\dot H^{2/\beta}\times\mathcal{P}$, with $\alpha,\beta\in[2,\infty]$, ie $L^\alpha\dot H^{2/\alpha}$ and $L^\beta\dot H^{2/\beta}$ are interpolated spaces between $L^\infty L^2$ and $L^2\dot H^1$.

1.3 Our approach of the weak-strong uniqueness problem

What we will do is try to find optimal conditions on $\mathcal{P}$ so that F be bounded from $L^\alpha\dot H^{2/\alpha}\times L^\beta\dot H^{2/\beta}\times\mathcal{P}$ to $\mathbb{R}$. We will use two important tools :

- The paraproduct of Bony : following Gallagher and Planchon [19], we will split the product defining the integrand of F into three terms, with the help of the paraproduct algorithm. In other words, we write

$$F(u,v,h) = \int_0^T \int_{\mathbb{R}^d} \sum_j (\Delta_j u \cdot \nabla v) S_j h \, dx \, ds$$
$$+ \int_0^T \int_{\mathbb{R}^d} \sum_j (S_j u \cdot \nabla v) \Delta_j h \, dx \, ds + \int_0^T \int_{\mathbb{R}^d} \sum_{|j-k| \le 1} (\Delta_j u \cdot \nabla v) \Delta_k h \, dx \, ds$$

  (see Section 3.2.1 for a definition of the Littlewood-Paley operators Δ_j, S_j, and a more consequent explanation of the paraproduct algorithm).

- Multiplier spaces : following Lemarié [25], we observe that, in order to give a meaning to the integral

$$\int_{\mathbb{R}^d} a \, b \, c \, dx \ ,$$

  where $a \in \dot{H}^\alpha$ and $b \in \dot{H}^\beta$ (α and β are real numbers), it suffices that $c \in \mathbf{M}(\dot{H}^\alpha, \dot{H}^{-\beta})$. We denote $\mathbf{M}(\dot{H}^\alpha, \dot{H}^{-\beta})$ the multiplier space

$$\mathbf{M}(\dot{H}^\alpha, \dot{H}^{-\beta}) = \{ f \in \mathcal{S}' \, , \, \| f \phi \|_{\dot{H}^{-\beta}} \le C \| \phi \|_{\dot{H}^\alpha} \} \ .$$

  This observation can be adapted to the functional F.

In order to combine the ideas of Lemarié and of Gallagher and Planchon, we will introduce paramultiplier spaces, ie spaces of functions f which make one of the mappings

$$\phi \mapsto \sum_j \Delta_j f S_j \phi$$
$$\phi \mapsto \sum_j S_j f \Delta_j \phi$$
$$\phi \mapsto \sum_{|j-k| \le 1} \Delta_j f \Delta_k \phi$$

bounded from one Sobolev space in another.
A refined study of these spaces is necessary, and requires tools of harmonic analysis. Once we have described these paramultiplier spaces, we will be able to apply our results to the Navier-Stokes equations. This will yield a criterion for weak-strong uniqueness which generalizes all results already known, and is, in a certain sense, optimal.

1.4 Organisation of the article

We recall first in Part 2 some results about the Navier-Stokes equations, and their weak and strong solutions ; we then proceed by reviewing known results about weak-strong uniqueness.

We state in Part 3 our main results.

Two of them concern the Navier-Stokes equations : Theorem 3.2 is an optimal criterion for weak-strong uniqueness for the Navier-Stokes equations, it is proved in Part 4.

Theorem 3.5 gives a condition on the initial value for local uniqueness of weak solutions of the Navier-Stokes equations. It is proved in Part 5.

The proofs of Theorems 3.2 and 3.5 rely on a result of harmonic analysis, namely the description of multiplier and paramultiplier spaces given in Theorem 3.9. Part 6 is dedicated to its proof. The reader only interested in the Navier-Stokes equations can skip this part and simply admit this result.

1.5 Notations

In order try to keep the notations as light as possible, we will use the following conventions :

- We write C for a universal constant in an expression ; but its value may change from one line to another.
- In Littlewood Paley type expressions, we will sometimes (when this does not elude a technical difficulty) not mention the indices shifts : for example, we write $\sum_j \Delta_j f \Delta_j g$ instead of $\sum_{|j-k|\leq 1} \Delta_j f \Delta_k g$.
- When dealing with vectors, we shall sometimes write only scalar expressions, when the multi-dimensionality does not play any role.
- If u is a function of x and t, defined on an interval $t \in [0,T]$ made clear by the context, and if B is a Banach space, we shall use the notations $L^p B$ and $\mathcal{C}B$ instead of, respectively, $L^p([0,T],B)$ and $\mathcal{C}([0,T],B)$.

2 Navier-Stokes equations and weak-strong uniqueness : review of known results

2.1 The Navier-Stokes system, weak solutions, strong solutions

Recall the Navier-Stokes system

$$(NS) \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u = \nabla p \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0 \ . \end{cases}$$

It is often easier to solve (NS) under its integral form (both formulations are equivalent for large classes of solutions, see [25])

$$(INS) \qquad u(t) = e^{t\Delta} u_0 + B(u,u) \ ,$$

where

$$B(u,u)(t) \overset{\text{def}}{=} \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u(s) \otimes u(s)) ds \ . \tag{2}$$

2.1.1 Leray's weak solutions and the energy space

Leray [26] proved in 1934 the existence of *weak* solutions, ie obtained by a weak limiting process and satisfying the equation in the distribution sense. These solutions are of finite energy : the initial data u_0 should belong to L^2, and the solution u itself to the energy space

$$\mathcal{L} \overset{\text{def}}{=} L^\infty([0,\infty), L^2) \cap L^2([0,\infty), \dot{H}^1) \ .$$

For later use, we also define

$$\mathcal{L}_t \overset{\text{def}}{=} L^\infty([0,t], L^2) \cap L^2([0,t], \dot{H}^1) \ .$$

Theorem 2.1 (Leray [26]) *Let* $u_0 \in L^2$*. Then* (NS) *has a solution* $u \in \mathcal{L}$*, global in time. Furthermore,* u *verifies the following energy inequality*

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2 \, ds \leq \ \|u_0\|_2^2 \ . \tag{3}$$

Definition 2.2 (Leray weak solutions) *We call* u *a Leray weak solution if* u *is a solution of* (NS) *for some* $u_0 \in L^2$ *and if* u *satisfies the energy inequality (3).*

If $d = 2$, Leray weak solutions are unique and continuous with values in L^2, but for $d > 2$, it is not known whether, in general, Leray weak solutions are unique and / or regular. Finally, the following result will be useful in the following

Lemma 2.3 (Foias [16], Serrin [41]) *Let* T *be a real number in* $(0,\infty]$*,* u_0 *a divergence-free function in* L^2*, and take* $u \in \mathcal{L}_T$ *a solution of* (NS)*. Then there exists a zero measure set* N *such that for any* $t \in [0,T) \setminus N$*, and any function* $\phi \in \mathcal{C}^\infty([0,t], \mathcal{S})$ *whose divergence identically vanishes,*

$$\int_0^t (\langle u, \partial_t \phi\rangle - \langle \nabla u, \nabla \phi\rangle - \langle u \cdot \nabla u, \phi\rangle) \, ds = \langle u(t), \phi(t)\rangle - \langle u_0, \phi(0)\rangle \ .$$

Furthermore, it is possible to change the definition of u *on* N *in such a way that the above equality holds for any* $t \in [0,T)$*, and that* u *is weakly* L^2 *continuous.*

In the following, we **always** assume that the Leray solutions we consider are weakly L^2 continuous.

2.1.2 Strong solutions and critical spaces

While the weak solutions of (NS) belong to the energy space $\mathcal{L}$, the strong (ie obtained by a fixed point argument) solutions of (NS) are a priori of infinite energy : it is natural to construct them in critical spaces, in other words spaces whose scaling is adapted to the Navier-Stokes equations. Let us be more precise : if $u(x,t)$ is a solution of (NS) associated to the initial data $u_0(x)$, then $\lambda u(\lambda(x - x_0), \lambda^2 t)$ is a solution associated to $\lambda u_0(\lambda(x - x_0))$ for any $\lambda > 0$ and $x_0 \in \mathbb{R}^d$. The following definition is now natural.

Definition 2.4 *We say that a Banach space* B *of distributions on* $\mathbb{R}^d$ *is critical for the initial conditions if its norm verifies for any* $\lambda \in \mathbb{R}$*, any* $x_0 \in \mathbb{R}^d$ *and any* $u \in B$*,*

$$\|u\| = \|\lambda u(\lambda(\cdot - x_0)\| \ .$$

A Banach space of distributions of $\mathbb{R}^d \times \mathbb{R}$ *is a critical path space if its norm verifies for any* $\lambda \in \mathbb{R}$ *and any* $u \in B$.

$$\|u\| = \|\lambda u(\lambda(\cdot - x_0), \lambda^2 \cdot)\| \ .$$

It must be emphasized that in the following, *all* the spaces (except the energy space) in which we will take u_0 (respectively u) will be critical spaces for the initial conditions (respectively critical path spaces).

Many works have been devoted to strong solutions of the Navier-Stokes equations, and we will mention only some of them.
Let us begin with the theorem of Koch and Tataru. Before stating it, recall that ∂BMO is the space of derivatives of functions of BMO (see [22]), and that $\partial BMO^{(0)}$ is the closure of the Schwartz class in ∂BMO. In the following theorem, the existence part is due to Koch and Tataru [22], and the uniqueness to Miura [35].

Theorem 2.5 (Koch and Tataru [22], Miura [35]) *If* $u_0 \in \partial BMO^{(0)}$, *there exists* $T > 0$ *such that the system* (NS) *admits a unique solution in*

$$\mathcal{C}([0,T), \partial BMO^{(0)}) \cap L^\infty_{\text{loc}}((0,T), L^\infty) \ .$$

This theorem is very important because the space ∂BMO enjoys a maximality property (see [1]) in the sense that any known critical space for the initial data for which the (NS) system is well posed is included in ∂BMO.
This shows the importance of the space BMO for the Navier-Stokes equations ; in our approach of the weak-strong uniqueness problem, we will mainly work with BMO-type spaces.

We will also need some results of Lemarié-Rieusset, who studied in particular shift invariant local measure spaces.

Definition 2.6 ([25]) *A Banach space* E *is a shift invariant Banach space of test functions if and only if*

- $\mathcal{S} \subset E \subset \mathcal{S}'$.
- E *and its norm are translation invariant.*
- $\mathcal{S}$ *is dense in* E.

A Banach space E *is a space of local measures if and only if*

- E *is the dual of a shift invariant Banach space of test functions.*
- E *is homogeneous of degree -1.*
- *If* $f \in E$, $g \in \mathcal{S}$, $\|fg\|_E \leq C\|f\|_E\|g\|_\infty$.

The spaces of local measures can be seen as generalisations of the classical Lebesgue spaces : the following theorems are generalisations of theorems previously known only in the framework of Lebesgue spaces. Besov spaces (which have been introduced in the study of the Navier-Stokes equations by Cannone, Meyer and Planchon [8], [34]) appear in both these theorems ; for a definition of these spaces, see the Appendix, Section 7.1.

Theorem 2.7 ([25] p.176) *Let E be a space of local measures embedded in $\dot{B}^{-1}_{\infty,\infty}$, and let $u_0 \in E^{(0)}$ (the closure of the Schwartz class in E). Then there exists $T > 0$ such that (NS) has a solution $u \in \mathcal{C}([0,T), E^{(0)})$.*

Theorem 2.8 ([25] p.200) *Let $\sigma \in (-1, 0)$, and set $\sigma = -\frac{2}{q}$.*
Suppose F is a space of local measures, q a real number in the set $(2, \infty)$, and define

$$E = \dot{B}^{-2/q}_{F,q} \ .$$

Suppose finally that F is embedded in $\dot{B}^{-1+2/q}_{\infty,\infty}$, which is equivalent to : E is embedded in $\dot{B}^{-1}_{\infty,\infty}$. Then if $u_0 \in E^{(0)}$ there exists $T > 0$ such that (NS) has a unique solution u such that

$$\|u\|_{L^q([0,T],F)} + \sup_{t\in[0,T)} t^{1/q}\|u(t)\|_F + \sup_{t\in[0,T)} \sqrt{t}\|u(t)\|_\infty < \infty \ .$$

2.2 Weak-strong uniqueness : known results

First of all, let us present the weak-strong uniqueness problem.

2.2.1 Presentation of the problem

Let $u_0 \in L^2$. Let g and h be such that

$$(4) \qquad \begin{array}{l} g \text{ and } h \text{ are Leray weak solutions of } (NS) \\ h \text{ is a strong solution of } (NS). \end{array}$$

Then one has **weak-strong uniqueness** if $g = h$ as long as h is defined.

Actually the weak strong uniqueness problem is twofold. We shall try to

- determine to which space $\mathcal{P}$ (*path space*) h must belong so that weak-strong uniqueness holds ;
- determine to which space $\mathcal{I}$ (*initial value space*) u_0 must belong so that there exists a strong solution $h \in \mathcal{P}$.

We will now give very quickly the idea [26] [41] generally used to prove weak-strong uniqueness ; we shall come back to this in Parts 3 and 4.
Consider g and h as in (4) and set $w = g - h$. Using the energy inequality (3) satisfied by g and h, we get formally that

$$(5) \qquad \|w(t)\|_2^2 + \int_0^t \|\nabla w(s)\|_2^2 \, ds \le \|w_0\|_2^2 + 2 \left| \int_0^t (w \cdot \nabla w | h)(s) \, ds \right| \ .$$

The crucial point is now to estimate the trilinear term

$$(6) \qquad F(u, v, h) \overset{\text{def}}{=} \int_0^T \int_{\mathbb{R}^d} (u \cdot \nabla v) \cdot h \, dx \, dt \ .$$

If F is continuous from $\mathcal{L}^2 \times \mathcal{P}$ to $\mathbb{R}$, then it becomes possible to justify the formal inequality (5), to apply to it the Gronwall lemma, and finally to get $w = 0$.

2.2.2 Known results

Let us review the results which have been, to our knowledge, obtained about the weak-strong uniqueness ; for each of them, we will indicate what are the path space $\mathcal{P}$ and the initial value space $\mathcal{I}$, and what method is used in the proof.
Many of the references cited below also address the question of the regularity of the weak solutions ; it is sometimes even their first motivation. As we shall see in Section 2.3, this problem is connected with the weak-strong uniqueness, but these two questions are not equivalent. Besides, the question of the initial value space for u_0 which gives solutions u in a given path space is often left aside by the authors, but we try as much as possible to give a couple $(\mathcal{I}, \mathcal{P})$.

- Prodi [38] and Serrin [41] showed that weak-strong uniqueness holds if h belongs to the path space

  $$(7) \qquad \mathcal{P} = L^q([0,T], L^p) \quad \text{with} \quad \frac{d}{p} + \frac{2}{q} = 1 \quad \text{and} \quad d < p < \infty \ .$$

  If q and p verify (7), a strong solution in $L^q([0,T], L^p)$ exists if and only if u_0 belongs to the initial value space
  $$\mathcal{I} = \dot{B}_{p,q}^{-1+d/p} \ ,$$
  because of Theorem 2.8.

  The idea of the proof of Prodi and Serrin is simple : if $u, v \in \mathcal{L}$, then ∇v belongs to $L^2 L^2$ and, by interpolation, u belongs to $L^r \dot{H}^{2/r}$ for any $p \in [2, \infty]$; hence by Sobolev injection
  $$u \in L^r L^{\frac{2rd}{rd-4}} \ .$$
  This implies that $u \cdot \nabla v$ belongs to $L^{\frac{2r}{2+r}} L^{\frac{2rd}{rd-4}}$. To make F bounded, it suffices to take h in the dual of this space : it is not hard to see that, taking $r \in]2, \infty[$, this yields precisely the relation (7).

- Von Wahl [45] extended the result of Serrin to the limit case
  $$\mathcal{P} = \mathcal{C}([0,T], L^d) \ ,$$
  which corresponds to
  $$\mathcal{I} = L^d \ ,$$
  see [21].

  The idea of the proof is the same as above (use Sobolev injections to see that the functional F is continuous), but one has to be more careful when applying the Gronwall lemma.

  Finally, this last result was improved by Kozono and Sohr [24], who proved that weak-strong uniqueness holds for
  $$\mathcal{P} = L^\infty([0,T], L^d) \ .$$

- The other limit case, namely
  $$\mathcal{P} = L^2([0,T], BMO)$$

was treated by Kozono and Taniuchi [23]. It is not clear what is the associated initial value space.

The method of the proof is still based on the continuity of F. Noticing that u belongs to $L^\infty L^2$, that ∇v belongs to $L^2 L^2$, and that

$$\begin{cases} \operatorname{div} u = 0 \\ \operatorname{curl} \nabla v = 0 \ , \end{cases}$$

one can conclude, using the div − curl lemma [10] that

$$u\nabla v \in L^2 \mathcal{H}^1 \ ,$$

where $\mathcal{H}^1$ is the Hardy space, whose dual is BMO. Therefore F is continuous from $\mathcal{L}^2 \times L^2 BMO$ to $\mathbb{R}$.

- The relation (7) obviously does not make sense if $q < 2$. But we observe that, as $q \geq 2$ gets smaller, down to 2, p gets larger, up to ∞, ie the required space regularity increases as the required time regularity decreases. If one wants to take $q < 2$, it seems logical to demand more space regularity than L^∞. This is the result obtained by Beirão da Vega [2] : weak-strong uniqueness holds for

$$\mathcal{P} = L^q W^{1,p} \quad \text{with} \quad \frac{2}{q} + \frac{d}{p} = 2 \quad \text{and} \quad p \in (1, \min(2, \frac{d}{d-2})) \ .$$

It is not clear what initial value space u_0 should belong to.

The method employed by Beirão da Vega is based on an L^p energy estimate.

Following the same approach, but with a proof relying on the continuity of the trilinear term, Ribaud [39] showed that weak-strong uniqueness holds for

$$\mathcal{P} = L^q W^{s,p} \quad \text{with} \quad \frac{2}{q} - s + \frac{d}{p} = 1 \quad \text{and} \quad p, q \in (1, \infty) \ , \ s \geq 0, \ .$$

Again, it is not clear what initial value space u_0 should belong to.

- In [19], Gallagher and Planchon studied a Besov spaces scale, which is much more refined than the Lebesgue spaces scale used by Serrin. They proved that weak-strong uniqueness holds for the path space

$$\mathcal{P} = L^q \dot{B}_{p,q}^{-1+\frac{d}{p}+\frac{2}{q}} \quad \text{with} \quad \frac{d}{p} + \frac{2}{q} > 1 \ .$$

The corresponding initial value space is

$$\mathcal{I} = \dot{B}_{p,q}^{-1+d/p} \ ,$$

see [9] [20].

Gallagher and Planchon proved the continuity of $F : \mathcal{L}^2 \times L^q \dot{B}_{p,q}^{-1+\frac{d}{p}+\frac{2}{q}} \to \mathbb{R}$ with the help of a paraproduct type decomposition of the three term product defining F in (6).

- Dubois [13] studied the case of Lorentz, Morrey, and Besov over Morrey spaces. She could find many new path spaces which grant weak strong uniqueness. We give below some of these path spaces, plus the corresponding initial value spaces, which are obtained with the help of Theorem 2.8.

$$\begin{cases} \mathcal{P} = L^q L^{p,\infty} & \text{with} \quad \frac{d}{p} + \frac{2}{q} = 1 \quad \text{and} \quad p \in (d, \infty] \\ \mathcal{I} = \dot{B}^{-1+d/p}_{L^{p,\infty},q} & \end{cases}$$

$$\begin{cases} \mathcal{P} = L^q M^{r,p} & \text{with} \quad \frac{d}{p} + \frac{2}{q} = 1 \quad \text{and} \quad p \in [d, \infty] \ , \ r \in (2, p] \\ \mathcal{I} = \dot{B}^{-1+d/p}_{M^{r,p},q} & \end{cases}$$

  The weak-strong uniqueness for all these classes is proved by the classical argument : one shows that the functional F is continuous from $\mathcal{P} \times \mathcal{L}^2$ to $\mathbb{R}$.

- Finally, Lemarié-Rieusset ([25], chapter 21) was able to generalise some of the previous results using multiplier spaces. He proved that weak-strong uniqueness holds for the path spaces

$$\begin{aligned} &\mathcal{P} = \mathcal{C} X_1^{(0)} \\ &\mathcal{P} = L^{\frac{2}{1-r}} X_r \qquad \text{with} \qquad r \in [0, 1) \ , \end{aligned} \tag{8}$$

  where, by definition, $X_s = \mathbf{M}(\dot{H}^s, L^2)$ is the space of distributions such that their pointwise product with a function in $\dot{H}^s$ belongs to L^2 (we will come back in greater detail to these spaces in Part 3, since they will play a very important role in our main theorem). The embeddings of Proposition 6.13 show that the first case above generalises the result of von Wahl, and the second one the result of Serrin.

  The initial value spaces for u_0 are, respectively, and due to Theorems 2.7 and 2.8,

$$\mathcal{I} = \dot{X}_1^{(0)} \quad \text{and} \quad \mathcal{I} = \dot{B}^{r-1}_{X_r, \frac{2}{1-r}} \ .$$

  Recall that the idea of Serrin was to make F continuous, by choosing an appropriate path space for w. More precisely, Serrin noticed that the two first arguments of F are such that $u \in L^p \dot{H}^{2/p}$ for any $p \in [2, \infty]$ and $\nabla v \in L^2 L^2$; then he used Sobolev injections to characterize the space to which w should belong. But Lemarié-Rieusset stops here, and the conditions (8) are then obvious.

2.3 Regularity of the weak solutions

We have been focusing in this paper on the uniqueness of the weak solutions, but another problem is still open, namely the regularity of these solutions. By regularity, we mean the belonging of a weak solution to the space $\mathcal{C}^\infty((0, T] \times \mathbb{R}^d)$.

Uniqueness and regularity of weak solutions are of course related problems. The uniqueness seems to be harder to establish, because it requires some kind of regularity at $t = 0$. Regularity for $t > 0$ does not a priori grant uniqueness because the bifurcation from u_0 might happen precisely at $t = 0$.

In most cases, milder conditions than the ones known to ensure weak-strong uniqueness suffice to imply the regularity of weak solutions. Recall that the classical condition for weak-strong uniqueness is the belonging of a weak solution u of (NS) to some space $L^q L^p$, with $\frac{2}{q}+\frac{d}{p}=1$. If one only simply wants regularity of weak solutions, this condition can be improved much more than if one is looking for uniqueness. For example, for $d=3$, Sohr [42] proved that if a weak solution u of (NS) verifies

$$u \in L^{s,r}([0,T],L^{q,\infty}) \ , \text{ with } T>0 \ , \ 3<q<\infty \ , \ 2<s\leq r<\infty \text{ and } \frac{2}{s}+\frac{3}{q}=1 \ ,$$

then u is regular. Montgomery-Smith [36] proved that the same result holds if

$$\int_0^T \frac{\|u(t)\|_q^p}{1+\log^+\|u(t)\|_q}\,dt<\infty \text{ with } T>0 \ , \ 3<q<\infty \text{ and } \frac{2}{p}+\frac{3}{q}=1 \ .$$

A more complete review of existing results can be found in [3].

Finally, let us note that the Hölder regularity of suitable (in the sense of Caffarelli, Kohn and Nirenberg [7]) weak solutions belonging to $L^\infty L^d$ has recently been proved by Escauriaza, Seregin and Šverák (see [14] [40] and references therein) using backward uniqueness theory for parabolic equations.

3 Statement of the main results

We will first state our results concerning the Navier-Stokes equations : Theorem 3.2 and Theorem 3.5 ; in order to do this, we simply need to define multiplier spaces.

However, the proof of these theorems will require the use of tools of harmonic analysis, in particular Theorem 3.9. These tools are presented in the following of this part.

3.1 Weak-strong uniqueness for the Navier-Stokes equations

In this section, we will state our main results concerning the Navier-Stokes equations, trying to keep the technical points as elementary as possible.

First, let us define the Calderón fractional derivation operator Λ^α : it equals, for any real number α, $|D|^\alpha$; in other words

$$\Lambda^\alpha f=\mathcal{F}^{-1}(|\xi|^\alpha\widehat{f}(\xi)) \ .$$

Secondly, we say that a function belongs to the multiplier space $\mathbf{M}(\dot{H}^s,L^2)$ if it maps, by pointwise multiplication, $\dot{H}^s$ in L^2 :

$$\mathbf{M}(\dot{H}^s,L^2)=\{f\in\mathcal{S}' \ , \ \|f\phi\|_2\leq C\|\phi\|_{\dot{H}^s}\}$$

(see Section 3.2.2 for details).

Finally, we denote Lip for the set of lipschitzian functions, ie of functions whose gradient belongs to L^∞.

We can now define the spaces X_s.

Definition 3.1

$$\begin{cases} X_s=\mathbf{M}(\dot{H}^s,L^2) & \textit{if } s\in(0,1] \\ X_s=\Lambda^s BMO & \textit{if } s\in(-1,0] \\ X_{-1}=\mathrm{Lip} & \end{cases}$$

Let us state the main theorem. We recall that for a functional space F, $F^{(0)}$ denotes the closure of the Schwartz class in F.

Theorem 3.2 *Let* u *and* v *be two Leray weak solutions of* (NS) *for a given* $u_0 \in L^2$*. Suppose furthermore that for some* $T > 0$*,* $u \in \mathcal{P}$*, where either*

$$\mathcal{P} = \mathcal{C}([0,T], X_1^{(0)})$$

or

$$\mathcal{P} = L^{\frac{2}{1-r}}([0,T], X_r) \qquad \text{for some } r \in [-1,1) \ .$$

Then $u = v$ *on* $[0,T]$*. Furthermore,* u *belongs to* $\mathcal{C}([0,T], L^2)$ *and the energy equality holds :*

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2 \, ds = \|u_0\|_2^2 \ .$$

The proof of Theorem 3.2 is given in Part 4

Remark 3.3

- *Our result is optimal, in a sense which will be made precise in Section 4.2.6.*
- *In particular, the above theorem encompasses all the results given in Section 2.2.2 (except for the result of Kozono and Sohr related to* $L^\infty L^d$*, but this is a limit case and its proof is atypical). This can be checked using the embeddings of Section 6.5. Or one can observe that our method gives the optimal space* $\mathcal{P}$ *(see Section 4.2) which makes* $F : L^{2/\alpha}\dot{H}^\alpha \times L^{2/\beta}\dot{H}^\beta \times \mathcal{P} \to \mathbb{R}$ *bounded, for some* $\alpha, \beta \in [0,1]$*. Since the proofs of all the results recalled in section 2.2.2 rely on the boundedness of* F *over* $L^{2/\alpha}\dot{H}^\alpha \times L^{2/\beta}\dot{H}^\beta \times \mathcal{P}$*, they are necessarily generalized by our criterion.*
- *Weak-strong uniqueness in the case* $r \in [0,1]$ *was already obtained by Lemarié [25] ; but for* $r < 0$*, our result is new.*
- *Theorem 3.2 remains true (i.e., weak-strong uniqueness holds for the path spaces* $\mathcal{P}$ *given in the theorem) if an exterior force* $f \in L^2\dot{H}^{-1}$ *is added, so that* (NS) *is replaced by*

$$(NSF)\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u = \nabla p + f \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0 \ . \end{cases}$$

- *Finally, let us remark that weak-strong uniqueness is only a particular case of stability. Proceeding as in [19] and changing slightly the proof of Theorem 3.2, we can prove the following result : let* u *and* v *be two Leray weak solutions of* (NS) *respectively associated to the initial data* u_0 *and* v_0*. Assume moreover that* u *belongs to one of the path spaces* $\mathcal{P}$ *appearing in the statement of Theorem 3.2, for some* $r \in [-1,1)$*. Then, denoting* $w = u - v$ *and* $w_0 = u_0 - v_0$*, we have*

$$\|w(t)\|_2^2 + 2\int_0^t \|\nabla w(s)\|_2^2 \, ds \le \|w_0\|_2^2 \exp\left(C \int_0^t \|u(s)\|_{X_r}^{\frac{2}{1-r}} \, ds \right)$$

  for any $t \in [0,T]$*, and for a constant* C *independent of* u *and* v*.*

Remark 3.4 *Three different assertions on* v *(uniqueness of* v*,* L^2 *continuity of* v*, energy equality for* v*) are contained in Theorem 3.2 ; each of these assertions represents a gain of regularity for the Leray weak solution* v*. Indeed*

- *It is not known whether Leray weak solutions are, in general, unique.*
- *A Leray weak solution is a priori only weakly L^2 continuous, see Lemma 2.3.*
- *A Leray weak solution a priori only satisfies the energy inequality (3).*

These three points are of course related.

It is natural to wonder now : what is the set of initial data which yields a solution in one of the path spaces $\mathcal{P}$ appearing in Theorem 3.2 ? The answer to this question yields a criterion for the local uniqueness of weak solutions relying on the initial data

Theorem 3.5 *Let $u_0 \in L^2 \cap X_1^{(0)}$. Then there exists $T > 0$ such that there exists a Leray solution u of (NS) which belongs to $\mathcal{C}([0,T], X_1^{(0)})$. Furthermore, this solution is unique, on [0,T], in the class of Leray solutions.*
Similarly, take $u_0 \in L^2 \cap \dot{B}^{r-1\,(0)}_{X_r, \frac{2}{1-r}}$, for some $r \in (0,1)$. Then there exists $T > 0$ such that there exists a Leray solution u of (NS) which belongs to $L^{\frac{2}{1-r}}([0,T], X_r)$. Furthermore, this solution is unique, on [0,T], in the class of Leray solutions.

Theorem 3.5 is proved in Part 5.

Remark 3.6 *The above theorem does not deal with the case $r \leq 0$; we did not succeed to treat it. One of the difficulties is that we are led to considering functions belonging to Lebesgue spaces in time, whose index is smaller than 2. It is difficult to give a meaning to the pointwise product of such functions, and hence to the nonlinear term in (NS).*

3.2 A result of harmonic analysis

In the previous section, we have stated our two main results concerning the Navier-Stokes equations. Their proofs require tools of harmonic analysis, which we present now.

3.2.1 Littlewood Paley decomposition and paraproduct algorithm

We shall give first basic elements of the Littlewood-Paley theory. A more detailed exposition can be found in [44].

Let us first define a homogeneous Littlewood-Paley decomposition

$$\begin{gathered}\Psi \in \mathcal{S}\\ \operatorname{Supp}(\widehat{\Psi}) \subset \mathcal{C}(0,3/4,8/3)\\ \Delta_j = \widehat{\Psi}(2^{-j}D)\\ \sum_{j\in\mathbb{Z}} \Delta_j = Id \text{ in } \mathcal{S}' \text{ modulo polynomials.}\end{gathered}$$

This definition has an inhomogeneous counterpart

$$\begin{gathered}\Phi \in \mathcal{S}\\ \operatorname{Supp}(\widehat{\Phi}) \subset B(0,8/3)\\ S_j = \widehat{\Phi}(2^{-j}D)\\ S_0 + \sum_{j>0} \Delta_j = Id \text{ in } \mathcal{S}' \;.\end{gathered}$$

Hence we have

$$S_j f \xrightarrow[j\to+\infty]{\mathcal{S}'} f \ \ \forall f \in \mathcal{S}'$$

and (modulo polynomials), any distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ may be decomposed as

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f \ \ .$$

We are particularly interested in the pointwise product of two distributions (if it exists). If $f \in \mathcal{S}'$, we can always define the multiplication operator M_f by

$$M_f(\phi) \overset{\text{def}}{=} f\phi \ ;$$

M_f maps $\mathcal{S}$ in $\mathcal{S}'$.

Using the Littlewood-Paley decomposition, one can split the classical pointwise product of two distributions into three terms : this is the paraproduct algorithm of Bony [4].

$$\begin{aligned} fg &= \sum_j \Delta_j f S_j g + \sum_j S_j f \Delta_j g + \sum_{|j-k|\le 1} \Delta_j f \Delta_k g \\ &\overset{\text{def}}{=} \ \Pi(f,g) \ \ + \ \ \widetilde{\Pi}(f,g) \ \ + \ \ R(f,g) \ . \end{aligned} \tag{9}$$

The two first sums are the paraproduct terms, and the last the remainder. The interest of this decomposition is that, in the two first sums $\Pi(f,g)$ and $\widetilde{\Pi}(f,g)$, due to the spectral localisation of the operators Δ_j and S_j, the spectrum of each of the summands lies in a dyadic annulus. The last sum is usually the hardest to handle, but it still has interesting properties, since the spectrum of each of the summands is supported in a dyadic ball.

3.2.2 Definitions of multiplier and paramultiplier spaces

We want to study how pointwise multiplication maps Sobolev spaces of various regularity indexes. We are led to considering the spaces of functions f for which (respectively) the applications

$$M_f\,,\ \Pi(f,\cdot)\,,\ \widetilde{\Pi}(f,\cdot)\,,\ R(f,\cdot) \ \ : \dot{H}^s \longrightarrow \dot{H}^{s+\alpha}$$

are bounded. The following definition makes this precise.

Definition 3.7 *Let* $s \in (-\frac{d}{2},\frac{d}{2})$ *and* $\alpha \in \mathbb{R}$ *such that* $s+\alpha \in (-\frac{d}{2},\frac{d}{2})$. *Then*

- $\mathbf{M}(\dot{H}^s,\dot{H}^{s+\alpha}) \overset{def}{=} \{f \in \mathcal{S}'\,,\ \|M_f(\phi)\|_{\dot{H}^{s+\alpha}} \le C\|\phi\|_{\dot{H}^s}\}$
- $\mathbf{\Pi}(\dot{H}^s,\dot{H}^{s+\alpha}) \overset{def}{=} \{f \in \dot{B}^{\alpha}_{\infty,\infty}\,,\ \|\Pi(f,\phi)\|_{\dot{H}^{s+\alpha}} \le C\|\phi\|_{\dot{H}^s}\}$
- *If* $\alpha < 0$, $\widetilde{\mathbf{\Pi}}(\dot{H}^s,\dot{H}^{s+\alpha}) \overset{def}{=} \{f \in \dot{B}^{\alpha}_{\infty,\infty}\,,\ \|\widetilde{\Pi}(f,\phi)\|_{\dot{H}^{s+\alpha}} \le C\|\phi\|_{\dot{H}^s}\}$ *;*
  if $\alpha = 0$, $\widetilde{\mathbf{\Pi}}(\dot{H}^s,\dot{H}^s) \overset{def}{=} L^\infty$.
- $\mathbf{R}(\dot{H}^s,\dot{H}^{s+\alpha}) \overset{def}{=} \{f \in \dot{B}^{\alpha}_{\infty,\infty}\,,\ \|R(f,\phi)\|_{\dot{H}^{s+\alpha}} \le C\|\phi\|_{\dot{H}^s}\}$

Remark 3.8 - *We define the space $\boldsymbol{M}(\dot{H}^s, \dot{H}^{s+\alpha})$ as the set of $f \in \mathcal{S}'$ such that M_f is bounded. But in order to define the paramultiplier spaces, we consider only functions in the Besov spaces $\dot{B}^{\alpha}_{\infty,\infty}$, which is consistent with the scaling, but a priori not completely general. We believe that any distribution f such that to $\Pi(f,\cdot)$, $R(f,\cdot)$ or $\widetilde{\Pi}(f,\cdot)$ maps $\dot{H}^s$ in $\dot{H}^{s+\alpha}$ also belongs to $\dot{B}^{\alpha}_{\infty,\infty}$, but we have not been able to prove it.*

- *The definition of $\widetilde{\boldsymbol{\Pi}}(\dot{H}^s, \dot{H}^{s+\alpha})$ poses a specific problem : indeed, $\dot{B}^{\alpha}_{\infty,\infty}$ does not embed in $\mathcal{S}'$ for $\alpha > 0$ (see the Appendix). Therefore, $S_j f$ cannot, for these values of α, be defined as an element of $\mathcal{S}'$, and it is not clear which meaning should be given to the pointwise product $S_j f \Delta_j \phi$ then. For this reason, and because $\widetilde{\Pi}(f,\cdot)$ is obviously continuous on $\dot{H}^s$ if $f \in L^\infty$, we chose to define $\widetilde{\boldsymbol{\Pi}}(\dot{H}^s, \dot{H}^{s+\alpha})$ as L^∞ if $\alpha = 0$. For $\alpha > 0$, we do not consider this space.*

- *The indexes of the Sobolev spaces, s and $s+\alpha$, are taken in $(-\frac{d}{2}, \frac{d}{2})$. The reason is of course that, with s lying in this interval, the following embeddings hold*

$$\mathcal{S} \hookrightarrow \dot{H}^s \hookrightarrow \mathcal{S}' \ ,$$

so there are no difficulties when handling Sobolev spaces. Besides, this range of values for s and $s+\alpha$ suffices for the application to the Navier-Stokes equations. A larger range of s and $s+\alpha$ is considered in Part 6.

3.2.3 Main result obtained in Part 6

In this section, we will state the main result proved in Part 6.

The spaces BMO^s have been introduced by Youssfi [46] [47] as

$$BMO^s \overset{\text{def}}{=} \boldsymbol{\Pi}(\dot{H}^s, \dot{H}^s) \ .$$

These spaces have the following properties (Proposition 6.1, see Part 6) :

- If $s > t$, $BMO^s \hookrightarrow BMO^t$.
- If $s = 0$, BMO^0 is the classical BMO space.
- If $s < 0$, $BMO^s = \dot{B}^0_{\infty,\infty}$.

As we will see, all the paramultiplier spaces can be described starting from the BMO^s, by (fractional) differentiation or integration. Recall the definition of the Calderón operator

$$\Lambda \overset{\text{def}}{=} |D| \ .$$

We now come to the main theorem proved in Part 6.

Theorem 3.9 *Take $s \in (-\frac{d}{2}, \frac{d}{2})$, and $\alpha \in \mathbb{R}$ such that $s+\alpha \in (-\frac{d}{2}, \frac{d}{2})$. One has then*

- $\boldsymbol{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha}) = \Lambda^{-\alpha} BMO^s$.
- $\mathbf{R}(\dot{H}^s, \dot{H}^{s+\alpha}) = \Lambda^{-\alpha} BMO^{-s-\alpha}$.
- *If $\alpha < 0$,* $\widetilde{\boldsymbol{\Pi}}(\dot{H}^s, \dot{H}^{s+\alpha}) = \dot{B}^{\alpha}_{\infty,\infty}$.

- *If* $\alpha > 0$, $\mathbf{M}(\dot{H}^s, \dot{H}^{s+\alpha}) = \{0\}$;
  if $\alpha = 0$, $\mathbf{M}(\dot{H}^s, \dot{H}^{s+\alpha}) = BMO^{|s|} \cap L^\infty$;
  if $\alpha < 0$, $\alpha \neq -2s$, $\mathbf{M}(\dot{H}^s, \dot{H}^{s+\alpha}) = \Lambda^{-\alpha} BMO^{\max(s,-s-\alpha)}$;
  if $s = 1/2$ *or* 1, $\mathbf{M}(\dot{H}^s, \dot{H}^{-s}) = \Lambda^{2s} BMO^s$.

The two following figures are an attempt to sum up the situation.

Value of s	$-d/2 < s < 0$	$s = 0$	$0 < s < \frac{d}{2}$
BMO^s	$\dot{B}^0_{\infty,\infty}$	BMO	BMO^s

Figure 1: The spaces BMO^s

Space considered	$\alpha > 0$	$\alpha = 0$	$\alpha < 0$
$\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$	$\Lambda^{-\alpha} BMO^s$	BMO^s	$\Lambda^{-\alpha} BMO^s$
$\mathbf{R}(\dot{H}^s, \dot{H}^{s+\alpha})$	$\Lambda^{-\alpha} BMO^{-s-\alpha}$	BMO^{-s}	$\Lambda^{-\alpha} BMO^{-s-\alpha}$
$\widetilde{\mathbf{\Pi}}(\dot{H}^s, \dot{H}^{s+\alpha})$		L^∞	$\dot{B}^{-\alpha}_{\infty,\infty}$
$\mathbf{M}(\dot{H}^s, \dot{H}^{s+\alpha})$	$\{0\}$	$L^\infty \cap BMO^{|s|}$	If $s \neq -\alpha/2$, $\Lambda^{-\alpha} BMO^{\max(s,-s-\alpha)}$

Figure 2: Multiplier and paramultiplier spaces ; s and $s+\alpha$ are supposed to lie in (-d/2,d/2)

3.2.4 Back to the X_s spaces

In this very short section, we take the opportunity to give a first hint that the result of harmonic analysis which has been stated above (Theorem 3.9) is indeed useful in the context of the Navier-Stokes equations. Using this theorem, one sees that the X_s spaces defined in Definition 3.1 can also be given by (we drop the case $s = -1$, which is exceptional)

$$\left\{ \begin{array}{ll} X_s = \Lambda^s BMO^s & \text{if } s \in (0,1] \\ X_s = \Lambda^s BMO & \text{if } s \in (-1,0] \ . \end{array} \right.$$

The two above equalities show that the lack of symmetry in Definition 3.1 between the cases $s > 0$ and $s \leq 0$ was actually artificial.
More generally, we shall see that Theorem 3.9 plays a central role in the proof of the two theorems about the Navier-Stokes equations which we have stated, Theorem 3.2 and Theorem 3.5.

4 Optimality and proof of Theorem 3.2

4.1 Proof of Theorem 3.2

This section is dedicated to the proof of Theorem 3.2. We will often distinguish two cases, $r \in [0,1]$ and $r \in [-1,0)$. Theorem 3.2 has already been proved by Lemarié [25] in the former case ; in the latter case $r \in [-1,0)$, it is new. We shall essentially follow the scheme of the proof of Lemarié, but, if $r \in [-1,0)$, new technical problems arise ; in particular, one has to make use of a paraproduct decomposition.

We need first some preparatory steps ; the following lemma is in some sense the key of the theorem, because it explains why, under the hypotheses of the theorem, the functional F is continuous (since $\mathcal{L} \hookrightarrow L^{2/\alpha}\dot{H}^{\alpha}$ for any $\alpha \in [0,1]$).

Lemma 4.1 *Let u belong to one of the path spaces $\mathcal{P}$ defined in Theorem 3.2, and v belong to $\mathcal{L}$ (defined in (1)), both being divergence free. Then $(v \cdot \nabla u)$ can be written as a (finite) sum of functions which belong to one of the spaces $L^{\frac{2}{2-\alpha}}\dot{H}^{-\alpha}$, for some $\alpha \in [0,1]$.*

Proof of the lemma : We have to distinguish two cases :

Case 1 : $r \in (0,1]$ We can simply write

$$v \cdot \nabla u = \nabla \cdot (v \otimes u) \ ,$$

and then observe that $u \in L^{\frac{2}{1-r}}\mathbf{M}(\dot{H}^{r}, L^2)$ and $v \in L^{2/r}\dot{H}^{r}$. It follows immediately that

$$\nabla \cdot (v \otimes u) \in L^2\dot{H}^{-1} \ .$$

Case 2 : $r \in [-1,0]$ We use the paraproduct decomposition (see (9)), forgetting the vectorial nature of u and v for a moment :

$$v \cdot \nabla u = \Pi(\nabla u, v) + R(\nabla u, v) + \widetilde{\Pi}(\nabla u, v) \ .$$

Recall that u belongs to $L^{\frac{2}{1-r}}([0,T], X_r)$, and v belongs to $\mathcal{L}$, hence to $L^{\frac{2}{r+1}}\dot{H}^{r+1}$ and $L^{\infty}L^2$. We will examine one by one the terms of the paraproduct decomposition.

- $\nabla u \in L^{\frac{2}{1-r}}\Lambda^{1+r}BMO = L^{\frac{2}{1-r}}\mathbf{\Pi}(L^2, \dot{H}^{-r-1})$ (see Theorem 3.9) and $v \in L^{\infty}L^2$ hence
$$\Pi(\nabla u, v) \in L^{\frac{2}{1-r}}\dot{H}^{-r-1} \ .$$
- $\nabla u \in L^{\frac{2}{1-r}}\Lambda^{1+r}BMO = L^{\frac{2}{1-r}}\mathbf{R}(\dot{H}^{r+1}, L^2)$ (see Theorem 3.9) and $v \in L^{\frac{2}{1+r}}\dot{H}^{r+1}$ hence
$$R(\nabla u, v) \in L^1L^2 \ .$$
- If $r \in (-1,0]$, $\nabla u \in L^{\frac{2}{1-r}}\dot{B}^{-1-r}_{\infty,\infty} = L^{\frac{2}{1-r}}\widetilde{\mathbf{\Pi}}(\dot{H}^{r+1}, L^2)$ (due to the embedding $BMO \hookrightarrow \dot{B}^{0}_{\infty,\infty}$ and to Theorem 3.9) and $v \in L^{\frac{2}{r+1}}\dot{H}^{r+1}$ hence
$$\widetilde{\Pi}(\nabla u, v) \in L^1L^2 \ .$$
If $r = -1$, $\nabla u \in L^1L^{\infty} = L^1\widetilde{\mathbf{\Pi}}(L^2, L^2)$ and $v \in L^{\infty}L^2$, hence
$$\widetilde{\Pi}(\nabla u, v) \in L^1L^2 \ .$$

In both cases, we have obtained the announced result : $(v \cdot \nabla u)$ is a (finite) sum of functions belonging to one of the spaces $L^{\frac{2}{2-\alpha}}\dot{H}^{-\alpha}$, for some $\alpha \in [0,1]$. ■

The following proposition is actually the second assertion of the theorem.

Proposition 4.2 *Let u satisfy the hypotheses of Theorem 3.2. It is then strongly L^2 continuous.*

Remark 4.3 *The classical way to prove the L^2 strong continuity of u under such hypotheses is to use the weak formulation of (NS), see [41]. We will use a different method, already used in [25] and based on the integral form of the Navier-Stokes equations.*

Proof of the proposition : Since u is a Leray weak solution, it is also (see [25]) a solution of the integral Navier-Stokes equations (INS) :

$$u(t) = e^{t\Delta}u_0 + B(u,u) \; .$$

The trend $e^{t\Delta}u_0$ is clearly L^2 continuous. The other term, $B(u,u)$, is defined as

$$B(u,u)(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}(u(s)\cdot\nabla u(s))ds \; .$$

But, using Lemma 4.1, all we have to show is actually that, if $z \in L^{\frac{2}{2-\alpha}}\dot{H}^{-\alpha}$ for some $\alpha \in [0,1]$,

$$h(t) \overset{\text{def}}{=} \int_0^t e^{(t-s)\Delta}\mathbb{P}z(s)\,ds$$

is continuous with values in L^2. We can forget from now on the projector $\mathbb{P}$, since it is bounded on L^2. First, let us show that h is well defined. We have

$$\begin{aligned}\|\Delta_j h(t)\|_2 &\le \int_0^t \|e^{(t-s)\Delta}\Delta_j z(s)\|_2 ds\\ &\le \int_0^t e^{-(t-s)2^{2j}}2^{j\alpha}\|\Delta_j z(s)\|_{\dot{H}^{-\alpha}}ds \; .\end{aligned}$$

The last integral is bounded independently of t due to Hölder's inequality, since $s \mapsto \|\Delta_j z(s)\|_{\dot{H}^{-\alpha}}$ belongs to $L^{\frac{2}{2-\alpha}}$ and $s \mapsto e^{-(t-s)2^{2j}}2^{j\alpha}$ has a norm in $L^{2/\alpha}(-\infty,t]$ bounded independently of j. In other words,

$$\|\Delta_j h(t)\|_2 \le C\|\Delta_j z\|_{L^{\frac{2}{2-\alpha}}\dot{H}^{-\alpha}} \; ,$$

so taking the square and summing over j we get

$$\|h(t)\|_2 \le C\|z\|_{L^{\frac{2}{2-\alpha}}\dot{H}^{-\alpha}} \; . \tag{10}$$

This proves that h belongs to $L^\infty L^2$; proving the continuity is now easy. Suppose for example $t' < t$, then

$$h(t) - h(t') = \int_{t'}^t e^{(t-s)\Delta}\mathbb{P}z(s)\,ds + \int_0^{t'} e^{(t'-s)\Delta}\mathbb{P}(e^{(t-t')\Delta} - Id)z(s)\,ds \; ,$$

and the estimate (10) yields the conclusion. ■

The following lemma is the key step in the proof of Theorem 3.2.

Lemma 4.4 *Let u and v as in Theorem 3.2. Set $w = u - v$. Then for any t in $[0,T]$,*

$$\langle u(t), v(t)\rangle + 2\int_0^t \langle\nabla u, \nabla v\rangle(s)\,ds = \|u_0\|_2^2 + \int_0^t \langle w\cdot\nabla u, w\rangle(s)\,ds \; .$$

Furthermore, the last term can be estimated by

$$\left|\int_0^t \langle w\cdot\nabla u, w\rangle(s)\,ds\right| \le C\|u\|_{\mathcal{P}}\|w\|_{\mathcal{L}_t}^2 \; ,$$

where $\mathcal{P}$ is defined in Theorem 3.2.

Remark 4.5 *We will hereafter give a proof of Lemma 4.4 which is quite natural (the idea of it goes back to Serrin [41]). It only works in space dimension $d \leq 4$; for greater space dimensions, Lemma 4.4 can be shown following the scheme of the proof of Lemarié, see [25], Chapter 21.*

PROOF : **1.** As mentioned above, we assume $d \leq 4$. Let us define first the family of mollifiers $\rho_n(t) = n\rho(nt)$, where ρ is a smooth and even function supported in $[-1, 1]$. Take u, v and t as in the statement of Theorem 3.2 ; we set for any $s \in [0, t]$

$$u_n(s) = \int_0^t \rho_n(s-\tau)u(\tau)d\tau \qquad\qquad v_n(s) = \int_0^t \rho_n(s-\tau)v(\tau)ds\tau \ .$$

We now observe that, for $d = 2, 3$ or 4, we can modify the statement of Lemma 2.3 by allowing ϕ to belong to $H^1([0, t], H^1)$. Indeed, using the Sobolev injection $\dot{H}^1 \hookrightarrow L^{\frac{2d}{d-2}}$, we see that for such d and ϕ it is possible to give a meaning to the term

$$\int_0^t \langle u \cdot \nabla u, \phi \rangle ds$$

and we conclude by a simple approximation argument. We can now apply Lemma 2.3 to u with v_n as a test function, and to v with u_n as a test function ; we get

$$\begin{aligned} &\int_0^t \left(\langle u, \partial_t v_n\rangle - \langle \nabla u, \nabla v_n\rangle - \langle u \cdot \nabla u, v_n\rangle\right) ds = \langle u(t), v_n(t)\rangle - \langle u_0, v_n(0)\rangle \\ &\int_0^t \left(\langle v, \partial_t u_n\rangle - \langle \nabla v, \nabla u_n\rangle - \langle v \cdot \nabla v, u_n\rangle\right) ds = \langle v(t), u_n(t)\rangle - \langle v_0, u_n(0)\rangle \ . \end{aligned} \tag{11}$$

All we have to do is now to sum these two equalities and pass to the limit $n \to \infty$.

2. We find first that

$$\int_0^t \langle u, \partial_t v_n\rangle\, ds + \int_0^t \langle v, \partial_t u_n\rangle\, ds = \int_0^t \int_0^t \partial_t \rho_n(\tau - s) \left[\langle v(\tau), u(s)\rangle + \langle v(s), u(\tau)\rangle\right] ds\, d\tau = 0 \tag{12}$$

since ρ is an even function. Besides, we have clearly

$$\int_0^t \left[\langle \nabla u, \nabla v_n\rangle + \langle \nabla v, \nabla u_n\rangle\right] ds \overset{n\to\infty}{\longrightarrow} 2\int_0^t \langle \nabla u, \nabla v\rangle\, ds \ . \tag{13}$$

Let us examine now

$$\langle u_0, v_n(0)\rangle - \frac{1}{2}\langle u_0, v_0\rangle = \int_0^1 \rho(s)\, \langle u_0, v\left(\frac{s}{n}\right) - v(0)\rangle\, ds \overset{n\to\infty}{\longrightarrow} 0$$

by weak L^2 continuity of v and the Lebesgue theorem. In other words,

$$\langle u_0, v_n(0)\rangle \overset{n\to\infty}{\longrightarrow} \frac{1}{2}\langle u_0, v_0\rangle \tag{14}$$

and, likewise,

$$\begin{aligned} \langle v_0, u_n(0)\rangle &\overset{n\to\infty}{\longrightarrow} \frac{1}{2}\langle u_0, v_0\rangle \\ \langle v(t), u_n(t)\rangle &\overset{n\to\infty}{\longrightarrow} \frac{1}{2}\langle u(t), v(t)\rangle \\ \langle u(t), v_n(t)\rangle &\overset{n\to\infty}{\longrightarrow} \frac{1}{2}\langle u(t), v(t)\rangle \ . \end{aligned} \tag{15}$$

3. We are left with the convection terms. Thanks to Lemma 4.1, we know that $u \cdot \nabla u$ can be written as a sum of functions each of which belongs to a space $L^{\frac{2}{2-\alpha}}\dot{H}^{-\alpha}$, for some α in $[0,1]$. On the other hand, we know that v belongs to $L^{\frac{2}{\alpha}}\dot{H}^{\alpha}$ for any $\alpha \in [0,1]$. It is therefore clear that

$$\int_0^t \langle u \cdot \nabla u, v_n \rangle \, ds \overset{n\to\infty}{\longrightarrow} \int_0^t \langle u \cdot \nabla u, v \rangle \, ds \ . \tag{16}$$

For the last term, an integration by parts (justified since $d \le 4$) yields

$$\begin{aligned} \int_0^t \langle v \cdot \nabla v, u_n \rangle \, ds &= -\int_0^t \langle v \cdot \nabla u_n, v \rangle \, ds \\ &= -\int_0^t \int_{n(s-t)}^{ns} \rho(\tau) \langle v(s) \cdot \nabla u \left(s - \frac{\tau}{n} \right), v(s) \rangle \, d\tau \, ds \ . \end{aligned}$$

Consequently,

$$\begin{aligned} &\int_0^t \langle v \cdot \nabla v, u_n \rangle \, ds + \int_0^t \langle v \cdot \nabla u, v \rangle \, ds \\ &\qquad = \int_{1/n}^{t-1/n} \int_{-1}^1 \rho(\tau) \langle v(s) \cdot \nabla \left[u(s) - u\left(s - \frac{\tau}{n} \right) \right], v(s) \rangle \, d\tau \, ds \\ &\qquad\qquad - \int_{[0,1/n]\cup[t-1/n,t]} \int_{-1}^1 \rho(\tau) \langle v(s) \cdot \nabla u \left(s - \frac{\tau}{n} \right), v(s) \rangle \, d\tau \, ds \\ &\qquad\qquad + \int_{[0,1/n]\cup[t-1/n,t]} \langle v \cdot \nabla u, v \rangle \, ds \\ &\qquad \overset{\text{def}}{=} I + II + III \ . \end{aligned}$$

Using the same arguments as in Lemma 4.1, we get

$$|I| \le C \|v\|_{\mathcal{L}_t}^2 \int_{-1}^1 \rho(\tau) \left\| u - u\left(\cdot - \frac{\tau}{n} \right) \right\|_{L^{\frac{2}{1-r}}([1/n,t-1/n],X_r)} d\tau \overset{n\to\infty}{\longrightarrow} 0 \ ,$$

and we see similarly that

$$\lim_{n\to\infty} II = \lim_{n\to\infty} III = 0 \ .$$

Finally, we have shown that

$$\int_0^t \langle v \cdot \nabla v, u_n \rangle \, ds \overset{n\to\infty}{\longrightarrow} -\int_0^t \langle v \cdot \nabla u, v \rangle \, ds \ . \tag{17}$$

4. To prove the first assertion of the lemma, it suffices to gather the equations (12), (13), (14), (15), (16) and (17), and to insert these limits in the sum of the two equalities of (11). The proof of the second assertion of the lemma (estimate on the trilinear term) is a repetition of arguments already given, in particular Lemma 4.1. ■

We now come to the proof of Theorem 3.2, but most of the work has already been done.

Proof of Theorem 3.2 : **1.** First remark that the energy equality can be proved using exactly the same technique as in Lemma 4.4. We will now prove the uniqueness.

2. Assume first $r \neq 1$.

Let T_0 be the largest real number smaller than T such that $u = v$ on $[0, T_0]$. We will suppose that $T_0 < T$ and obtain a contradiction. By weak continuity $u(T_0) = v(T_0)$, so we can take $T_0 = 0$.
On the one hand, u and v verify the energy inequality

$$\begin{aligned}
\|u\|_{\mathcal{L}_t}^2 &= \|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2 ds \le \|u_0\|_2^2 \\
\|v\|_{\mathcal{L}_t}^2 &= \|v(t)\|_2^2 + 2\int_0^t \|\nabla v(s)\|_2^2 ds \le \|v_0\|_2^2 \ ;
\end{aligned}$$

on the other hand, lemma 4.4 yields

$$\|u_0\|_2^2 - \langle u, v\rangle(t) - 2\int_0^t \langle \nabla u, \nabla v\rangle(s) ds = -\int_0^t \langle w \cdot \nabla u, w\rangle(s)\, ds \le C\|u\|_{L^{\frac{2}{1-r}}([0,t],X_r)} \|w\|_{\mathcal{L}_t}^2 \ .$$

Combining these two estimates, we get that

$$\begin{aligned}
\|w\|_{\mathcal{L}_t}^2 &= \|w(t)\|_2^2 + 2\int_0^t \|\nabla w(s)\|_2^2 ds \\
&= \|u(t)\|_2^2 + \|v(t)\|_2^2 - 2\langle u, v\rangle(t) + 2\int_0^t \|\nabla u(s)\|_2^2 ds + 2\int_0^t \|\nabla v(s)\|_2^2 ds \\
&\qquad - 4\int_0^t \langle \nabla u, \nabla v\rangle(s) ds \\
&\le C\|u\|_{L^{\frac{2}{1-r}}([0,t],X_r)} \|w\|_{\mathcal{L}_t}^2 \ .
\end{aligned}$$

We now simply have to choose $t > 0$ such that $C\|u\|_{L^{\frac{2}{1-r}}([0,t],X_r)} < 1$; then $w = 0$ on $[0, t]$, ie $u = v$ on $(0, t]$: this is the contradiction we were looking for.
3. Assume now $r = 1$, ie $u \in \mathcal{C}([0, T], X_1^{(0)})$. If $\epsilon > 0$, we can choose λ and ζ such that

$$\begin{aligned}
&u = \lambda + \zeta \\
&\|\lambda\|_{L^\infty([0,T],X_1)} < \epsilon \\
&\|\zeta\| \in L^\infty([0, T], L^\infty) \ .
\end{aligned}$$

Then the trilinear term can be estimated as follows for $t < T$

$$\begin{aligned}
\left|\int_0^t \langle w \cdot \nabla u, w\rangle(s)\, ds\right| &= \left|\int_0^t \langle w \cdot \nabla w, \lambda + \zeta\rangle(s)\, ds\right| \\
&\le C\epsilon \int_0^t \|\nabla w(s)\|_2^2 ds + C\|\zeta\|_{L^\infty([0,\tau],L^\infty)} \left(\int_0^t \|w(s)\|_2^2 ds\right)^{1/2} \left(\int_0^t \|\nabla w(s)\|_2^2 ds\right)^{1/2} \\
&\le 2C\epsilon \int_0^t \|\nabla w(s)\|_2^2 ds + \frac{C}{\epsilon}\int_0^t \|w(s)\|_2^2 ds \ .
\end{aligned}$$

We choose ϵ such that $2C\epsilon < 1$; proceeding as in the case $r \neq 1$, we get

$$\|w(s)\|_2^2 ds \le \frac{C}{\epsilon}\int_0^t \|w(s)\|_2^2 ds \ .$$

If we now apply the Gronwall lemma, we obtain that $u = v$ on $[0, T]$. ■

4.2 Continuity of the trilinear term : a heuristic approach

We would like in this section to investigate the optimality of Theorem 3.2. All the manipulations we will perform will be rather formal, but could be justified.

As we have seen, the continuity of the trilinear term is the crucial point to prove weak-strong uniqueness, and we could almost say that weak-strong uniqueness holds for a path space $\mathcal{P}$ if and only if F is continuous on $\mathcal{L}^2 \times \mathcal{P}$.

4.2.1 Splitting of the trilinear term

To study the continuity of F, we are going to split it into three terms, using the paraproduct algorithm, as was already done in [19].

To simplify the notations, we will consider that F operates on real functions, and not on vector fields. The reader can check that we are perfectly elicited to do so. With this convention we have

(18)

$$\begin{aligned} F(u,v,h) &= \int_0^T \int_{\mathbb{R}^d} (u\cdot\nabla v)\cdot h\,dx\,dt \\ &= \int_0^T \int_{\mathbb{R}^d} \Pi(h,u)\nabla v\,dx\,dt + \int_0^T \int_{\mathbb{R}^d} \widetilde{\Pi}(h,u)\nabla v\,dx\,dt + \int_0^T \int_{\mathbb{R}^d} R(h,u)\nabla v\,dx\,dt \\ &\overset{\text{def}}{=} F_\Pi(u,v,h) + F_{\widetilde{\Pi}}(u,v,h) + F_R(u,v,h)\ . \end{aligned}$$

If one drops the exceptional case $s=-1$, it is actually, as was noted in [19], the term F_Π which determines the continuity of F. Indeed, we will see that we have to require less regularity on h in order to make continuous the two other terms $F_{\widetilde{\Pi}}$ and F_R.

4.2.2 Continuity of F_Π

Since, for any α and β in $[2,\infty]$, $u\in L^\alpha \dot{H}^{2/\alpha}$ and $\nabla v \in L^\beta \dot{H}^{2/\beta-1}$, and since

$$F_\Pi(u,v,h) = \int_0^T \int_{\mathbb{R}^d} \Pi(h,u)\nabla v dx dt\ ,$$

a straightforward computation shows that a sufficient condition for F_Π to be continuous is

$$h \in L^{\frac{\alpha\beta}{\alpha\beta-\alpha-\beta}} \mathbf{\Pi}(\dot{H}^{2/\alpha}, \dot{H}^{1-2/\beta}) \text{ for some } \alpha,\beta\in[2,\infty]\ .$$

Finally, by the embedding property of the spaces $\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$ (Lemma 6.2), we see that the above criterion is implied by the following one

$$\begin{cases} h \in L^{\frac{2}{1-r}} \mathbf{\Pi}(\dot{H}^r, L^2) \text{ for some } r\in[0,1] \\ \text{or } h\in L^{\frac{2}{1+t}} \mathbf{\Pi}(L^2, \dot{H}^t) \text{ for some } t\in[0,1]\ , \end{cases}$$

which we can also write as

$$h \in L^{\frac{2}{1-r}} X_r \text{ for some } r\in[-1,1]\ . \tag{19}$$

4.2.3 Continuity of $F_{\widetilde{\Pi}}$

To study the continuity of $F_{\widetilde{\Pi}}$, we will use the well-known identity

$$\int_0^T \int_{\mathbb{R}^d} (u \cdot \nabla v) \cdot h \, dx \, dt = - \int_0^T \int_{\mathbb{R}^d} (u \cdot \nabla h) \cdot v \, dx \, dt \ . \tag{20}$$

This identity is simply the result of an integration by parts, where we use the fact that a solution of the Navier-Stokes equations is divergence free. Applying the same idea to $F_{\widetilde{\Pi}}$, we obtain

$$\begin{aligned} F_{\widetilde{\Pi}}(u,v,h) &= \sum_{i,j=1}^{d} \sum_{k \in \mathbb{Z}} \int_0^T \int_{\mathbb{R}^d} \Delta_k u_i \, \partial_i v_j \, S_k h_j \, dx \, dt \\ &= - \sum_{i,j,k} \int \int \Delta_k u_i \, v_j \, S_k \partial_i h_j \\ &= - \int \int \widetilde{\Pi}(\nabla h, u) v \ . \end{aligned}$$

Since, for any α and β in $[2, \infty]$, $u \in L^\alpha \dot{H}^{2/\alpha}$ and $v \in L^\beta \dot{H}^{2/\beta}$, $F_{\widetilde{\Pi}}$ will be continuous if, for some α and β,

$$\nabla h \in L^{\frac{\alpha\beta}{\alpha\beta-\alpha-\beta}} \widetilde{\mathbf{\Pi}}(\dot{H}^{2/\alpha}, \dot{H}^{-2/\beta}) = L^{\frac{\alpha\beta}{\alpha\beta-\alpha-\beta}} \dot{B}^{-2/\beta-2/\alpha}_{\infty,\infty} \ .$$

In other words, it suffices that

$$\begin{cases} h \in L^1 \, \mathrm{Lip} \\ \text{or} \quad h \in L^{\frac{2}{1-r}} \dot{B}^{-r}_{\infty,\infty} \quad \text{for some } r \text{ in (-1,1]} \end{cases}$$

and this last space contains $L^{\frac{2}{1-r}} X_r$.

4.2.4 Continuity of F_R

We now come to the last term

$$F_R(u,v,h) = \int_0^T \int_{\mathbb{R}^d} R(h,u) \nabla v \, dx \, dt \ .$$

As in the two last subsections, a sufficient condition for F_R to be continuous is that, for some α and β in $[-1, 1]$

$$h \in L^{\frac{\alpha\beta}{\alpha\beta-\alpha-\beta}} \mathbf{R}(\dot{H}^{2/\alpha}, \dot{H}^{1-2/\beta}) \ .$$

Thanks to Theorem 3.9, we can reformulate this criterion as

$$\begin{aligned} & h \in L^\infty \Lambda BMO \\ & \text{or, for some } r \in [-1, 1), \ h \in L^{\frac{2}{1-r}} \dot{B}^{-r}_{\infty,\infty} \ , \end{aligned}$$

and the first space above contains $L^\infty X_1$, and the latter $L^{\frac{2}{1-r}} X_r$.

4.2.5 Making use of "div-curl" type lemmas

We have tried to exploit as far as possible the belonging of u and v to certain functional spaces in order to find optimal criteria on $\mathcal{P}$ that make F bounded. But we can try to make use of another piece of information about u and v : that both are divergence-free. Since, furthermore, their gradient has a vanishing curl, it seems natural to apply "div-curl" lemmas (see [10], [47]). These lemmas state that, if $E = (E_1...E_d)$ is divergence free and $B = (B_1...B_d)$ is curl free, and if furthermore $E \in \dot{H}^s$, $B \in \dot{H}^{-s}$, then their inner product $E \cdot B$ belongs to a space which is a predual of BMO^s (in the case $s = 0$, it is the Hardy space $\mathcal{H}^1$).

If one tries to apply these lemmas, the resulting conditions on $\mathcal{P}$ that make F bounded are not better than the ones we have already found. For example, consider

$$F(u, v, h) = \int_0^T \int_{\mathbb{R}^d} (u \cdot \nabla v) \cdot h \, dx \ .$$

We notice that $u \in L^\alpha \dot{H}^{2/\alpha}$ and is divergence free, and $\nabla v \in L^\beta \dot{H}^{2/\beta - 1}$ and is curl free. In order to apply a div-curl lemma, we must assume that

$$\frac{2}{\alpha} = 1 - \frac{2}{\beta} \ .$$

Then F is bounded if h belongs to $L^2 BMO^{2/\alpha}$. Since this space is embedded in $L^2 BMO$ for $\alpha \geq 0$, this boundedness criterion was already established in the analysis of the splitting of F into three terms.

4.2.6 In which sense is the criterion (19) optimal ?

Conclusion 4.6 *Our aim in this section was to find conditions on the path space* $\mathcal{P}$ *so that* F *be continuous from* $\mathcal{L}^2 \times Y$ *to* $\mathcal{R}$*. As a conclusion, Theorem 3.2 (which is equivalent to (19 except in the limit case* $r = 1$ *for technical reasons) is optimal provided :*

- *One does not decompose more finely the functional* F *than with the paraproduct we have used - but we do not see how to achieve a finer decomposition.*

- *One does not use simultaneously more knowledge about the functions* u *and* v *(the arguments of* F*) than the vanishing of their divergence and their belonging of* u *to* $L^\alpha \dot{H}^{2/\alpha}$ *and of* v *to* $L^\beta \dot{H}^{2/\beta}$ *for some* α *and* β *- but we do not see how to exploit genuinely the fact that* $u, v \in L^\infty L^2 \cap L^2 \dot{H}^1$*. Really taking advantage of the fact that* u *and* v *are solutions of the Navier-Stokes equations seems out of reach.*

5 The initial value problem

Theorem 3.2 gives path spaces $\mathcal{P}$ such that : if u is a solution of (NS) with the initial value u_0, and $u \in \mathcal{P}$, then weak-strong uniqueness holds. Recall that these path spaces $\mathcal{P}$ are $\mathcal{C}X_1^{(0)}$ or $L^{\frac{2}{1-r}} X_r$ with $r \in [-1, 1)$).

In this part, we would like to address the following question : to what initial value space $\mathcal{I}$ should u_0 belong to, so that the solution (or at least, one solution) u of (NS) is in $\mathcal{P}$?

5.1 The trend $e^{t\Delta}u_0$

A classical procedure to solve (NS) is to set up a fixed point argument for the integral equation (INS). Given an initial value space for u_0, the first step is to find a path space $\widetilde{\mathcal{P}}$ such that

$$u_0 \in \mathcal{I} \implies e^{t\Delta}u_0 \in \widetilde{\mathcal{P}} \ .$$

Then, one can solve (INS) in $\widetilde{\mathcal{P}}$ using Picard's theorem. We are going to apply this procedure backwards, ie for each one of the spaces $\mathcal{P} = \mathcal{C}X_1^{(0)}$ or $L^{\frac{2}{1-r}}X_r$, we will find $\mathcal{I}$ such that

$$u_0 \in \mathcal{I} \iff e^{t\Delta}u_0 \in \mathcal{P} \ .$$

Proposition 5.1 *Let $u_0 \in \mathcal{S}'$. Then*
(i) $e^{t\Delta}u_0 \in \mathcal{C}([0,\infty), X_1)^{(0)} \iff u_0 \in X_1^{(0)}$
(ii) If $r \in [0,1)$, $e^{t\Delta}u_0 \in L^{\frac{2}{1-r}}([0,\infty), X_r) \iff u_0 \in \dot{B}^{r-1}_{X_r,\frac{2}{1-r}} = \dot{B}^{-1}_{BMO^r,\frac{2}{1-r}}$
(iii) If $r \in (-1,0]$, $e^{t\Delta}u_0 \in L^{\frac{2}{1-r}}([0,\infty), X_r) \iff u_0 \in \dot{B}^{-1}_{\infty,\frac{2}{1-r}}$

Proof : (i) is obvious.

(ii) This assertion is classical, and we shall prove only $\Rightarrow$. Suppose first

$$e^{t\Delta}u_0 \in L^{\frac{2}{1-r}}([0,\infty), X_r) \ .$$

The idea is then to write

$$\Delta_j u_0 = \Delta_j e^{-t\Delta}e^{t\Delta}u_0 \ .$$

The symbol of $\Delta_j e^{-t\Delta}$ reads $\psi\left(\frac{\xi}{2^j}\right)e^{t|\xi|^2}$, therefore the convolution kernel of this operator is bounded in L^1 (independently of j) if $\frac{1}{2}4^{-j} < t < 2\cdot 4^{-j}$. Thus, if t lies in this interval, we have, since X_r is a shift-invariant Banach space,

$$\|\Delta_j f\|_{X_r} \le C\|e^{t\Delta}f\|_{X_r} \ ,$$

for a constant independent of j, and for any $f \in \mathcal{S}'$. Using this last inequality, we get

$$\left(2^{j(r-1)}\|\Delta_j u_0\|_{X_r}\right)^{\frac{2}{1-r}} = 2^{-2j}\|\Delta_j u_0\|_{X_r}^{\frac{2}{1-r}} \le C\int_{\frac{1}{2}4^{-j}}^{2\cdot 4^{-j}} \|e^{t\Delta}u_0\|_{X_r}^{\frac{2}{1-r}}\,dt$$

and summing over j, we find

$$\|u_0\|_{\dot{B}^{r-1}_{X_r,\frac{2}{1-r}}} = \left[\sum_j \left(2^{j(r-1)}\|\Delta_j u_0\|_{X_r}\right)^{\frac{2}{1-r}}\right]^{\frac{1-r}{2}} \le C\|e^{t\Delta}u_0\|_{L^{\frac{2}{1-r}}([0,\infty),X_r)} \ .$$

This proves $\Rightarrow$. The converse case is classical and left to the reader.

(iii) can be proved following the same lines as (ii). But one has to prove the boundedness of the kernels of Littlewood-Paley type operators in the Hardy space $\mathcal{H}^1$ rather than in L^1, because of the duality between $\mathcal{H}^1$ and BMO. This can be done using the criterion in Stein [43], page 128, 5.2. ■

5.2 Construction of solutions and bicontinuity of B : the case $0 < r < 1$

In this section, we will construct local solutions of (NS) which belong to the path spaces arising in Theorem 3.2, in the case $0 < r < 1$.
A simple application of Theorem 2.8 gives the following proposition.

Proposition 5.2 *Let $r \in (0,1)$ and $u_0 \in \dot{B}^{r-1\,(0)}_{X_r,\frac{2}{1-r}}$. Then there exists a time $T > 0$ and a solution u of (NS) such that $u \in L^{\frac{2}{1-r}}([0,T], X_r)$.*

This proposition is not completely satisfying : it applies only if u_0 is in the closure of the Schwartz class in $\dot{B}^{r-1}_{X_r,\frac{2}{1-r}}$. To improve on this result, we need to examine the bicontinuity of the bilinear operator B defined in (2). Recall first the result of Fabes, Jones and Riviere [15].

Proposition 5.3 (Fabes, Jones and Riviere [15]) *If $\frac{2}{p} + \frac{d}{q} = 1$, with $q \in (d, \infty)$, and if $T > 0$,*

$$B \, : \, (L^p([0,T], L^q))^2 \to L^p([0,T], L^q)$$

is a bounded operator. Furthermore, its operator norm does not depend on T.

It is proved in Proposition 6.13 that $L^{d/s} \hookrightarrow X_s$; this embedding and the above result make the following proposition natural.

Proposition 5.4 *First, set $r \in (0,1)$ such that $\mathbf{M}(\dot{H}^r, \dot{H}^{-r}) = \Lambda^{2r} BMO^r$. Then for $T > 0$,*

$$B : (L^{\frac{2}{1-r}}([0,T], X_r))^2 \to L^{\frac{2}{1-r}}([0,T], X_r)$$

(B is defined in (2)) is a bounded operator. Furthermore, its operator norm does not depend on T.

Remark 5.5 *It is known that $\mathbf{M}(\dot{H}^r, \dot{H}^{-r}) = \Lambda^{2r} BMO^r$ only in the cases $r = 1/2$ and $r = 1$, see Theorem 6.10. For the other $r > 0$, the problem is open.*

Proof : Let $u, v \in L^{\frac{2}{1-r}}([0,T], X_r) = L^{\frac{2}{1-r}}([0,T], \mathbf{M}(\dot{H}^r, L^2))$. Using Proposition 6.7 we get that

$$w = uv \in L^{\frac{1}{1-r}}([0,T], \mathbf{M}(\dot{H}^r, \dot{H}^{-r})) \ .$$

Since $\mathbf{M}(\dot{H}^r, \dot{H}^{-r}) = \Lambda^{2r} BMO^r$, we actually have

$$\widetilde{w} = \Lambda^{-r} w \in L^{\frac{1}{1-r}}([0,T], X_r) \ .$$

So it turns out that

$$B(u,v)(t) = \mathbb{P} \int_0^t \int_{\mathbb{R}} e^{(t-s)\Delta} \nabla \Lambda^r (\widetilde{w}(s)) dx ds \ ,$$

with $\widetilde{w} \in L^{\frac{1}{1-r}}([0,T], X_r)$.
Thanks to Theorem 6.11, we know that the Riesz transforms are bounded on X_r if $r \in (0, d/2)$, therefore we can forget from now on the operator $\mathbb{P}$. We now need the following classical lemma (see for example [25] p.21).

Lemma 5.6 *If $r \in (0,1)$, the kernel K of the convolution operator $e^{\Delta}\nabla\Lambda^r$ belongs to L^1.*

We notice that for $\tau > 0$, the kernel of $e^{\tau\Delta}\nabla\Lambda^r$ reads

$$\tau^{-\frac{1+r+d}{2}} K\left(\frac{x}{\sqrt{\tau}}\right) ,$$

where according to Lemma 5.6, $K \in L^1$. We have then, using the fact that X_r is a shift invariant Banach space and that $\mathbb{P} \in \mathcal{L}(X_r)$,

$$\begin{aligned} \|B(u,v)(t)\|_{X_r} &\leq \int_0^t \left\| (t-s)^{-\frac{1+r+d}{2}} K\left(\frac{x}{t-s}\right) \right\|_1 \|\widetilde{w}(s))\|_{X_r} ds \\ &\leq \int_0^t (t-s)^{-\frac{1+r}{2}} \|\widetilde{w}(s)\|_{X_r} ds \ . \end{aligned}$$

Since $\widetilde{w} \in L_T^{\frac{1}{1-r}} X_r$, it now suffices to apply the Hardy-Littlewood-Sobolev theorem to conclude the proof. ■

Now that the boundedness of B over $(L^{\frac{2}{1-r}}([0,T],X_r))^2$ is established, the study of the solutions of (NS) in that space is easy.

Theorem 5.7 *First, set $r \in (0,1)$ such that $\mathbf{M}(\dot{H}^r, \dot{H}^{-r}) = \Lambda^{2r} BMO^r$.*
Let $u_0 \in \dot{B}^{r-1}_{X_r, \frac{2}{1-r}}$. Then there exists a solution u of (NS) with the initial data u_0, such that $u \in L^{\frac{2}{1-r}}([0,T],X_r)$, for a time $T > 0$.
Conversely, suppose that, for a given initial data u_0, $u \in L^{\frac{2}{1-r}}([0,\infty),X_r)$ is a solution of (NS). Then $u_0 \in \dot{B}^{r-1}_{X_r, \frac{2}{1-r}}$.

Proof : Take $u_0 \in \dot{B}^{r-1}_{X_r, \frac{2}{1-r}}$. Using Proposition 5.1 and Proposition 5.4, it is easy to solve (INS) in $L^{\frac{2}{1-r}}([0,T],X_r)$, for $T > 0$ small enough, with the help of a fixed point theorem (see for example [20]).

Conversely, assume that $u \in L^{\frac{2}{1-r}}([0,\infty),X_r)$ is a solution of (NS) for some u_0. Then, according to Proposition 5.4, $B(u,u)$ also belongs to $L^{\frac{2}{1-r}}([0,\infty),X_r)$, and hence

$$e^{t\Delta}u_0 = u + B(u,u)$$

as well. By Proposition 5.1, this implies that $u_0 \in \dot{B}^{r-1}_{X_r, \frac{2}{1-r}}$. ■

5.3 Construction of solutions and bicontinuity of B : the case $r = 1$

5.3.1 Construction of solutions in $\mathcal{C}([0,T],X_1^{(0)})$

Proposition 5.8 *We consider the system (NS) for an initial value u_0. Then $u_0 \in X_1^{(0)}$ if and only if there exists a $T > 0$ and a solution $u \in \mathcal{C}([0,T],X_1^{(0)})$.*

Proof : The "if" part is obvious. To prove the "only if" part, we use Theorem 2.7. ■

This proposition answers the question of the initial value space corresponding to the path space $\mathcal{C}([0,T),X_1^{(0)})$.
However, it is interesting to investigate further the properties of the space X_1 : we will see in the following section that it is a fully adapted space in the sense of Meyer.

5.3.2 Fully adapted spaces

Spaces fully adapted to the Navier-Stokes equations have been introduced by Meyer in [34].The aim of these spaces is to provide a functional analytic framework for the Navier-Stokes equations such that every term of the equations has the same regularity. Let us be more explicit : if the pressure is eliminated, the Navier-Stokes equations read

$$\begin{cases} \partial_t u - \Delta u + \mathbb{P}\nabla \cdot (u \otimes u) = 0 \\ \operatorname{div} u = 0 \end{cases}$$

plus an initial condition. Supposing that u belongs to $L^\infty X$, and fixing t at a given value, the three terms of the first equation above will belong to the same functional space if the mapping

$$(f, g) \mapsto \Delta^{-1}\mathbb{P}\nabla \cdot (f \otimes g)$$

is bounded from $X \times X$ to X. This yields the following definition.

Definition 5.9 (Meyer [34]) *The Banach space X is said to be fully adapted to the Navier-Stokes equations if*

- *The Riesz transforms act boundedly on X.*
- *The following inequality holds : $\|\Lambda^{-1}(fg)\|_X \le C\|f\|_X\|g\|_X$.*

Some examples of fully adapted spaces are given in [34] : L^3, $L^{3,\infty}$, $\Lambda^{1-d}PM$ (where PM is the set of pseudo-measures, i.e. of functions whose Fourier transform belongs to L^∞), and $\dot{B}^{-1+d/p}_{p,\infty}$ for $p \in [1, d)$.
All these spaces are included in X_1 : for $\Lambda^{1-d}PM$, it can be easily established using product and convolution rules in Lorentz spaces ; for the other spaces, it is proved in Proposition 6.13.
Besides, using Theorem 3.9 and Proposition 6.11, we see easily that

Proposition 5.10 *X_1 is a fully adapted space.*

As a conclusion, to our knowkledge, X_1 is the largest known fully adapted space.

5.4 Proof of Theorem 3.5

To prove Theorem 3.5, it suffices to put together some of the results obtained above.
Suppose first $u_0 \in L^2\dot{B}^{r-1\,(0)}_{X_r,\frac{2}{1-r}}$. Due to Theorem 2.8, there exists a solution u of (NS) belonging to the space $L^{\frac{2}{1-r}}([0,T], X_r)$.
We can now apply a result proved in [12], Chapter 4 : since u is built up using a fixed point method, in a space whose norm includes a term of the form $\sup_{t\in[0,T]} \sqrt{t}\|u(t)\|_\infty$, u is actually a Leray solution.
It now suffices to apply Theorem 3.2 to obtain that u is the only Leray solution on $[0, T]$. This concludes the proof of the theorem in case $u_0 \in L^2 \cap \dot{B}^{r-1\,(0)}_{X_r,\frac{2}{1-r}}$.
The case $u_0 \in L^2 \cap X_1^{(0)}$ is very similar. ■

5.5 A summarizing picture

Figure 3 illustrates for which initial data u_0 it is known that there exists a strong solution in one of the spaces $\mathcal{P}$ which yield weak-strong uniqueness, according to Theorem 3.2. We say then that weak-strong uniqueness holds for u_0. For these initial data u_0, we have local uniqueness of the Leray weak solutions.
To describe the regularity of u_0, we use the classical scale of Besov critical spaces $\dot{B}^{-1+d/p}_{p,q}$ for $p, q \in [1, \infty]$. The vertical axis represents d/p, and the horizontal one $2/q$.

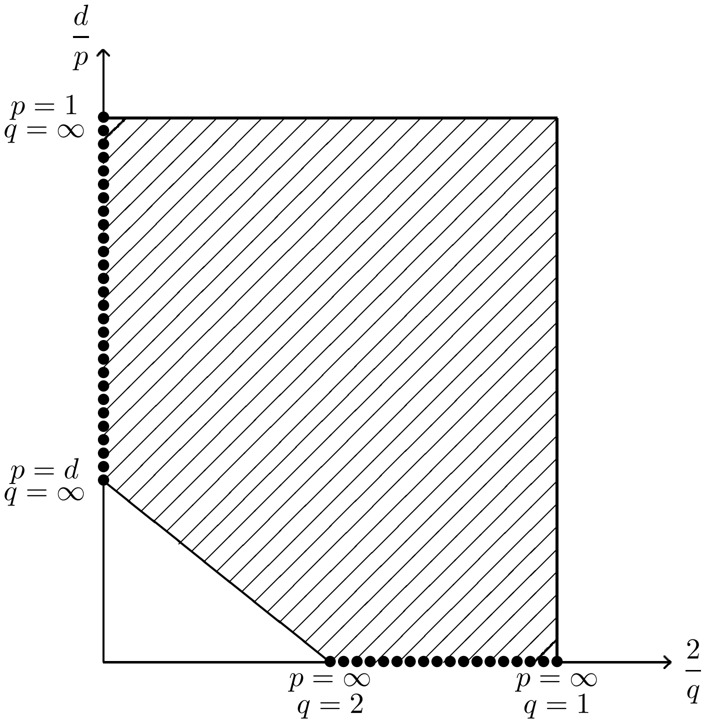

Figure 3: Initial value spaces $\dot{B}^{-1+d/p}_{p,q}$ for which weak-strong uniqueness holds

The area for which $p, q \in [1, \infty]$ is divided into five different sets, which we will examine one by one :

- If (p, q) is (strictly) in the shaded region, then for $u_0 \in \dot{B}^{-1+d/p}_{p,q}$, we have weak-strong uniqueness,see Gallagher and Planchon [19]. Observe that, due to the classical embedding

  $$\dot{B}^{-1+d/p}_{p,q} \hookrightarrow \dot{B}^{-1+d/\widetilde{p}}_{\widetilde{p},\widetilde{q}} \text{ for } \widetilde{p} \geq p \text{ and } \widetilde{q} \geq q \ \ ,$$

  if weak strong uniqueness holds for $(\widetilde{p}, \widetilde{q})$, it holds as well for any (p, q) which, in the above picture, lies in the top right quarter of the plane whose bottom left corner is $(\widetilde{p}, \widetilde{q})$. So the case of the shaded region is actually settled by the study of the points lying on the diagonal $\frac{d}{p} + \frac{2}{q} = 1$, which is the object of the next item.

- If (p, q) is on the diagonal, ie verifies $\frac{d}{p} + \frac{2}{q} = 1$, then we have weak-strong uniqueness for $u_0 \in \dot{B}^{-1+d/p}_{p,q}$. We have even proved a better result : weak-strong uniqueness holds for $u_0 \in \dot{B}^{-1+d/p\ (0)}_{X_{d/p},q} = \dot{B}^{-1\ (0)}_{BMO^{d/p},q}$ (Theorem 3.5) and we have

$$\dot{B}^{-1+d/p}_{p,q} \hookrightarrow \dot{B}^{-1+d/p\ (0)}_{X_{d/p},q} \ .$$

- The above embedding is optimal : one can prove that, for $\widetilde{p} > p$ or $\widetilde{q} > q$, the embedding
$$\dot{B}^{-1+d/\widetilde{p}}_{\widetilde{p},\widetilde{q}} \hookrightarrow \dot{B}^{-1+d/p}_{X_{d/p},q}$$
does not hold. For this reason, the results we have proved do not say anything about the white region, and we do not know whether Leray weak solutions are locally unique for $u_0 \in \dot{B}^{-1+d/p}_{p,q}$, and (p,q) lying in that region.

- We have not been able to settle the case of initial data in $\dot{B}^{-1}_{\infty,q}$ for any $q \in [1,2]$ (on the picture, this is the horizontal dotted line). Indeed, according to Proposition 5.1, initial data in $\dot{B}^{-1}_{\infty,\frac{2}{1-r}}$ with $r \in [-1,0]$ correspond to a trend $e^{t\Delta}u_0$ in $L^{\frac{2}{1-r}}X_r$. Therefore, because of the general principle that the solution u belongs to the same functional space as the trend $e^{t\Delta}u_0$, there *should* exist a strong solution belonging to $L^{\frac{2}{1-r}}X_r$ for $u_0 \in \dot{B}^{-1}_{\infty,\frac{2}{1-r}}$. However, matters may be more complicated in this case, and we have not been able to prove anything.

- The last set of initial data we have to consider corresponds to the vertical dotted line $q = \infty$, $p \in [1,d]$. The following embedding is proved in Proposition 6.13
$$\dot{B}^{-1+d/p}_{p,\infty} \hookrightarrow X_1 \text{ for } p < d \ \ .$$
On the other hand, we have been able to prove weak-strong uniqueness for $u_0 \in X_1^{(0)}$ (Theorem 3.5), and we can deduce that weak-strong uniqueness holds for $u_0 \in \dot{B}^{-1+d/p\ (0)}_{p,\infty}$ with $p \in [1,d)$.

6 Multipliers and paramultipliers

Multiplier and paramultiplier spaces are defined in Section 3.2.2 ; they are described in Theorem 3.9.
We shall in the next subsection prove a proposition which describes, for any s and α, the spaces $\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$. This is the main step in the proof of Theorem 3.9. In the three following sections, we describe the spaces $\mathbf{R}(\dot{H}^s, \dot{H}^{s+\alpha})$, $\tilde{\mathbf{\Pi}}(\dot{H}^s, \dot{H}^{s+\alpha})$ and $\mathbf{M}(\dot{H}^s, \dot{H}^{s+\alpha})$, and this completes the proof of Theorem 3.9. In the sequel of the present part, we examine the embeddings between paramultiplier spaces and more classical functional spaces, and finally give other possible points of view on these spaces.

Our multipliers are distributions which, by pointwise multiplication, map a Sobolev space based on L^2 of a given index on another one. Multiplier spaces are studied in a more general framework in [28] and in [5], Chapter III.
Gala and Lemarié [18] have obtained independently from us a result close to ours : they focused on multipliers and considered therefore only the case $\alpha \leq 0$, i.e. the case when the pointwise multiplication operator M_f, or the operator $\Pi(f,\cdot)$, maps a given Sobolev space in a Sobolev space of lower regularity. Their method, which is based on duality, is completely different from ours.

6.1 The spaces $\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$

In this section we intend to study the spaces $\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$ which have already been defined in Section 3.2.2.
We shall relax here this definition, by allowing s and α to be any real numbers. This will permit us to state a more general result without any supplementary effort.
So, if s and α belong to $\mathbb{R}$, we set

$$\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha}) \overset{\text{def}}{=} \{f \in \dot{B}^{\alpha}_{\infty,\infty}\,,\ \|\Pi(f,\phi)\|_{\dot{H}^{s+\alpha}} \leq C\|\phi\|_{\dot{H}^s} \text{ for any } \phi \in \mathcal{S}_\infty\}\ ,$$

see the Appendix for the definition of $\mathcal{S}_\infty$. This definition makes sense because of the density of $\mathcal{S}_\infty$ in $\dot{H}^s$.

It is well-known (see [43]) that one of the definitions of the space BMO is

$$BMO = \mathbf{\Pi}(L^2, L^2)\,.$$

This definition can be generalized to other Sobolev spaces, as has been done by Youssfi [46] [47] :

$$BMO^s \overset{\text{def}}{=} \mathbf{\Pi}(\dot{H}^s, \dot{H}^s)\,.$$

Let us first give a few properties of the BMO^s spaces.

Proposition 6.1 *Let $s, t \in \mathbb{R}$. Then*
(i) If $s > t$, $BMO^s \hookrightarrow BMO^t$.
(ii) If $s > \dfrac{d}{2}$, $BMO^s = \{0\}$.
(iii) If $s = 0$, BMO^0 is the classical BMO space.
(iv) If $s < 0$, $BMO^s = \dot{B}^0_{\infty,\infty}$.

Proof : (i) is proved in [47] ; it is a particular case of Lemma 6.2.
(ii) is Corollary 1 of [46].
(iii) is true by definition.
(iv) can be proved by a simple computation : let $s < 0$, $f \in \dot{B}^0_{\infty,\infty}$, and $\phi \in \mathcal{S}$, then we have

$$\sum_{j\in\mathbb{Z}} 4^{js}\|\Delta_j f S_j\phi\|_2^2 \leq C\sum_{j\in\mathbb{Z}} 4^{js}\|S_j\phi\|_2^2 \leq C\|\phi\|^2_{\dot{H}^s}\ .$$

■

We now want to consider the case $\alpha \neq 0$, and will first prove that the spaces $\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$ are decreasing when s increases.

Lemma 6.2 *Let $\alpha, s, t \in \mathbb{R}$ with $s > t$. Then*

$$\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha}) \hookrightarrow \mathbf{\Pi}(\dot{H}^t, \dot{H}^{t+\alpha})\ .$$

Proof : Let $f \in \mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$ and $\phi, \psi \in \mathcal{S}$; we intend to prove that

$$|<\Pi(f,\psi), \phi>| = \left|<\sum_{j\in\mathbb{Z}} \Delta_j f S_j\psi\,,\ \phi>\right| \leq C\|\psi\|_{\dot{H}^t}\|\phi\|_{\dot{H}^{-\alpha-t}}\ .$$

The Fourier transform of $\Delta_j f S_j \psi$ is localised in an annulus $\mathcal{C}(0, A2^j, B2^j)$. To use this fact, we define a new operator $\widetilde{\Delta}_j$ by

$$\widetilde{\Psi} \in \mathcal{S}$$
$$\widehat{\widetilde{\Psi}} \text{ is supported in an annulus and equal to 1 on } \mathcal{C}(A2^j, B2^j)$$
$$\widetilde{\Delta}_j = \widehat{\widetilde{\Psi}}(2^{-j}D) \ .$$

With this new definition, we get that

$$\begin{aligned} | < \Pi(f,\psi), \phi > | &\leq \sum_{j \in \mathbb{Z}\,,\, k<j} \left| < \Delta_j f \Delta_k \psi \,,\, \widetilde{\Delta}_j \phi > \right| \\ &\leq \left(\sum_{j\,,\, k<j} \|\Delta_j f \Delta_k \psi\|_2^2 4^{(t-s)k} 4^{(s+\alpha)j} \right)^{1/2} \left(\sum_{j\,,\, k<j} \|\widetilde{\Delta}_j \phi\|_2^2 4^{(s-t)k} 4^{(-s-\alpha)j} \right)^{1/2} \\ &\leq \left(\sum_k 4^{(t-s)k} \sum_{j>k} \|\Delta_j f \Delta_k \psi\|_2^2 4^{(s+\alpha)j} \right)^{1/2} \left(\sum_j 4^{(-s-\alpha)j} \|\widetilde{\Delta}_j \phi\|_2^2 \sum_{k<j} 4^{(s-t)k} \right)^{1/2} \ . \end{aligned}$$

So far, we have only used the Schwarz inequality. We will now exploit the fact that $f \in \mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$. Since $s > t$, we have

$$\begin{aligned} &| < \Pi(f,\psi), \phi > | \\ &\qquad \leq C \left(\sum_k 4^{(t-s)k} \|\Pi(f, \Delta_k \psi)\|_{\dot{H}^{s+\alpha}} \right)^{1/2} \left(\sum_j 4^{(-s-\alpha)j} \|\widetilde{\Delta}_j \phi\|_2^2 4^{(s-t)j} \right)^{1/2} \\ &\qquad \leq C \left(\sum_k 4^{(t-s)k} \|\Delta_k \psi\|_2^2 4^{sk} \right)^{1/2} \left(\sum_j 4^{(-t-\alpha)j} \|\widetilde{\Delta}_j \phi\|_2^2 \right)^{1/2} \\ &\qquad \leq C \|\psi\|_{\dot{H}^t} \|\phi\|_{\dot{H}^{-\alpha-t}} \ . \end{aligned}$$

This concludes the proof of the lemma. ■

We are now in a position to prove a technical lemma, analogous to Proposition 3 of [47]. The idea is to show that if a function belongs to $\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$, then the operator

$$\phi \mapsto \sum_j R_j f S_j \phi$$

inherits the boundedness property (from $\dot{H}^s$ to $\dot{H}^{s+\alpha}$) of $\phi \mapsto \sum_j \Delta_j f S_j \phi$, provided the R_j are smooth Fourier multipliers supported inside annuli.
This lemma will then enable us to describe all the spaces $\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$, using only the BMO^s spaces.

Lemma 6.3 *Let $h \in \mathcal{S}$ be such that* $\operatorname{Supp} h \subset \{\gamma^{-1} \leq |\xi| \leq \gamma\}$ *with $\gamma > 1$. Define for all $j \in \mathbb{Z}$*

$$R_j = h(2^{-j}D) \ .$$

Consider also $f \in \mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$*, with* s *and* α *in* $\mathbb{R}$*. There exists then a constant* C *such that for any* $\phi \in \mathcal{S}_\infty$*,*

$$\sum_{j\in\mathbb{Z}} \|R_j f S_j \phi\|_2^2 4^{j(s+\alpha)} \leq C\|\phi\|_{\dot{H}^s}^2 \ .$$

Proof : **1.** First, take f in $\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$ and let $N \geq 0$ be such that

$$R_j = R_j\left(\sum_{\nu=-N}^{N} \Delta_{j+\nu}\right) \quad \text{and} \quad R_j\left(\sum_{\nu=-N}^{N} \Delta_{j+\nu} f S_{j+\nu}\phi\right) = R_j\left(\sum_{\nu=-\infty}^{\infty} \Delta_{j+\nu} f S_{j+\nu}\phi\right) \ ,$$

for any $\phi \in \mathcal{S}_\infty$. Choosing N which fulfills the two above conditions is possible, due to the spectral localisation of the operators Δ_j, S_j and R_j.
We define then

$$\begin{cases} X_j(\phi) = R_j f S_j \phi \\ Y_j(\phi) = \displaystyle\sum_{\nu=-N}^{N} R_j(\Delta_{j+\nu} f S_{j+\nu}\phi) = R_j(\Pi(f,\phi)) \ . \end{cases}$$

Still using the spectral localisation of the R_j on the one hand, and the belonging of $\Pi(f,\cdot)$ to $\mathcal{L}(\dot{H}^s, \dot{H}^{s+\alpha})$ on the other hand, we obtain

$$\sum_{j\in\mathbb{Z}} 4^{j(s+\alpha)} \|Y_j(\phi)\|_2^2 \leq C\|\phi\|_{\dot{H}^s}^2 \ ,$$

and consequently we just have to show that

$$\sum_{j\in\mathbb{Z}} 4^{j(s+\alpha)} \|X_j(\phi) - Y_j(\phi)\|_2^2 \leq C\|\phi\|_{\dot{H}^s}^2 \ .$$

Finally, we can write $Y_j(\phi) - X_j(\phi) = A_j(\phi) + B_j(\phi)$, with

$$\begin{cases} A_j(\phi) = \displaystyle\sum_{\nu=-N}^{N} \left[R_j(\Delta_{j+\nu} f S_{j+\nu}\phi) - R_j(\Delta_{j+\nu} f) S_{j+\nu}\phi\right] \\ B_j(\phi) = \displaystyle\sum_{\nu=-N}^{N} R_j(\Delta_{j+\nu} f)\left[S_{j+\nu}\phi - S_j\phi\right] \ . \end{cases}$$

2. The term B_j is the easier to treat : since $f \in \dot{B}^\alpha_{\infty,\infty}$ and $S_{j+\nu}\phi - S_j\phi = \displaystyle\sum_{k=j+1}^{j+\nu} \Delta_k \phi$ (for $\nu > 0$, with a symmetrical formula in the case $\nu < 0$), we have, for a constant C depending on N and f,

$$\|B_j(\phi)\|_2^2 \leq C 4^{-j\alpha} \sum_{\nu=-N}^{N} \|\Delta_{j+\nu}\phi\|_2^2 \ ,$$

which implies

$$\sum_j 4^{j(s+\alpha)} \|B_j\phi\|_2^2 \leq C\|\phi\|_{\dot{H}^s}^2 \ .$$

3. Writing

$$\phi_j = S_j\phi \ , \ f_j = \Delta_j f \ , \ H_j = 2^{dj}\widehat{h}(2^j\cdot)$$

we get that

$$\begin{aligned}
A_j(\phi)(x) &= \sum_{\nu=-N}^{N} \int_{\mathbb{R}^d} H_j(x-y)(\phi_{j+\nu}(x) - \phi_{j+\nu}(y)) f_{j+\nu}(y)dy \\
&= \sum_{\nu=-N}^{N} \int_{\mathbb{R}^d} H_j(x-y)\left(\phi_{j+\nu}(x) - \phi_{j+\nu}(y) - \sum_{i=1}^{d}(x_i - y_i)\partial_i\phi_{j+\nu}(y)\right) f_{j+\nu}(y)dy \\
&\qquad + \int_{\mathbb{R}^d} H_j(x-y)\sum_{i=1}^{d}(x_i-y_i)\partial_i\phi_{j+\nu}(y) f_{j+\nu}(y)dy \\
&\overset{\text{def}}{=} \sum_{\nu=-N}^{N} I_{j,\nu}(x) + II_{j,\nu}(x) \ .
\end{aligned}$$

4. First, to estimate I_j, we will consider only the case $\nu = 0$; the other cases are identical. Besides, we will only treat the case where $s < 2$; the cases where s is larger than 2 can be handled in the same way, but the Taylor expansion we need to use has then to be developped up to terms including derivatives of larger order. If $s < 2$, applying Taylor's formula of order 2, we see that

$$\begin{aligned}
|I_j(x)| &= \left| \int_{\mathbb{R}^d}\int_0^1 H_j(x-y)\sum_{i,k=1}^{d}(x_i-y_i)(x_k-y_k)(1-t)\partial_{ik}\phi_j(y+t(x-y))f_j(y)\,dt\,dy \right| \\
&\le \|f_j\|_\infty \int_{\mathbb{R}^d}\int_0^1 \left| H_j(z)\sum_{i,k=1}^{d} z_i z_k (1-t)\partial_{ik}\ \phi_j(x+(t-1)z)\right| dt\,dz \ .
\end{aligned}$$

Therefore, recalling that $f \in \dot{B}^{\alpha}_{\infty,\infty}$, we find

$$\begin{aligned}
\|I_j\|_2 &\le C2^{-j\alpha}\int_{\mathbb{R}^d}\int_0^1 \left| H_j(z)\sum_{i,k=1}^{d} z_i z_k\right| \|\partial_{ik}\ \phi_j(x+(t-1)z)\|_{L^2_x}\,dt\,dz \\
&\le C2^{j(-\alpha-2)}\|\partial^2\phi_j\|_2
\end{aligned}$$

since $\int_{\mathbb{R}^d} |H_j(z)z_iz_k|\,dz = C2^{-2j}$. Now we can sum over j :

$$\begin{aligned}
\sum_j 4^{j(s+\alpha)}\|I_j\|_2^2 &\le C\sum_j 4^{j(s-2)}\|\partial^2\phi_j\|_2^2 \le C\sum_{j\,,\,k<j} 4^{j(s-2)}4^{2k}\|\Delta_k\phi\|_2^2 \\
&= C\sum_k 4^{2k}\|\Delta_k\phi\|_2^2\sum_{j>k}4^{j(s-2)} = C\sum_k 4^{ks}\|\Delta_k\phi\|_2^2 \le C\|\phi\|^2_{\dot{H}^s} \ .
\end{aligned}$$

5. We are left with $\sum_\nu II_{j,\nu}$. It may be written as

$$\sum_\nu II_{j,\nu}(x) = 2^{-j}\sum_{\nu=-N}^{N}\sum_{i=1}^{d}\widetilde{\Delta}^i_j(S_{j+\nu}\partial_i\phi\Delta_{j+\nu}f) \ ,$$

where $\widetilde{\Delta}_j^i$ is the convolution operator whose kernel reads $x \mapsto 2^j x_i H_j(x)$ and whose symbol is of the form $F_i(2^{-j}\xi)$, with $F_i \in \mathcal{S}$. In the following, we shall forget the index i, in order to keep notations as light as possible.
Summing over j, and keeping in mind that, according to lemma 6.2, $\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha}) \hookrightarrow \mathbf{\Pi}(\dot{H}^{s-1}, \dot{H}^{s+\alpha-1})$, we get

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 4^{j(s+\alpha)} \left\| \sum_{\nu=-N}^{N} II_{j,\nu} \right\|_2^2 &= \sum_{j\in\mathbb{Z}} 4^{j(s+\alpha-1)} \left\| \widetilde{\Delta}_j \left(\sum_{\nu=-N}^{N} \Delta_{j+\nu} f S_{j+\nu} \partial_i \phi \right) \right\|_2^2 \\ &= \sum_j 4^{j(s+\alpha-1)} \|\widetilde{\Delta}_j \Pi(f, \partial_i\phi)\|_2^2 \le C \|\Pi(f,\partial_i\phi)\|^2_{\dot{H}^{s+\alpha-1}} \\ &\le C\|f\|_{\mathbf{\Pi}(\dot{H}^{s-1},\dot{H}^{s+\alpha-1})} \|\partial_i \phi\|^2_{\dot{H}^{s-1}} \le C\|\phi\|^2_{\dot{H}^s}\end{aligned}$$

This ends the proof of the lemma. ■

The following theorem is a straightforward consequence of the previous lemma. It shows that all the $\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$ spaces can be deduced from the BMO^s spaces.

Theorem 6.4 *Let s and α in $\mathbb{R}$. We denote by Λ the Calderón operator, $\Lambda = |D|$. Then*

$$\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha}) = \Lambda^{-\alpha} BMO^s \ .$$

Proof : Recall that $\mathbf{\Pi}(\dot{H}^s, \dot{H}^s) = BMO^s$. Hence, it suffices to prove that

$$\Lambda^\alpha \mathbf{\Pi}(\dot{H}^s, \dot{H}^t) \hookrightarrow \mathbf{\Pi}(\dot{H}^s, \dot{H}^{t-\alpha})$$

for any s, t and α in $\mathbb{R}$. Let $f \in \mathbf{\Pi}(\dot{H}^s, \dot{H}^t)$. Let

$$R_j = \Delta_j \Lambda^\alpha 2^{-j\alpha} \ ;$$

the symbol of this operator reads $\psi(2^{-j}\xi)2^{-j\alpha}|\xi|^\alpha$. We can apply Lemma 6.3 to get

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 4^{j(t-\alpha)} \|\Delta_j \Lambda^\alpha f S_j \phi\|_2^2 &= \sum_{j\in\mathbb{Z}} 4^{jt} \|R_j f S_j \phi\|_2^2 \\ &\le C\|\phi\|^2_{\dot{H}^s} \ ,\end{aligned}$$

which implies the theorem. ■

6.2 Study of the spaces $\mathbf{R}(\dot{H}^s, \dot{H}^{s+\alpha})$

From now on, we let s and $s+\alpha$ belong to $(-\frac{d}{2}, \frac{d}{2})$.
The study of the spaces $\mathbf{R}(\dot{H}^s, \dot{H}^{s+\alpha})$ reduces to the study of the spaces $\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha})$ since the operator $R(f, \cdot)$ is almost the transpose operator of $\Pi(f, \cdot)$. This observation gives the following proposition.

Proposition 6.5 *Let s and t belong to $(-\dfrac{d}{2}, \dfrac{d}{2})$. One has then*

$$\mathbf{R}(\dot{H}^s, \dot{H}^t) = \mathbf{\Pi}(\dot{H}^{-t}, \dot{H}^{-s}) \ .$$

In other words,

$$\mathbf{R}(\dot{H}^s, \dot{H}^t) = \Lambda^{s-t} BMO^{-t} \ .$$

Proof : Let $f \in \dot{B}^{t-s}_{\infty,\infty}$. By definition,

(21) $$f \in \mathbf{R}(\dot{H}^s, \dot{H}^t) \iff \langle R(f,\phi), \psi\rangle \leq C\|\phi\|_{\dot{H}^s}\|\psi\|_{\dot{H}^{-t}} \text{ for any } \phi \text{ and } \psi \text{ in } \mathcal{S} \ .$$

Let us denote $\widetilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$. Due to the spectral localisation of the Littlewood-Paley operators, there exists a $N \geq 0$ such that

$$\langle R(f,\phi), \psi\rangle = \langle \sum_{j\in\mathbb{Z}} \Delta_j f \widetilde{\Delta}_j \phi \, , \, \psi\rangle = \sum_j \langle \Delta_j f \widetilde{\Delta}_j \phi \, , \, S_{j+N}\psi\rangle = \sum_j \langle \Delta_j f S_{j+N}\psi \, , \, \widetilde{\Delta}_j \phi\rangle \ .$$

From now on, we denote by $A(\phi,\psi)$ any bilinear operator such that

$$|A(\phi,\psi)| \leq C\|\phi\|_{\dot{H}^s}\|\psi\|_{\dot{H}^{-t}} \ .$$

It is easy to see that, for any integers M and N,

$$\left|\sum_j \langle \Delta_j f (S_{j+N} - S_{j-M})\psi \, , \, \widetilde{\Delta}_j \phi\rangle\right| \leq C\|f\|_{\dot{B}^{t-s}_{\infty,\infty}}\|\phi\|_{\dot{H}^s}\|\psi\|_{\dot{H}^{-t}} \ ,$$

therefore, if M is an integer,

$$\langle R(f,\phi), \psi\rangle = \sum_j \langle \Delta_j f S_{j-M}\psi \, , \, \widetilde{\Delta}_j \phi\rangle + A(\phi,\psi) \ .$$

Using once again the spectral localisation of the Littlewood-Paley operators, we see that for M large enough, $\langle \Delta_j f S_{j-M}\psi \, , \, \widetilde{\Delta}_j \phi\rangle = \langle \Delta_j f S_{j-M}\psi \, , \, \phi\rangle$, which implies

$$\langle R(f,\phi), \psi\rangle = \langle \sum_j \Delta_j f S_{j-M}\psi \, , \, \phi\rangle + A(\phi,\psi) \ .$$

We observe now that, still because of the spectral localisation of the Δ_j and S_j,

$$\left|\langle \sum_j \Delta_j f (S_j - S_{j-M})\psi \, , \, \phi\rangle\right| \leq C\|f\|_{\dot{B}^{t-s}_{\infty,\infty}}\|\phi\|_{\dot{H}^s}\|\psi\|_{\dot{H}^{-t}} \ ,$$

and this implies

$$\langle R(f,\phi), \psi\rangle = \langle \sum_j \Delta_j f S_j\psi \, , \, \phi\rangle + A(\phi,\psi) = \langle \Pi(f,\psi), \phi\rangle + A(\phi,\psi) \ .$$

This proves the first assertion of the proposition. The second assertion is a consequence of Theorem 6.4. ■

6.3 Study of the spaces $\widetilde{\mathbf{\Pi}}(\dot{H}^s, \dot{H}^{s+\alpha})$

These spaces are easily described. Recall that they are defined only for $\alpha \leq 0$, and that they are equal to L^∞ for $\alpha = 0$.

Proposition 6.6 *Let $s \in \mathbb{R}$ and $\alpha < 0$. Then*

$$\widetilde{\mathbf{\Pi}}(\dot{H}^s, \dot{H}^{s+\alpha}) = \dot{B}^{\alpha}_{\infty,\infty} \ .$$

Proof : Consider $f \in \dot{B}^{\alpha}_{\infty,\infty}$, and $\phi \in \mathcal{S}$.

$$\sum_{j\in\mathbb{Z}} 4^{j(s+\alpha)}\|S_j f \Delta_j \phi\|_2^2 \leq C\sum_{j\in\mathbb{Z}} 4^{j\alpha}\|S_j f\|_\infty^2 4^{js}\|\Delta_j \phi\|_2^2 \leq C\|\phi\|^2_{\dot{H}^s} \ ,$$

which proves the proposition. ■

6.4 Study of the spaces $\mathbf{M}(\dot H^s, \dot H^{s+\alpha})$

We shall study in this paragraph functions f such that the operator M_f maps a given Sobolev space into another given Sobolev space.

Proposition 6.7 (Elementary properties of multipliers) *Let $r \geq s \geq t$ be real numbers in $(-\frac{d}{2}, \frac{d}{2})$, and α be a non-positive real number. One has then*

(i) $\mathbf{M}(\dot H^r, \dot H^s) = \mathbf{M}(\dot H^{-s}, \dot H^{-r})$.

(ii) Let furthermore $f \in \mathbf{M}(\dot H^r, \dot H^s)$, $g \in \mathbf{M}(\dot H^s, \dot H^t)$ *Then* $h = fg \in \mathbf{M}(\dot H^r, \dot H^t)$

(iii) If $r \geq s \geq -\alpha/2 \geq 0$, $\mathbf{M}(\dot H^r, \dot H^{r+\alpha}) \hookrightarrow \mathbf{M}(\dot H^s, \dot H^{s+\alpha})$.

(iv) If $s+\alpha \in (-\frac{d}{2}, \frac{d}{2})$, $\mathbf{M}(\dot H^s, \dot H^{s+\alpha}) \hookrightarrow \dot B^{\alpha}_{\infty,\infty}$.

(v) If $f \in M(\dot H^r, \dot H^s)$, *then* $\|\chi * f\|_{\mathbf{M}(\dot H^r, \dot H^s)} \leq C\|\chi\|_1 \|f\|_{\mathbf{M}(\dot H^r, \dot H^s)}$.

PROOF :
(i) follows by duality

(ii) is obvious.

(iii) To prove this point, take $f \in \mathbf{M}(\dot H^r, \dot H^{r+\alpha})$. By (i), we know that f belongs also to $\mathbf{M}(\dot H^{-r-\alpha}, \dot H^{-r})$. In other words, $M_f \in \mathcal{L}(\dot H^r, \dot H^{r+\alpha}) \cup \mathcal{L}(\dot H^{-r-\alpha}, \dot H^{-r})$. Since $r \geq s \geq -\alpha/2 \geq 0$, we have

$$-r-\alpha \leq s \leq r \qquad \text{and} \qquad -r \leq s+\alpha \leq r+\alpha \ ,$$

so by complex interpolation, $M_f \in \mathcal{L}(\dot H^s, \dot H^{s+\alpha})$.

(iv) To prove this embedding, observe that if $\phi \in \mathcal{S}$, it may be written as

$$\phi = \sum_{n\in\mathbb{Z}} g_n h_n \text{ with } \sum_{n\in\mathbb{Z}} \|g_n\|_{\dot H^s} \|h_n\|_{\dot H^{-s-\alpha}} < \infty \ .$$

(To obtain such a decomposition, it suffices to consider a smooth non-homogeneous partition of unity in dyadic annuli $1 = \sum_{j\geq 0} \lambda_j$ and then to write $\phi = \sum_{j\geq 0} \lambda_j \widetilde{\lambda_j} \phi$, where $\lambda_j \widetilde{\lambda_j} = \lambda_j$.)
Now let $f \in \mathbf{M}(\dot H^s, \dot H^{s+\alpha})$, and ϕ be the convolution kernel associated with the Littlewood-Paley decomposition, which we decompose as a sum as above. Then if $x \in \mathbb{R}^d$,

$$\begin{aligned}|\Delta_j f(x)| = \left| \int_{\mathbb{R}^d} 2^{jd} f(x-y)\phi(2^j y)\, dy \right| &\leq 2^{jd} \left\| f(x-\cdot) \sum_{n\in\mathbb{Z}} g_n(2^j\cdot) h_n(2^j\cdot) \right\|_1 \\ &\leq 2^{jd} \|f\|_{\mathbf{M}(\dot H^s, \dot H^{s+\alpha})} \sum_n \|g_n(2^j\cdot)\|_{\dot H^s} \|h_n(2^j\cdot)\|_{\dot H^{-s-\alpha}} \leq C2^{-j\alpha} \ .\end{aligned}$$

(v) Let $f \in \mathbf{M}(\dot H^r, \dot H^s)$, and $\chi, \phi, \psi \in \mathcal{S}$. Then

$$\langle \chi * f, \phi\psi \rangle = -\langle f * (\phi\psi), \chi(-\cdot) \rangle \ .$$

By definition of $\mathbf{M}(\dot H^r, \dot H^s)$, we find that

$$|\langle \chi * f, \phi\psi \rangle| \leq \|f\|_{\mathbf{M}(\dot H^r, \dot H^s)} \|\phi\|_{\dot H^r} \|\psi\|_{\dot H^{-s}} \|\chi\|_1 \ .$$

This proves (v) and the proposition. ■

We would like now to describe the multiplier spaces $\mathbf{M}(\dot H^s, \dot H^{s+\alpha})$ with the help of the results that we have proved about the paramultiplier spaces $\mathbf{\Pi}(\dot H^s, \dot H^{s+\alpha})$, $\mathbf{R}(\dot H^s, \dot H^{s+\alpha})$, and $\widetilde{\mathbf{\Pi}}(\dot H^s, \dot H^{s+\alpha})$. Matters would be easy if we could affirm

$$\text{“}M_f \in \mathcal{L}(\dot H^s, \dot H^{s+\alpha}) \quad \Longleftrightarrow \quad \Pi(f,\cdot),\ R(f,\cdot) \text{ and } \widetilde{\Pi}(f,\cdot) \in \mathcal{L}(\dot H^s, \dot H^{s+\alpha})\text{”} \ .$$

Unfortunately, we do not know whether this statement is always true or not. The $\Leftarrow$ part is obviously true, which gives the embedding

$$\mathbf{\Pi}(\dot H^s, \dot H^{s+\alpha}) \cap \mathbf{R}(\dot H^s, \dot H^{s+\alpha}) \cap \widetilde{\mathbf{\Pi}}(\dot H^s, \dot H^{s+\alpha}) \subset \mathbf{M}(\dot H^s, \dot H^{s+\alpha}) \ , \tag{22}$$

but the converse embedding is not clear : there might be compensations between the operators $\Pi(f,\cdot)$, $R(f,\cdot)$ and $\widetilde{\Pi}(f,\cdot)$ which make their sum bounded while each of them is not bounded.
The following lemma, which we borrow from Gala and Lemarié [18], will enable us to prove the converse embedding in (22) in most cases.

Lemma 6.8 ([18]) *Take $s \in (0, d/2)$, and $t \in (-s, s)$. Then*

$$\mathbf{M}(\dot H^s, \dot H^t) \hookrightarrow \mathbf{\Pi}(\dot H^s, \dot H^t) \ .$$

PROOF : Take $f \in \mathbf{M}(\dot H^s, \dot H^t)$. Since by Theorem 6.4 $\mathbf{M}(\dot H^s, \dot H^t) \hookrightarrow \dot B^{t-s}_{\infty,\infty}$, we have $f \in \widetilde{\Pi}(\dot H^s, \dot H^t)$. On the other hand, if $\phi \in \mathcal{S}$,

$$\begin{aligned}\|\Pi(f,\phi)\|^2_{\dot H^{-s}} &\le C \sum_{j\in\mathbb{Z}} 4^{-js} \|\Delta_j f S_j \phi\|_2^2 \le C \sum_j 4^{-j(s+t)} \|\Delta_j f S_j\phi\|^2_{\dot H^t} \\ &\le C \sum_j 4^{-j(s+t)} \|S_j \phi\|^2_{\dot H^s} \ ,\end{aligned}$$

since, by Proposition 6.7 point (v), $\|\Delta_j f\|_{\mathbf{M}(\dot H^s, \dot H^t)} \le C \|f\|_{\mathbf{M}(\dot H^s,\dot H^t)}$. It is now easy to end the computation.

$$\|\Pi(f,\phi)\|^2_{\dot H^{-s}} \le C \sum_{j\in\mathbb{Z}} 4^{-j(s+t)} \sum_{k<j} 4^{ks} \|\Delta_k \phi\|_2^2 \le C \|\phi\|^2_{\dot H^{-t}} \ ,$$

because $s+t>0$. Hence $f \in \Pi(\dot H^{-t}, \dot H^{-s})$, and Proposition 6.5 gives that $f \in R(\dot H^s, \dot H^t)$.
We can now conclude :

$$\Pi(f,\cdot) = M_f - R(f,\cdot) - \tilde\Pi(f,\cdot)$$

belongs to $\mathcal{L}(\dot H^s, \dot H^t)$. ■

And as a consequence of this lemma and of other results proved above, we can describe the spaces $\mathbf{M}(\dot H^s, \dot H^{s+\alpha})$ in most cases.

Proposition 6.9 *Let α and s be two real numbers such that s and $s+\alpha$ belong to $(-\frac{d}{2}, \frac{d}{2})$. One has then*

(i) $\mathbf{M}(\dot H^s, \dot H^{s+\alpha}) = \{0\}$ *if $\alpha > 0$.*

(ii) $\mathbf{M}(\dot H^s, \dot H^s) = BMO^{|s|} \cap L^\infty$.

(iii) $\mathbf{M}(\dot H^s, \dot H^{s+\alpha}) = \Lambda^{-\alpha} BMO^{\max(s,-s-\alpha)}$ *if $\alpha < 0$ and $2s+\alpha \neq 0$.*

PROOF : *(i)* We shall prove this assertion by contradiction. Take $s \in \mathbb{R}$, $\alpha > 0$, and $f \neq 0$ in $\mathbf{M}(\dot{H}^s, \dot{H}^{s+\alpha})$.
By point *(v)* of Proposition 6.7, if one convolves f with a function of $\mathcal{D}$, one obtains a $\mathcal{C}^\infty$ function, different of 0, and belonging to $\mathbf{M}(\dot{H}^s, \dot{H}^{s+\alpha})$. So we might as well suppose that f belongs to $\mathcal{C}^\infty$.
Furthermore, $\mathcal{D}$ is included in $\mathbf{M}(\dot{H}^s, \dot{H}^s)$, thus, by point *(ii)* of Proposition 6.7, the pointwise product of f with a function of $\mathcal{D}$ still belongs to $\mathbf{M}(\dot{H}^s, \dot{H}^s)$. For this reason, we can assume that f belongs to $\mathcal{D}$.
Finally, by translation invariance, we can suppose that $f(0) = 1$.
Now take ϕ in $\mathcal{C}^\infty$ such that

$$\phi(\xi) = |\xi|^{-s-d/2-\epsilon} \quad \text{if } |\xi| > 1 \ ,$$

and consider $\lambda = \widehat{\phi}$, which belongs obviously to $\dot{H}^s$. The pointwise product $f\lambda$ is well defined, and its Fourier transform reads $\widehat{f} * \phi$. Since $\widehat{f}$ belongs to $\mathcal{S}$ and verifies $\displaystyle\int \widehat{f} = 1$, the following equivalent holds, for $\xi \to \infty$,

$$\mathcal{F}(f\lambda)(\xi) = (\widehat{f} * \phi)(\xi) \sim |\xi|^{-s-d/2-\epsilon} \ .$$

If we choose $\epsilon < \alpha$, this proves that $f\lambda$ does not belong to $\dot{H}^{s+\alpha}$, yielding a contradiction.

(ii) This point is proved in [46].

(iii) Suppose first $2s + \alpha > 0$. We know that

$$\mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha}) \cap \mathbf{R}(\dot{H}^s, \dot{H}^{s+\alpha}) \cap \widetilde{\mathbf{\Pi}}(\dot{H}^s, \dot{H}^{s+\alpha}) \hookrightarrow \mathbf{M}(\dot{H}^s, \dot{H}^{s+\alpha}) \ ,$$

and we have proved in Theorems 6.4, Proposition 6.5 and Proposition 6.6 that, for $\alpha < 0$,

$$\begin{aligned} \mathbf{\Pi}(\dot{H}^s, \dot{H}^{s+\alpha}) &= \Lambda^{-\alpha} BMO^s \\ \mathbf{R}(\dot{H}^s, \dot{H}^{s+\alpha}) &= \Lambda^{-\alpha} BMO^{-s-\alpha} \\ \widetilde{\mathbf{\Pi}}(\dot{H}^s, \dot{H}^{s+\alpha}) &= \dot{B}^{\alpha}_{\infty,\infty} \ . \end{aligned} \tag{23}$$

We now just have to remember that $BMO^\alpha \hookrightarrow BMO^\beta$ for $\alpha > \beta$ to prove that

$$\Lambda^{-\alpha} BMO^s \hookrightarrow \mathbf{M}(\dot{H}^s, \dot{H}^{s+\alpha}) \ ;$$

the converse embedding is provided by Lemma 6.8.

The case $2s + \alpha < 0$ is nothing but the dual of the case $2s + \alpha > 0$. ∎

As appears in the statement of the preceding theorem, there were some cases where we have not satisfactorily described the space $\mathbf{M}(\dot{H}^s, \dot{H}^{s+\alpha})$, namely if $2s + \alpha = 0$.

For certain values of s, this case has been settled by Maz'ya and Verbitsky, using methods of potential theory.

Theorem 6.10 (Maz'ya, Verbitsky [30] [31]) *If* $s = 1/2$ *or* 1,

$$\mathbf{M}(\dot{H}^s, \dot{H}^{-s}) = \Lambda^{2s} BMO^s \ .$$

Putting together the results of Theorem 6.4, Proposition 6.5, Proposition 6.6, Proposition 6.9 and Theorem 6.10, we get Theorem 3.9.

Finally, we state another result of Maz'ya and Verbitsky [29] (lemma 3.1.), which we use in our treatment of the Navier-Stokes equations.

Theorem 6.11 (Maz'ya, Verbitsky [29]) *The Riesz transforms* $R_j = \partial_j(-\Delta)^{-1/2}$ *are bounded on* X_s *for* $s \in (0, d/2)$.

6.5 Comparison of multiplier and paramultiplier spaces with more classical functional spaces

In this section, we study the embeddings between multiplier and paramultiplier spaces, and other functional spaces : Lebesgue, Lorentz, Sobolev, Besov, and Morrey spaces. A few facts about Morrey spaces are recalled in the appendix.
We begin with the space $X_0 = BMO$, for which the following results are classical (see [5] and [44]).

Proposition 6.12 *The following embeddings hold :*
(i) (Lebesgue spaces) $L^\infty \hookrightarrow BMO$
(ii) (Sobolev spaces) If $p \in (1, \infty)$, $W^{d/p,p} \hookrightarrow BMO$
(ii) (Besov spaces) $\dot{B}^0_{\infty,2} \hookrightarrow BMO \hookrightarrow \dot{B}^0_{\infty,\infty}$

We now come to the X_s, with $s > 0$.

Proposition 6.13 *Let* $s \in (0, \frac{d}{2})$*. The following embeddings hold*
(i) (Lebesgue and Lorentz spaces) $L^{d/s} \hookrightarrow L^{d/s,\infty} \hookrightarrow X_s$
$$X_s \hookrightarrow L^2_{\mathrm{loc}}$$
(ii) (Morrey spaces) If $p \in (2, \dfrac{d}{s}]$, $M^{p,d/s} \hookrightarrow X_s \hookrightarrow M^{2,d/s}$ *and the last embedding is strict.*
(iii) (Besov spaces) $\dot{B}^{-s+d/p}_{p,\infty} \hookrightarrow X_s$ *provided* $p < \frac{d}{s}$*.*
$\dot{B}^0_{d/s,2} \hookrightarrow X_s$
$\dot{B}^0_{d/s,\infty}$ *and* X_s *are not comparable.*
$X_s \hookrightarrow \dot{B}^{-s}_{\infty,\infty}$

Proof : (i) The first embedding follows from the sharp Sobolev embedding $\dot{H}^s \hookrightarrow L^{\frac{2d}{d-2s},2}$. To prove the second embedding, take a compact set K, and $\phi \in \mathcal{S}$ such that $\phi = 1$ on K. Then if $f \in X_s$, $\phi f \in L^2$, and this yields the result.

(ii) The first embedding is a deeper result ; it can be easily deduced by duality of the lemma 7.9 given in the appendix. The second (strict) embedding is proved in [25].

(iii) Let $f \in \dot{B}^{-s+d/p}_{p,\infty}$, with $p < d/s$. Thanks to Theorem 3.9, the first embdeding of (iii) will be proved if we show that $\Pi(f, \cdot) \in \mathcal{L}(\dot{H}^s, L^2)$. Take $\phi \in \dot{H}^s$; by Sobolev embedding, $\dot{H}^{d/p} \hookrightarrow L^{\frac{2p}{p-2}}$, so we have

$$\begin{aligned}
\|\Pi(f,\phi)\|^2_{L^2} &\le C \sum_{j \in \mathbb{Z}} \|\Delta_j f S_j \phi\|_2^2 \le C \sum_j \|\Delta_j f\|_p^2 \|S_j f\|^2_{\frac{2p}{p-2}} \\
&\le C \sum_j 4^{j(s-d/p)} \|S_j \phi\|^2_{\dot{H}^{d/p}} \\
&\le C \sum_{j,k<j} 4^{j(s-d/p)} 4^{kd/p} \|\Delta_k \phi\|_2^2 \\
&\le C \|\phi\|^2_{\dot{H}^s} \ .
\end{aligned}$$

Let now $f \in \dot{B}^0_{d/s,2}$; to prove the second embedding of (iii), we will also show that

$\Pi(f,\cdot) \in \mathcal{L}(\dot{H}^s, L^2)$. Using the Sobolev embedding $\dot{H}^s \hookrightarrow L^{\frac{2d}{d-2s}}$, we have

$$\begin{aligned} \|\Pi(f,\phi)\|_{L^2}^2 &\leq C \sum_{j\in\mathbb{Z}} \|\Delta_j f S_j \phi\|_2^2 \leq C \sum_j \|\Delta_j f\|_{d/s}^2 \|S_j \phi\|_{\frac{2d}{d-2s}}^2 \\ &\leq C\|\phi\|_{\dot{H}^s}^2 \sum_j \|\Delta_j f\|_{d/s}^2 \leq C\|\phi\|_{\dot{H}^s}^2 \end{aligned}$$

Remark that the embedding $\dot{B}^0_{d/s,2} \hookrightarrow X_s$ could also have been proved using (i) and the Littlewood-Paley theorem : $\dot{B}^0_{d/s,2} \hookrightarrow L^{d/s}$.

To see that $X_s \hookrightarrow \dot{B}^0_{d/s,\infty}$ does not hold, it suffices to construct a function belonging to $L^{d/s,\infty} \setminus \dot{B}^0_{d/s,\infty}$ and to use (i). How can we build up such an example ?
It actually suffices to construct a function belonging to $L^{d/s,\infty}$ with its Fourier transform supported in a compact set disjoint from 0, and this is not hard to do.

Finally, we would like to show that $\dot{B}^0_{d/s,\infty} \hookrightarrow X_s$ does not hold either. We will follow an argument of Bourdaud [6]. Define

$$f = \sum_{j\geq 0} e^{i2^j x_1} \phi(x) \ ,$$

where ϕ belongs to the Schwartz class, and has a Fourier transform supported in $B(0,1)$. It is well-known that f is not a Radon measure ; and it is obvious that $f \in \dot{B}^0_{d/s,\infty}$. Since f is not a Radon measure, it cannot belong to L^2_{loc}, and therefore not to X_s.

The last embedding of the proposition has already been proved in Proposition 6.7. ■

Remark 6.14 *Since for $s > 0$ Λ^s is an isomorphism from BMO^s to X_s, the above embeddings imply corresponding embeddings for the BMO^s.*

6.6 Another approach : link with singular integrals and pseudodifferential operators

So far, we have studied multipliers and paramultipliers spaces using very basic tools ; this elementary approach has enabled us to describe all the relevant spaces, and to derive all the results which we needed in the application to the Navier-Stokes equations.
However, it is interesting to gain a wider prospect by connecting these spaces to the theory of pseudo-differential and of singular integral operators. We will first describe this connection, and then show how some of the results about paramultiplier spaces can be obtained with the help of the singular integral operators theory.

The problem we have been investigating in this part was to find conditions on f so that the operators $R(f,\cdot)$, $\Pi(f,\cdot)$, $\widetilde{\Pi}(f,\cdot)$, M_f be bounded from a Sobolev space $\dot{H}^s$ in $\dot{H}^{s+\alpha}$. The operators M_f are simple pointwise multiplication operators, not much more can be said about them ; the case of the operators $\widetilde{\Pi}(f,\cdot)$ has been settled very quickly ; and we have seen that $\Pi(f,\cdot)$ is almost the transpose of $R(f,\cdot)$. So let us concentrate on this last operator.
We would like to study the boundedness of this operator from $\dot{H}^s$ in $\dot{H}^{s+\alpha}$; but most of the results in the literature concern the boundedness of operators mapping some Sobolev space in itself. We therefore introduce

$$T = R(f,\cdot)\Lambda^\alpha$$

and the problem reduces to study the boundedness of T on $\dot{H}^{s+\alpha}$. If we suppose that $f \in \dot{B}^{\alpha}_{\infty,\infty}$, one can check easily that T is a pseudo-differential operator whose symbol σ verifies

$$|\partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta} |\xi|^{|\beta|-|\alpha|} .$$

Recall that σ would belong to the exotic symbol class $S^0_{1,1}$ (see [43]) if the right side of the above equality was replaced by $C(1+|\xi|)^{|\beta|-|\alpha|}$; so T belongs to a kind of homogeneous version of $S^0_{1,1}$. These estimates are not enough to grant the $\dot{H}^s$ continuity, but they imply that T is a singular integral operator (see the book of Meyer [32], page 294, for a proof of this statement).
We can therefore apply theorems similar to the $T1$ theorem of David and Journé, which have been proved by Meyer [32] and Youssfi [46] [47] ; these theorems essentially state that, for $t \in (0,1)$, T is bounded over $\dot{H}^t$ provided $T1 \in BMO^t$. With the help of these theorems, it is not hard to deduce a criterion of boundedness of T over $\dot{H}^{s+\alpha}$, and then to describe $\mathbf{R}(\dot{H}^s, H^{s+\alpha})$ for $-1 < s+\alpha < 1$. However, treating the general case (ie, for any s and α) with this method seems to be difficult ; indeed, the hypotheses of the $T1$ theorems, would then have to be supplemented with commutator conditions (see [46]) which are not very easy to manipulate.

6.7 A proof of $\mathbf{M}(\dot{H}^s, L^2) \hookrightarrow \mathbf{\Pi}(\dot{H}^s, L^2)$ based on the $\mathcal{H}^1$ - BMO duality

We have already proved above that

$$\mathbf{M}(\dot{H}^s, L^2) \hookrightarrow \mathbf{\Pi}(\dot{H}^s, L^2) ,$$

see Lemma 6.8 for the embedding and Lemma 6.3 for the equality. Notice that the converse embedding follows from the description of the paramultiplier spaces.
We shall give in this section another proof of the embedding above, based on a duality method.

Recall that if $s > 0$, we denote

$$X_s \stackrel{\text{def}}{=} \mathbf{M}(\dot{H}^s, L^2) .$$

Lemma 6.15 *Let $s \in [0,d)$. Then*

$$X_s \hookrightarrow \Lambda^s BMO .$$

Proof : **1.** The case $s = 0$ is obvious ; we suppose from now on that $s > 0$.
First, the predual of X_s can be described (see [25] p.211) : it is the space

$$Y_s = \{h \text{ such that } h = \sum_{n\in\mathbb{N}} \alpha_n \beta_n \text{ with } \sum_{n\in\mathbb{N}} \|\alpha_n\|_2 \|\beta_n\|_{\dot{H}^s} < \infty\} ,$$

with the natural norm $\|h\|_{Y_s} = \inf_{\sum_n \alpha_n\beta_n = h} \sum_n \|\alpha_n\|_2 \|\beta_n\|_{\dot{H}^s}$.
From now on, we fix $f \in X^s$. If $g \in \mathcal{S}_0$, using lemma 7.6 of the appendix (since $X_s \hookrightarrow \dot{B}^{-s}_{\infty,\infty}$), we have

$$(24) \qquad |\langle \Lambda^{-s} f, g\rangle| = |\langle f, \Lambda^{-s} g\rangle| \leq \|f\|_{X^s} \|\Lambda^{-s} g\|_{Y_s} .$$

We would like to show that $\Lambda^{-s} f \in BMO$, which is equivalent to

$$|\langle \Lambda^{-s} f, g \rangle| \leq \|g\|_{\mathcal{H}^1} \ .$$

Because of (24), it actually suffices to prove that Λ^{-s} continuously maps $\mathcal{H}^1$ in Y_s, ie that for any g in $\mathcal{H}^1$,

$$\|\Lambda^{-s} g\|_{Y_s} \leq \|g\|_{\mathcal{H}^1} \ .$$

Any function G in $\mathcal{H}^1$ can be decomposed as a sum of atoms $G = \sum_j \lambda_j m_j$; a function m is said to be an atom if for a certain $r > 0$ it satisfies

$$\begin{aligned} & \int m = 0 \\ & \operatorname{Supp} m \subset B(0,r) \\ & \|m\|_\infty \leq r^{-d} \ . \end{aligned} \tag{25}$$

We have then $\|G\|_{\mathcal{H}^1} = \sum_j |\lambda_j|$ (see [43]). In the following, we will take m to be one such atom, with $r = 1$, and will try to prove that $\Lambda^{-s} m \in Y_s$.

2. We will first examine $\Lambda^{-s} m$ near $B(0,1)$. The convolution kernel of Λ^{-s} is, since $s \in (0,d)$, $x \mapsto \dfrac{1}{|x|^{d-s}}$, which is integrable near 0. This implies that

$$\|\Lambda^{-s} m\|_\infty \leq C \ .$$

Define $\beta \in \dot{H}^r$, such that $\beta = 1$ on $B(0,2)$, and

$$\alpha = \chi_{B(0,2)} \Lambda^{-s} m \ .$$

We have then

$$\|\chi_{B(0,2)} \Lambda^{-s} m\|_{Y_r} \leq \|\alpha\|_{L^2} \|\beta\|_{\dot{H}^s} \leq C \cdot \tag{26}$$

3. Now we have to take care of $\Lambda^{-s} m$ outside $B(0,2)$. We observe first that if $x \notin B(0,2)$ and $y \in B(0,1)$, there exists a constant C such that

$$\left| \frac{1}{|x-y|^{d-s}} - \frac{1}{|x|^{d-s}} \right| \leq \frac{C}{|x|^{d-s+1}} \ .$$

Using this estimate, the zero integral of m and once again the fact that the convolution kernel of Λ^{-s} is $x \mapsto \dfrac{1}{|x|^{d-s}}$, we get

$$\begin{aligned} |\Lambda^{-s} m(x)| &= \left| \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-s}} m(y) dy \right| = \left| \int_{\mathbb{R}^d} \left(\frac{1}{|x-y|^{d-s}} - \frac{1}{|x|^{d-s}} \right) m(y) dy \right| \\ &\leq C \|m\|_\infty \frac{1}{|x|^{d-s+1}} \leq C \frac{1}{|x|^{d-s+1}} \ , \end{aligned} \tag{27}$$

for $|x| \geq 2$. We will now need a kind of space-Littlewood Paley decomposition. Let $\lambda, \widetilde{\lambda} \in \mathcal{C}^\infty$ such that

$$\begin{aligned} & \operatorname{Supp} \lambda \subset \{1 \leq |x| \leq 4\} \\ & \operatorname{Supp} \widetilde{\lambda} \subset \{0.9 \leq |x| \leq 4.1\} \\ & \lambda \widetilde{\lambda} = \lambda \\ & \text{if } \lambda_k \overset{\text{def}}{=} \lambda\left(\frac{\cdot}{2^k}\right) \text{ then } \sum_{k \in \mathbb{Z}} \lambda_k(x) = 1 \text{ for } x \neq 0 \ . \end{aligned}$$

Using λ and $\widetilde{\lambda}$, we can decompose $\Lambda^{-s}m$ outside $B(0,2)$ into functions whose space support are dyadic annuli,

$$\Lambda^{-s}m\left(1-\chi_{B(0,2)}\right)=\sum_{k\in\mathbb{N}}\lambda_k\left(\widetilde{\lambda}_k\Lambda^{-s}m\left(1-\chi_{B(0,2)}\right)\right)$$

and obtain :

$$\begin{aligned}\|\Lambda^{-s}m\left(1-\chi_{B(0,2)}\right)\|_{Y_s}&\leq\sum_{k\in\mathbb{N}}\left\|\lambda_k\widetilde{\lambda}_k\Lambda^{-s}m\left(1-\chi_{B(0,2)}\right)\right\|_{Y_s}\\&\leq\sum_{k\in\mathbb{N}}\|\lambda_k\|_{\dot{H}^s}\left\|\widetilde{\lambda}_k\Lambda^{-s}m\left(1-\chi_{B(0,2)}\right)\right\|_{L^2}.\end{aligned}\tag{28}$$

This last term is easily estimated : we have obviously

$$\|\lambda_k\|_{\dot{H}^s}=C2^{k(d/2-s)}$$

and (27) yields

$$\left\|\widetilde{\lambda}_k\Lambda^{-s}m\left(1-\chi_{B(0,2)}\right)\right\|_{L^2}\leq C2^{k(s-\frac{d}{2}-1)}\ .$$

Combining the two last estimates with (28), we see that

$$\|\Lambda^{-s}m\left(1-\chi_{B(0,2)}\right)\|_{Y_s}\leq C$$

and with the help of (26), we conclude that

$$\|\Lambda^{-s}m\|_{Y_s}\leq C\ .\tag{29}$$

4. The case of general atoms (ie, with $r\neq 1$) is easily deduced by homogeneity of (29). So the proof of the lemma is complete. ∎

As claimed at the beginning of this section, the previous lemma enables one to show that $\mathbf{M}(\dot{H}^s,L^2)\hookrightarrow\mathbf{\Pi}(\dot{H}^s,L^2)$. This is done in the following corollary.

Corollary 6.16 *Let* $s\in(0,d/2)$. *Then*

$$\mathbf{M}(\dot{H}^s,L^2)\hookrightarrow\mathbf{\Pi}(\dot{H}^s,L^2)$$

PROOF : Take $f\in\mathbf{M}(\dot{H}^s,L^2)$. By Lemma 6.15, we also have $f\in\Lambda^sBMO$. Propositions 6.5 and 6.6 now imply that

$$f\in\mathbf{R}(\dot{H}^s,L^2)\cap\widetilde{\mathbf{\Pi}}(\dot{H}^s,L^2)\ ,$$

so as a conclusion (recall that we denote M_f for the multiplication operator $\phi\mapsto f\phi$)

$$\Pi(f,\cdot)=M_f-R(f,\cdot)-\widetilde{\Pi}(f,\cdot)$$

is bounded from $\dot{H}^s$ to L^2. In other words, $f\in\mathbf{\Pi}(\dot{H}^s,L^2)$. ∎

7 Appendix

7.1 Some facts about Besov spaces

We will recall here some basic definitions and results whose proof can be found in [44], [25].
Besov spaces must in general be defined modulo polynomials. To be more precise, we need to introduce the space of tempered distributions modulo polynomials.

Definition 7.1 (Distributions modulo polynomials) *Let $\mathcal{S}_\infty$ be the space of Schwartz class functions which have vanishing moments of any order. The dual of this space is $\mathcal{S}'_\infty$, the space of tempered distributions modulo polynomials.*

Observe that the Fourier transform of a distribution of $\mathcal{S}'_\infty$ is defined anywhere except in 0. We can therefore apply a dyadic Fourier decomposition to an element of this space.

Definition 7.2 (Besov spaces)

1. *Let F be a Banach space, s a real number, and q a real number in the interval $[1,\infty]$. Consider the Δ_j operators defined in section 3.2.1. A distribution of $\mathcal{S}'_\infty$ is said to belong to $\dot{B}^s_{F,q}$ if and only if*

   $$\left(\sum_{j\in\mathbb{Z}}\left[2^{js}\|\Delta_j f\|_F\right]^q\right)^{1/q} < \infty \ .$$

   With the above quantity as norm, $\dot{B}^s_{F,q}$ is a Banach space.

2. *In case F is a Lebesgue space L^p, we denote $\dot{B}^s_{p,q}$ instead of $\dot{B}^s_{L^p,q}$.*

In case F is a Lebesgue space, we can define Besov spaces in a more simple way.

Definition 7.3 (Besov spaces over Lebesgue spaces) *Suppose $s < \frac{d}{p}$, or $s = \frac{d}{p}$ and $q = 1$; we define then $\dot{B}^s_{p,q}$ as the set of functions f of $\mathcal{S}'$ such that*

$$\left(\sum_{j\in\mathbb{Z}}\left[2^{js}\|\Delta_j f\|_F\right]^q\right)^{1/q} < \infty \ . \tag{30}$$

and

$$f = \sum_j \Delta_j f \quad in \quad \mathcal{S}' \ . \tag{31}$$

With (30) as a norm, this definition makes of $\dot{B}^s_{p,q}$ a Banach space.
If $s \in [\frac{d}{p}+n, \frac{d}{p}+n+1)$, or $s = \frac{d}{p}$ and $q = 1$, we replace (31) by

$$f = \sum_j \Delta_j f \quad in\ \mathcal{S}' \quad modulo\ polynomials\ of\ order\ at\ most\ n.$$

An important point is the density of $\mathcal{S}_\infty$ in $\dot{B}^s_{p,q}$.

Lemma 7.4 *The space $\mathcal{S}_\infty$ is embedded in $\dot{B}^s_{p,q}$ for any s, p and q. It is a dense subspace provided $q < \infty$.*

Sobolev spaces can be defined as a particular class of Besov spaces.

Definition 7.5 (Sobolev spaces) *Let $s \in \mathbb{R}$. We define $\dot{H}^s = B^s_{2,2}$.*

One cannot a priori define the duality bracket between $f \in \dot{B}^0_{\infty,\infty}$ and $\phi \in \mathcal{S}$; it is necessary to take ϕ in

$$\mathcal{S}_0 = \{\phi \in \mathcal{S} \text{ such that } \int \phi = 0\} \ .$$

Consider now for $s \in \mathbb{R}$ the Calderón operators $\Lambda^s = |D|^s$. Obviously,

$$\Lambda^s \ : \ \dot{B}^0_{\infty,\infty} \mapsto \dot{B}^{-s}_{\infty,\infty} \ .$$

On the other hand, $\Lambda^s \phi$ makes sense for $\phi \in \mathcal{S}$.

The following lemma is used in the proof of Lemma 6.15.

Lemma 7.6 *Let $s \in \mathbb{R}$. If $\phi \in \dot{B}^0_{1,1}$ and $f \in \dot{B}^{-s}_{\infty,\infty}$ we have*

$$\langle f, \Lambda^{-s}\phi \rangle = \langle \Lambda^{-s} f, \phi \rangle \ ,$$

where the brackets on the left hand side are the duality brackets between $\dot{B}^{-s}_{\infty,\infty}$ and $\dot{B}^s_{1,1}$, and the brackets on the right hand side are the duality brackets between $\dot{B}^0_{\infty,\infty}$ and $\dot{B}^0_{1,1}$

Proof : It is almost a tautology. We have, for $\phi \in \dot{B}^0_{1,1}$ and $f \in \dot{B}^{-s}_{\infty,\infty}$,

$$\begin{aligned}
\langle \Lambda^{-s} f, \phi \rangle &= \sum_{j, |k-j|<1} \langle \Delta_j \Lambda^{-s} f, \Delta_k \phi \rangle \\
&= \sum_{j, |k-j|<1} \langle \Delta_j f, \Lambda^{-s} \Delta_k \phi \rangle \\
&= \langle f, \Lambda^{-s} \phi \rangle \ .
\end{aligned}$$

7.2 Some facts about Morrey spaces

First of all, Morrey spaces are defined as follows.

Definition 7.7 (Homogeneous Morrey spaces) *If $1 < p \leq q \leq \infty$, $f \in M^{p,q}$ if and only if f is locally L^p and $\|f\|_{M^{p,q}} < \infty$ where*

$$\|f\|_{M^{p,q}} = \sup_{x \in \mathbb{R}^d \, , \, R>0} R^{d(\frac{1}{q} - 1p)} \|f\|_{L^p(B(x,R)} \ .$$

The next proposition recalls some basic properties of the Morrey spaces.

Proposition 7.8 *(i) If $p \in (1, \infty]$, $M^{p,p} = L^p$.*
(ii) If $p_1 > p_2$, $M^{p_1,q} \hookrightarrow M^{p_2,q}$.
(iii) $M^{p,q}$ has the homogeneity of L^q.
(iv) The dual space of $M^{p,q}$ is $N^{p',q'}$ (conjugate exponents), which can be defined as

$$N^{p,q} = \left\{ h = \sum_{k \in \mathbb{N}} g_k \ \text{with} \ \sum_k \text{diam}(\text{Supp}(g_k))^{d\left(\frac{1}{q} - \frac{1}{p}\right)} \|g_k\|_{L^p} \right\} \ .$$

The next lemma is crucial if one wants to compare Morrey spaces and multiplier or paramultiplier spaces. It was first proved by Lemarié [25], see also Dubois [12].

Lemma 7.9 *Let* $2 < p \leq q \leq \infty$ *; define* $s = \frac{2}{q}$ *; take* $\phi \in L^2$, $\psi \in \dot{H}^s$. *Then*

$$\|\phi\psi\|_{N^{p',q'}} \leq \|\phi\|_{L^2}\|\psi\|_{\dot{H}^s} \, .$$

References

[1] P. Auscher, S. Dubois, P. Tchamitchian, *On the stability of global solutions to Navier-Stokes equations in the space*, Journal de Mathématiques Pures et Appliquées **83**, 673-697 (2004)

[2] H. Beirão da Veiga, *A new regularity class for the Navier-Stokes equations in* $\mathbb{R}^N$, Chinese Annals of Mathematics, Series B **16**, 407-412 (1995)

[3] L. Berselli, R. Manfrin, *On a theorem by Sohr for the Navier-Stokes equations*, Journal of Evolution Equations **4**, 193-211 (2004)

[4] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Annales scientifiques de l'Ecole normale supérieure **14**, 209-246 (1981)

[5] G. Bourdaud, *Analyse fonctionnelle dans l'espace euclidien*, Publications mathématiques de l'université Paris VII, **23** (1987)

[6] G. Bourdaud, *Une remarque sur les multiplicateurs de* $\dot{H}^s$ *dans* $\dot{B}^0_{2,\infty}$, personnal communication (2004)

[7] L. Caffarelli, R.-V. Kohn, L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Communications in Pure and Applied Mathematics **XXXV**, 771-831 (1982)

[8] M. Cannone, *Ondelettes, paraproduits, et Navier-Stokes*, Diderot Editeur, Paris (1995)

[9] J.-Y. Chemin, *Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel*, Journal d'Analyse Mathématique, **77**, 22-50 (1999)

[10] R. Coifman, P.-L. Lions, Y. Meyer, S. Semmes, *Compensated compactness and Hardy spaces*, Journal de Mathématiques Pures et Appliquées **72**, 247-286 (1993)

[11] G. David, J.-L. Journé *A boundedness criterion for generalized Calderón-Zygmund operators*, Annals of Mathematics **120**, 371-397 (1984)

[12] S. Dubois, *Thèse de doctorat. Equations de Navier-Stokes dans l'espace : espaces critiques et solutions d'énergie finie*, Université de Picardie Jules Verne (2002)

[13] S. Dubois, *Uniqueness for some Leray-Hopf solutions to the Navier-Stokes equations*, Journal of Differential Equations **189**, 99-147 (2003)

[14] L. Escauriaza, G.A. Seregin, V. Šverák, $L_{3,\infty}$*-solutions of Navier-Stokes equations and backward uniqueness*, Russian Mathematical Surveys **58**, 211-250 (2003)

[15] E. B. Fabes, B. F. Jones, N. M. Riviere, *The initial value problem for the Navier-Stokes equations with data in L^p*, Archive for Rational and Mechanical Analysis **45**, 222-240 (1972)

[16] C. Foias, *Une remarque sur l'unicité des solutions des équations de Navier-Stokes en dimension n*, Bulletin de la Société Mathématique de France **89**, 1-8 (1961)

[17] H. Fujita, T. Kato, *On the Navier-Stokes initial value problem*, Archive for Rational and Mechanical Analysis **16**, 260-315 (1964)

[18] S. Gala, P.-G. Lemarié-Rieusset, *Multipliers between Sobolev spaces and fractional differentiation*, prépublication de l'Université d'Evry Val d'Essonne **203** (2005)

[19] I. Gallagher, F. Planchon, *On global infinite energy solutions to the Navier-Stokes equations*, Archive for Rational and Mechanical Analysis **161**, 307-337 (2002)

[20] I. Gallagher, D. Iftimie, F. Planchon, *Comportement asymptotique et stabilité des solutions globales des équations de Navier-Stokes*, Annales de l'Institut Fourier **53**, 1387-1424 (2003)

[21] T. Kato, *Strong L^p solutions of the Navier-Stokes equations in $\mathbb{R}^m$ with applications to weak solutions*, Mathematische Zeitschrift **187**, 471-480 (1984)

[22] H. Koch, D. Tataru, *Well-posedness for the Navier-Stokes equations*, Advances in Mathematics **157**, 22-35 (2001)

[23] H. Kozono, Y. Taniuchi, *Bilinear estimates in BMO and the Navier-Stokes equations*, Mathematische Zeitschrift **235**, 173-194 (2000)

[24] H. Kozono, H. Sohr, *Remark on uniqueness of weak solutions to the Navier-Stokes equations*, Analysis **16**, 255-271 (1996)

[25] P.-G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem*, Chapman-Hall (2003)

[26] J. Leray, *Sur le mouvement d'un fluide visqueux remplissant l'espace*, Acta Mathematica **63**, 193-248 (1934)

[27] R. May, *Existence, unicité et régularité des solutions faibles des équations de Navier-Stokes*, Thèse, Université d'Evry Val d'Essonne (2002)

[28] V. G. Maz'ya, T. O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*, Pitman, Monographs and studies in Mathematics **23**, Boston-London (1985)

[29] V. G. Maz'ya, I. E. Verbitsky, *Capacitary estimates for fractional integrals, with applications to partial differential equations and Sobolev multipliers*, Arkiv för Matematik **33**, 81-115 (1995)

[30] V. G. Maz'ya, I. E. Verbitsky, *The Schrödinger operator on the energy space: Boundedness and compactness criteria*, Acta Mathematica **188**, 263-302 (2002)

[31] V. G. Maz'ya, I. E. Verbitsky, *The form boundedness criterion for the relativistic Schrödinger operator*, preprint (2004)

[32] M. Meyer, *Une classe d'espaces fonctionnels de type BMO. Application aux intégrales singulières*, Arkiv för Matematik **27**, 305-318 (1989)

[33] Y. Meyer, *Ondelettes et opérateurs II : Opérateurs de Calderón-Zygmund*, Hermann, Paris (1990)

[34] Y. Meyer, *Wavelets, paraproducts and Navier-Stokes equations*, Current developments in mathematics, Cambridge, MA (1999)

[35] H. Miura, *Remark on uniqueness of mild solutions to the Navier-Stokes equations*, Journal of Functional Analysis **218**, 110-129 (2005)

[36] S. Montgomery-Smith, *Conditions implying regularity of the three dimensional Navier-Stokes equations*, preprint

[37] F. Oru, *Rôle des oscillations dans quelques problèmes d'analyse non-linéaire*, Thèse, Ecole Normale Supérieure de Cachan (1998)

[38] G. Prodi, *Un teorema di unicitá per le equazioni di Navier-Stokes*, Annali di Matematica Pura e Applicata **48**, 173-182 (1959)

[39] F. Ribaud, *A remark on the uniqueness problem for the weak solutions of Navier-Stokes equations*, Annales de la Faculté des sciences de Toulouse, volume **XI**, 225-238 (2002)

[40] G. Seregin, V. Šverák, *On smoothness of suitable weak solutions to the Navier-Stokes equations*, preprint available at

`http://www.pdmi.ras.ru/~seregin/Recent%20Publications/recentpubl.htm`

[41] J. Serrin, *The initial value problem for the Navier-Stokes equations*, in *Nonlinear problems*, University of Wisconsin Press (R. E. Langer editor), 69-98 (1963)

[42] H. Sohr, *A regularity class for the Navier-Stokes equations in Lorentz spaces*, Journal of Evolution Equations **1**, 441-467 (2001)

[43] E. M. Stein, *Harmonic analysis*, Princeton University press (1993)

[44] H. Triebel, *Theory of function spaces II*, Birkhaüser Verlag (1992)

[45] W. von Wahl, *The equations of Navier-Stokes and abstract parabolic equations*, Friedrich Vieweg & Sohn, Braunschweig (1985)

[46] A. Youssfi, *Function spaces related to singular integral operators, in Function spaces, differential operators and nonlinear analysis*, B. G. Teubner Verlagsgesellschaft, Stuttgart, Leipzig 1993

[47] A. Youssfi, *Duality of type $\mathcal{H}^1$ - BMO and bilinear operators*, Journal of Mathematical Analysis and Applications **190**, 780-794 (1995)