SELF-SIMILAR SOLUTIONS FOR THE SCHRÖDINGER MAP EQUATION

PIERRE GERMAIN, JALAL SHATAH, AND CHONGCHUN ZENG

Abstract. We study in this article the equivariant Schrödinger map equation in dimension 2, from the Euclidean plane to the sphere. A family of self-similar solutions is constructed; this provides an example of regularity breakdown for the Schrödinger map. These solutions do not have finite energy, and hence do not fit into the usual framework for solutions. For data of infinite energy but small in some norm, we build up associated global solutions.

1. Introduction

1.1. Schrödinger maps: well-posedness and blow-up.

Let us first present the equation.

Given a map u from $\mathbb{R}^d$ to a manifold N with Christoffel symbols $\Gamma^i_{j,k}$, we define the covariant derivatives on $u^{-1}TN$ by

$$D_k X^i = \partial_k X^i + \Gamma^i_{j,l} \partial_k u^j X^l \ .$$

Suppose that an almost complex structure J on N is given. Then the Schrödinger map equation reads

$$(SM) \qquad\qquad \partial_t u = J D_k \partial^k u \ .$$

The critical space for this equation is $\dot{H}^{d/2}$, and the conserved quantity is the H^1 norm. The problem is therefore critical in dimension 2, and supercritical in higher dimensions.

Local well-posedness was proved for smooth and small data by Kenig, Ponce and Vega [4]. It was improved to global well-posedness for small data (under an equivariance assumption) by Chang, Shatah and Uhlenbeck [3]. Finally, in dimension $d \geq 4$, Benjenaru, Ionescu and Kenig [1] could prove global existence for data at the scaling of the equation, ie in $\dot{H}^{d/2}$.

A fundamental question, which is currently completely open, is that of the possible blow up of solutions of (SM). If one starts up with smooth data in the critical space $\dot{H}^{d/2}$, does a singularity appear?

Schrödinger maps are somewhat to linear Schrödinger equations what wave maps are to linear wave equations; therefore, the theory of blow up for wave maps, which has reached some important results, should be enlightening for us. We restrict the discussion to the case of the sphere. In dimension greater than 3, it was proved by Shatah [9] that self-similar blow-up can occur, namely there exists a solution of the form $f\left(\frac{x}{t}\right)$, where f is smooth and decays at infinity. In dimension 2, recent works of Krieger, Schlag and Tataru [6] and Sterbenz and Rodnianski [8] proved that blow up can occur, by concentration of a harmonic map ; the blow-up rate, however, has to be slower than the self-similar rate $\frac{1}{t}$.

1.2. **Equivariant maps.** Before stating our results, let us make the framework in which we will work a little more precise. We choose a target manifold N such that in the coordinates $(s,\theta) \in [0, S^*) \times \mathbb{S}^1$, its metric reads

$$ds^2 + \Gamma(s)^2 d\theta^2 \ ,$$

and we endow it with the almost complex structure

$$J\partial_s = \frac{1}{\Gamma(s)}\partial_\theta$$
$$J\partial_\theta = -\Gamma(s)\partial_s \ .$$

Let us denote (r,θ) for the polar coordinates on $\mathbb{R}^2$. We shall say that a map $u : \mathbb{R}^2 \to N$ is k-equivariant if it can be written under the form

$$u(r,\theta) = R(k\theta)W(r)$$

where W is a map from $\mathbb{R}^+$ to N and $R : N \to N$ the rotation operator defined by $R(\alpha)(s,\theta) = (s, \alpha + \theta)$. The equation (SM) becomes, for a k-equivariant map

$$\partial_t W = J\left(D_r\partial_r W + \frac{1}{r}\partial_r W - \frac{\Gamma'(W)\Gamma(W)k^2}{r^2}\partial_s\right) \ . \qquad (1.1)$$

1.3. **Smooth self-similar solution.** Our first result asserts the existence of smooth self-similar solutions to the Schrödinger map equation in dimension 2.

Theorem 1.1. *In dimension* 2 *there exists for any* k *a family of* k*-equivariant self-similar solutions with smooth profiles, in other words, solutions of the form*

$$R(k\theta)W\left(\frac{r}{\sqrt{t}}\right) \qquad \textit{with} \quad W \in \mathcal{C}^\infty \ .$$

These solutions are parameterized by $A \in \mathbb{R}$ *such that as* $r \to 0$

$$S = s(W(r)) \sim Ar^k$$
$$\langle W_r \,,\, \frac{1}{\Gamma}\partial_\theta \rangle \sim \frac{k}{4(k+1)} A r^{k+1} \ .$$

Also, there exists $S_0 \in [0, S^*)$ *and* $\theta_0 \in [0, 2\pi)$ *such that*

$$S = s(W(r)) \to S_0 \quad \text{as} \quad r \to \infty$$
$$\theta(W(r)) \to \theta_0 \quad \text{as} \quad r \to \infty \ .$$

Finally, we have the following asymptotics as $r \to \infty$*: there exists a function* γ *converging to* γ_0 *at infinity such that*

$$\langle W_r \,,\, \partial_s \rangle \sim \frac{\Gamma(S_0)}{r} \cos\left(\gamma(r) - \frac{r^2}{4}\right)$$
$$\langle W_r \,,\, \frac{1}{\Gamma}\partial_\theta \rangle \sim -\frac{\Gamma(S_0)}{r} \sin\left(\gamma(r) - \frac{r^2}{4}\right) \ .$$

The solution of the previous theorem gives an example of regularity breakdown: it is $\mathcal{C}^\infty$ for any time except 0.

However, it does not belong to the energy class $\dot{H}^1$, leaving the question of blow up for this class open.

1.4. **Complex notation.** Our next aim will be to build up, for small (self-similar) initial data, self-similar solutions of (SM). In order to do so, we suppose now that the manifold N has constant Gaussian curvature κ.

Following Chang, Shatah and Uhlenbeck [3], we will now write the equation one gets when using the stereographic projection. The metric becomes $\lambda(z\bar{z})\,dz\,d\bar{z}$, the coordinates are given by a complex number, and the complex structure is of course multiplication by i. If w is the coordinate of W, the k-equivariance condition becomes

$$w(r, \theta) = e^{ik\theta} w(r)$$

for some function $w : \mathbb{R}^+ \to \mathbb{C}$.

We take as the unknown function $V = \partial_r W - \dfrac{J}{r}\partial_\theta W$, whose complex coordinate reads

$$v = w_r + \frac{k}{r} w \ .$$

We now choose (Z, JZ) a frame along W such that $D_r Z = 0$ and $Z(0) = Z_0$. Denoting ζ the coordinate of Z, the coordinate of V in this frame is q, with

$$q = \zeta^{-1} v \ .$$

If we eventually set

$$u = e^{i(k-1)\theta} q \ ,$$

it satisfies the equation (we refer to [3])

$$u_t = i \left(\Delta u + \frac{\kappa}{2} |u^2| u + uR \right) \ , \tag{1.2}$$

with

$$|R| \leq C \left(\left| \frac{w}{r} u \right| + \left| \left(\frac{w}{r} \right)^2 \right| + \int_r^\infty \left(\frac{|u|^2}{r'} + \frac{w^2}{r'^3} \right) dr' \right) \ .$$

1.5. **A framework for self-similar solutions.** The next theorem shows that, for small (self-similar) data, it is possible to build up self-similar solutions; this theorem of course also applies to more general data, see below for the precise functional setting.

Theorem 1.2. *Let*

$$X = L_t^{4,\infty}(\mathbb{R}, L_x^{4,\infty})$$

and

$$X_0 = \left\{ q_0 \ \ k\text{-equivariant} \ \ , \|q_0\|_{X_0} = \|e^{it\Delta} q_0\|_X < \infty \right\} \ .$$

If $\|q_0\|_{X_0}$ is small enough, there exists a global solution u belonging to X. This solution is unique in a ball of sufficiently small radius of X.

Solutions for small data in spaces that contain self-similar solutions have already been built up for non-linear Schrödinger equations: the present theorem is close to the results obtained by Cazenave, Vega and Vilela [2]; Planchon [7] considered a different range of nonlinearities.

Also, observe that the spaces X and X_0 are at the scaling of the equation.

The next proposition will show that the solution built up in Theorem 1.1 fits in the framework of Theorem 1.2.

Proposition 1.3. *Any solution W built up in Theorem 1.1, when converted into the complex notation, belongs to X. The initial data for this solution is*

$$e^{i(k-1)\theta)} \zeta_1^{-1} \frac{\widetilde{S}_0}{r} \ ,$$

where ζ_1 is a complex number of modulus 1, and $\widetilde{S}_0$ the stereographic coordinate corresponding to S_0.

Remark 1.4. That the data for W is of the form $\frac{C}{r}$ is not surprising: since W is self-similar, it was to be expected that the data itself would be self-similar; also, for scaling reasons, a δ function at 0 was unlikely. So the main interest of the above proposition is to make clear what the constant C is.

Notice that the above proposition is remarkable in that it shows that for singular data one gets a smooth solution; this is of course not in general the case for the solutions provided by Theorem (1.2).

2. Proof of Theorem 1.1

2.1. **The self-similarity equation.** Using the self-similar ansatz

$$u(r,\theta) = R(k\theta)W\left(\frac{r}{\sqrt{t}}\right) \ ,$$

we see that (1.1) becomes

$$D_r W_r + \frac{1}{r}W_r - \frac{r}{2}JW_r - \frac{\Gamma'\Gamma k^2}{r^2}\partial_s = 0 \ . \tag{2.1}$$

We will denote in the following

$$S = s(W(r)) \qquad \text{and} \qquad \alpha = \langle JW_r \, , \, \partial_s \rangle = -\Gamma\theta(W(r))_r \ .$$

By projecting the above equation on ∂_s and ∂_θ we get

$$\begin{aligned} S_{rr} + \frac{S_r}{r} &= \frac{r\alpha}{2} + \frac{\Gamma'\Gamma k^2}{r^2} + \frac{\Gamma'}{\Gamma}\alpha^2 \\ \alpha_r + \frac{\alpha}{r} &= -\frac{rS_r}{2} - \frac{\Gamma'}{\Gamma}\alpha S_r \end{aligned}$$

We prescribe the initial conditions

$$\begin{aligned} S &\sim Ar^k \\ \alpha &\sim Br^{k+1} \ . \end{aligned}$$

By plugging them into the equation we see that the following compatibility condition has to be satisfied:

$$B = -\frac{k}{4(k+1)}A \ .$$

2.2. **The energy estimate.** Taking the scalar product of (2.1) with $r^2 W_r$, we get

$$\begin{aligned} 0 &= \langle D_r W_r + \frac{1}{r}W_r \, , \, r^2 W_r \rangle - \langle \frac{r}{2}JW_r \, , \, r^2 W_r \rangle - \langle \Gamma'\Gamma k^2 \partial_s \, , \, W_r \rangle \\ &= \langle D_r(rW_r) \, , \, rW_r \rangle - \Gamma'\Gamma k^2 S_r \\ &= \frac{1}{2}\partial_r \left(|rW_r|^2 - k^2\Gamma^2 \right) \\ &= \frac{1}{2}\partial_r \left(r^2 S_r^2 + r^2\alpha^2 - k^2\Gamma^2 \right) \ . \end{aligned}$$

Since rW_r vanishes at 0, we find that

$$\text{for } r \geq 0 \ , \qquad r^2 S_r^2 + r^2\alpha^2 = k^2\Gamma^2 \ . \tag{2.2}$$

This proves that the solution can be continued for all times. Also, we can define a function $\beta(r)$ such that

$$\begin{aligned} r\Gamma S_r &= \Gamma^2\cos(\beta) \\ r\Gamma\alpha &= \Gamma^2\sin(\beta) \ . \end{aligned} \tag{2.3}$$

2.3. **The equation on β.** The second equation of (2.2) can be written as

$$(r\Gamma\alpha)_r = -\frac{r}{2}(r\Gamma S_r) \ ,$$

which, expressed in terms of β, gives

$$\beta_r = -\frac{r}{2} - \frac{2\Gamma'\sin(\beta)}{r} \ . \tag{2.4}$$

Let us now integrate S_r starting from r_0 big enough. Using (2.3), we have

$$\int_{r_0}^R S_r\,dr = \int_{r_0}^R \frac{\Gamma\cos(\beta)}{r}\,dr = \int_{r_0}^R \frac{\Gamma\cos(\beta)\beta_r}{rg}\,dr$$

where we have set, due to (2.4), $g(r) = -\frac{r}{2} - \frac{2\Gamma'}{r}\sin(\beta)$. Integrating by parts, we get

$$\int_{r_0}^R S_r\,dr = -\int_{r_0}^R \frac{\sin(\beta)\Gamma' S_r}{rg}dr + \int_{r_0}^r \frac{\sin(\beta)(rg)_r}{(rg)^2}dr + \left[\frac{\sin(\beta)\Gamma}{g}\right]_{r_0}^r \ .$$

The equations (2.4) and (2.3) give the convergence of the right-hand side of the above expression as $r \to \infty$. This implies that S has a limit, that we denote S_0.

2.4. **The asymptotics.** An argument similar to the above one, plus the equation (2.4), shows that

$$\int_0^\infty \frac{2\Gamma'\sin(\beta)}{r}dr < \infty \ .$$

Therefore, there exists a constant γ such that

$$\beta(r) - r^2 \longrightarrow \gamma \quad \text{as} \quad r \to \infty \ .$$

The asymptotics of the theorem are then easily proved.

3. Proof of Theorem 1.2

Recall that the equation we want to solve is

$$u_t = i\left(\Delta u + \frac{\kappa}{2}|u^2|u + uR\right) \ ,$$

with

$$|R| \le C\left(\left|\frac{w}{r}u\right| + \left|\left(\frac{w}{r}\right)^2\right| + \int_r^\infty \left(\frac{|u|^2}{r'} + \frac{|w|^2}{r'^3}\right) dr'\right) \ .$$

We write it in integral form

$$u(t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-s)\Delta}\left(\frac{\kappa}{2}|u^2(s)|u(s) + R(s)u(s)\right) ds = e^{it\Delta}u_0 + N(u) \ .$$

By assumption, $e^{it\Delta}u_0$ is small in X. In order to apply a fixed point argument and prove the theorem, all we have to do is to show that the norm of $N(u)$ in X can be controlled by the norm of u in X.

We begin with a small lemma.

Lemma 3.1. *The following estimate holds*

$$\left\|\int_0^t e^{i(t-s)\Delta} f(s)\, ds\right\|_{L_t^{4,\infty} L_x^{4,\infty}} \le C\, \|f\|_{L_t^{4/3,\infty} L_x^{4/3,\infty}} \ .$$

Proof of the Lemma: Recall the classical dispersion estimate

$$\|e^{it\Delta}\|_{L^p \to L^{p'}} \le C t^{-d\left(\frac{1}{p}-\frac{1}{2}\right)} \qquad \text{for} \qquad 1 \le p \le 2 \ .$$

By real interpolation,

$$\|e^{it\Delta}\|_{L^{4/3,\infty} \to L^{4,\infty}} \le C t^{-1/2} \ .$$

Therefore

$$\begin{aligned}\left\|\int_0^t e^{i(t-s)\Delta} f(s)\, ds\right\|_{L^{4,\infty} L^{4,\infty}} &\le C\left\|\int_0^t \frac{1}{\sqrt{t-s}}\|f(s)\|_{L^{4/3,\infty}}\, ds\right\|_{L^{4,\infty}} \\ &\le C\|f\|_{L^{4/3,\infty} L^{4/3,\infty}} \ ,\end{aligned}$$

where we used in the last line the estimate for convolution in Lorentz spaces. ■

We begin with the first part of $N(u)$, that is the term involving $|u|^2 u$. It suffices to apply the previous lemma and the product estimate in Lorentz spaces to get

$$\left\|\int_0^t e^{i(t-s)\Delta}|u^2(s)|u(s)ds\right\|_{L^{4,\infty} L^{4,\infty}} \le C\|u^3\|_{L^{4/3,\infty} L^{4/3,\infty}} \le C\|u\|^3_{L^{4,\infty} L^{4,\infty}} \ .$$

Next we have to evaluate the terms contained in R. We begin with the one involving only u. By the same token as above, we have

$$\begin{aligned}\left\| \int_0^t e^{i(t-s)\Delta} u(s) \int_0^s \frac{|u|^2}{r'} dr' \right\|_{L^{4,\infty} L^{4,\infty}} &\leq \|u\|_{L^{4,\infty} L^{4,\infty}} \left\| \int_0^s \frac{|u|^2}{r'} dr' \right\|_{L^{2,\infty} L^{2,\infty}} \\ &\leq C \|u\|^3_{L^{4,\infty} L^{4,\infty}} ,\end{aligned}$$

where we used in the last line the following lemma.

Lemma 3.2. *There holds*

$$\left\| \int_0^s \frac{|u|^2}{s} ds \right\|_{L^{2,\infty}(\mathbb{R}^2} \leq C \|u\|^2_{L^{4,\infty}} .$$

Proof of the lemma: We observe that it suffices to show that the function that we want to estimate in $L^{2,\infty}$ can be bounded by $\frac{C}{r}$. Using the duality between Lorentz spaces, and denoting χ for the characteristic function of an interval, we have

$$\begin{aligned}\int_r^\infty \frac{|u|^2}{s} ds &\leq \|u^2\|_{L^{2,\infty}} \left\| \chi_{[r,\infty)} \frac{1}{s^2} \right\|_{L^{2,1}} \\ &\leq \frac{C}{r} \|u\|^2_{L^{4,\infty}}\end{aligned}$$

since

$$\left\| \chi_{[r,\infty)} \frac{1}{s^2} \right\|_{L^{2,1}} \leq \frac{C}{r} .$$

This concludes the proof of the lemma. ■

All the terms remaining to estimate in $N(u)$ are equal to one of the already treated terms, with one or more of the u's replaced by $\frac{w}{r}$. Since

$$|u| = |w_r + \frac{k}{r} w| ,$$

real interpolation and Lemma 3.6 of [3] give

$$\left\| \frac{w}{r} \right\|_{L^{4,\infty}} \leq C \|u\|_{L^{4,\infty}} .$$

We thus conclude that

$$\|N(u)\|_{L^{4,\infty} L^{4,\infty}} \leq C \|u\|^3_{L^{4,\infty} L^{4,\infty}} ,$$

proving the theorem.

4. Proof of proposition 1.3

That the solution W built up in Theorem 1.1 belongs to X is clear due to the asymptotics given in the Theorem. What remains to be proved is the convergence when $t \to 0$ of $u = e^{i(k-1)\theta)}\zeta^{-1}\left(w_r + \frac{k}{r}w\right)$ to $e^{i(k-1)\theta}\zeta_0^{-1}\frac{\widetilde{S}_0}{r}$. The first step is to find a formula for ζ; it will be provided by the following lemma.

Lemma 4.1. *Let ζ be defined by (the covariant derivatives are taken along $W(1,\cdot)$)*

$$D_r\zeta = 0 \quad \textit{and} \quad \zeta(0) = \zeta_0$$

for some constant phase factor ϕ. Then ζ is given for any r by

$$\zeta = e^{i\int_0^r \alpha\frac{\Gamma'}{\Gamma}\,ds}\zeta_0 \ .$$

There exists ζ_1 of modulus one such that

$$\zeta \longrightarrow \zeta_1 \quad \textit{as} \quad r \to \infty$$

Proof: We come back to the non-complex notation. Writing Z in coordinates $Z = x\partial_s + \frac{y}{\Gamma}\partial_\theta$, the equation $D_r Z = 0$ becomes

$$x' = -\alpha\frac{\Gamma'}{\Gamma}y$$
$$y' = \alpha\frac{\Gamma'}{\Gamma}x \ .$$

Integrating this equation gives the first result. For the limit, we just observe that $\alpha\frac{\Gamma'}{\Gamma} = \frac{\Gamma'}{r}\sin(\beta)$, so this is the integral studied in section 2.4.

We can now write the expression for u:

$$u\left(\frac{r}{\sqrt{t}}\right) = e^{i(k-1)\theta)}e^{-i\int_0^{\frac{r}{\sqrt{t}}}\alpha\frac{\Gamma'}{\Gamma}\,ds}\zeta_0^{-1}\left(\frac{1}{\sqrt{t}}w_r\left(\frac{r}{\sqrt{t}}\right) + \frac{k}{r}w\left(\frac{r}{\sqrt{t}}\right)\right) = I + II \ .$$

Due to the asymptotics of Theorem 1.1 and Lemma 4.1, it is clear that,

$$\text{as} \quad t \to 0 \qquad II \longrightarrow e^{i(k-1)\theta)}\zeta_1^{-1}\frac{\tilde{S}_0}{r} \ .$$

On the other hand, I can be written

$$I = \frac{1}{r}e^{i(k-1)\theta}\zeta_1^{-1}e^{i\epsilon_1}\Gamma(S_0 + \epsilon_2)e^{i(\gamma_0+\epsilon_3)}e^{-i\frac{r^2}{4t}} \ ,$$

with

$$\epsilon_1 = \int_{\frac{r}{\sqrt{t}}}^{\infty} \frac{\Gamma'}{s} \sin(\beta)\, ds$$
$$\epsilon_2 = \int_{\frac{r}{\sqrt{t}}}^{\infty} \frac{\Gamma}{s} \cos(\beta)\, ds$$
$$\epsilon_3 = \int_{\frac{r}{\sqrt{t}}}^{\infty} 2\frac{\Gamma'}{s} \sin(\beta)\, ds \ .$$

We would like to find the limit as t goes to 0 of I. We write

$$I = \chi_\rho I + (1-\chi_\rho) I \ ,$$

where χ is supported in the ball of center 0, radius 2ρ, and equal to 1 in the ball of center 0, radius ρ.

We use a non-stationnary phase argument to prove that $(1-\chi_\rho)I$ goes to 0: outside 0, the derivative (in r) of the phase of $e^{-i\frac{r^2}{4t}}$ does not vanish, whereas the derivatives of ϵ_1, ϵ_2, and ϵ_3 are bounded uniformly in t. So for any $\rho > 0$, $(1-\chi_\rho)I$ goes to 0 as t goes to 0.

On the other hand, since $|\chi_\rho I| \leq \chi_\rho \dfrac{C}{r}$ for any t, we have

$$\lim_{\rho \to 0} \sup_t \|\chi_\rho I\|_{L^1} \to 0 \ .$$

Combining the two preceding arguments, we see that

$$I \longrightarrow 0 \quad \text{in} \quad \mathcal{S}' \quad \text{as} \quad t \to 0 \ . \ \blacksquare$$

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Courant Institute of Mathematical Sciences, New York University, New York, NY 10012-1185

E-mail address: pgermain@math.nyu.edu

Courant Institute of Mathematical Sciences, New York University, New York, NY 10012-1185

E-mail address: shatah@math.nyu.edu

School of Mathematics, 686 Cherry Street, Georgia Institute of Technology, Atlanta, GA 30332-0160

E-mail address: zengch@math.gatech.edu