# ON THE OPEN SEA PROPAGATION OF WATER WAVES GENERATED BY A MOVING BED

ADRIAN CONSTANTIN AND PIERRE GERMAIN

ABSTRACT. Within the framework of linear theory, applicable far from the shore, we investigate the two-dimensional propagation of waves generated in the ocean by a sudden seabed deformation.

## 1. Introduction

Several recent investigations are devoted to the generation of tsunami waves by a sudden seabed deformation due to an earthquake [9, 10, 12, 15, 20]. In contrast to earlier approaches, which assume that the free-surface deformation at initiation is identical to the seabed deformation after the occurence of earthquake, these studies take into account the dynamics of seafloor displacement over a short period of time and do not consider the initial wave form as instantly generated. Instead, the water is still prior to the earthquake and a transient coupling occurs between the free surface displacement and the seabed deformation. Within the framework of linear irrotational theory, explicit integral formulas were obtained using Fourier transforms (or combined Fourier and Laplace transforms) and asymptotic analysis and/or numerical computations were performed. However, while the setting of linear theory is appropriate for waves of small amplitude, the hypothesis of irrotational flow is only justified if the length scales of the displacements in the horizontal and vertical directions are comparable. While such a scenario is definitely realistic, it is of interest to investigate the more general setting in which these length scales are different. The aim of this paper is to describe an approach that is applicable for two-dimensional waves, with emphasis on the wave propagation after initiation. The restriction to two-dimensional flows offers the advantage of mathematical simplicity but is also anchored in practical considerations. Namely, many tsunami source regions — in particular, those for the major three tsunami events since modern record-keeping began in the 20'th century — are elliptical, with a major axis in excess of hundreds of $km$ corresponding to the active part of the fault. The fact that most of the tsunami energy is transmitted at right angles to the major axis allows us to regard in this context the propagation of the tsunami in the open sea as being two-dimensional.

## 2. The governing equations

The fluid domain, representing the sea near the fault area, is bounded above by the water's free surface $Y = d + F(X,T)$, $d$ being the average depth of the sea, and below by the rigid seabed $Y = H(X,T)$, and is considered to be unbounded in the horizontal $X$-direction. At time $T = 0$ the seabed is flat, given by $Y = 0$. For some short time $T_0 > 0$, the sea bed moves in the region $X \in [-L, L]$ so that $H(X,T) = 0$ for $T \geq T_0$ and $X \notin (-L, L)$. Our aim is to describe the deformation of the free surface $Y = F(X,T)$ that is induced by this sudden motion of the sea bed. We assume that the water is inviscid and the resulting flow is a two-dimensional irrotational flow. In rectangular Cartesian coordinates $(X,Y)$, the equations of motion are the incompressible Euler equations

$$\begin{cases} U_X + V_Y = 0, \\ \rho(U_T + UU_X + VU_Y) = -P_X, \\ \rho(V_T + UV_X + VV_Y) = -P_Y - \rho g \end{cases} \tag{2.1}$$

where $\rho$ is the constant density, $g$ is constant acceleration of gravity, $(U, V)$ is the velocity field of the flow, and $P$ is the pressure. Since the effects of surface tension are negligible for wave lengths greater than a few centimeters (cf. [2, 14]), the major factor governing the wave motion is the balance between gravity and the inertia of the system. The corresponding kinematic boundary conditions

$$V = F_T + UF_X \quad \text{on the free surface} \quad Y = d + F(X, T), \tag{2.2}$$

and

$$V = H_T + UH_X \quad \text{on the rigid bed} \quad Y = H(X, T), \tag{2.3}$$

ensure that both fluid boundaries are interfaces: particles on these boundaries are confined to them at all times. In addition, we also have the dynamic boundary condition

$$P - P_{atm} = 0 \quad \text{on the free surface} \quad Y = d + F(X, T), \tag{2.4}$$

where $P_{atm}$ is the (constant) atmospheric pressure, which decouples the motion of the water from that of the air above it (see [13]). The above system (2.1)-(2.2)-(2.3)-(2.4) is to be solved with the initial conditions expressing the fact that the water is at rest at time $T = 0$:

$$F(X, 0) = 0, \qquad U(X, Y, 0) = 0, \qquad V(X, Y, 0) = 0, \tag{2.5}$$

the surface waves $F(X, T)$ being generated by the prescribed motion $H(X, T)$ of the bed. Since an incompressible inviscid flow that is irrotational initially remains so at later times, the initial conditions (2.5) ensure that throughout the wave propagation the vorticity of the flow vanishes, that is,

$$U_Y = V_X. \tag{2.6}$$

For a discussion of the local-well-posedness (existence, uniqueness, and continuous dependence on initial data for sufficently small time intervals) of the governing equations we refer to [1] (see also [11] and [21]).

Let us now briefly describe the process of non-dimensionalising and scaling the governing equations (2.1)-(2.2)-(2.3)-(2.4)-(2.6). In this process we introduce the two fundamental parameters $\varepsilon$ and $\delta$ that describe the regimes used to identify different general types of water wave propagation. To measure the change in pressure as a wave propagates at the water's surface, it is convenient to introduce the non-dimensional excess pressure $p$ relative to the hydrostatic pressure distribution by

$$P = P_{atm} + \rho g(d - Y) + \rho g d\, p.$$

To generate the non-dimensional version of the governing equations we introduce the average or typical wavelength $\lambda$ of the wave, using $\sqrt{dg}$ as a scale for the wave speed and $\lambda/\sqrt{gd}$ as a time scale. The change of variables

$$X = \lambda x, \quad Y = dy, \quad U = \sqrt{gd}\, u, \quad T = \frac{\lambda}{\sqrt{gd}}\, t, \quad V = \frac{d\sqrt{gd}}{\lambda}\, v, \tag{2.7}$$

coupled with the scaling

$$H(X, T) = a\, h(x, t), \qquad F(X, T) = a\, f(x, t), \tag{2.8}$$

where $a$ is a typical — perhaps maximum — amplitude of the wave, introduces non-dimensional dependent and independent variables. Note that (2.8) should be interpreted as expressing the fact that the variations of the wave and of the seabed are of comparable size. Introducing the amplitude and shallowness parameters by

$$\varepsilon = \frac{a}{d}, \qquad \delta = \frac{d}{\lambda}, \tag{2.9}$$

which measure the relative size of the amplitude to average water depth, respectively of the average water depth to wavelength, we obtain the non-dimensional version of the governing equations in the physical variables as

$$\begin{cases} \begin{cases} u_x + v_y = 0, \\ u_t + uu_x + vu_y = -p_x, \\ \delta^2\left(v_t + uv_x + vv_y\right) = -p_y, \\ u_y - \delta^2 v_x = 0, \end{cases} \quad \text{in} \quad \varepsilon h(x,t) < y < 1 + \varepsilon f(x,t)\,, \\ p = \varepsilon f \quad \text{on} \quad y = 1 + \varepsilon f(x,t), \\ v = \varepsilon f_t + \varepsilon u f_x \quad \text{on} \quad y = 1 + \varepsilon f(x,t), \\ v = \varepsilon h_t + \varepsilon u h_x \quad \text{on} \quad y = \varepsilon h(x,t)\,. \end{cases} \tag{2.10}$$

The magnitudes of $\varepsilon$ and $\delta$ can be used to identify different general types of wave problems. For example, the limits $\delta \to 0$ and $\delta \to \infty$ produce the shallow water, respectively deep water regime, while $\varepsilon \to 0$ corresponds to the linear regime of waves of small amplitude. Before proceeding with the linearisation approach, we would like to point out that the different nondimensionalisation of the horizontal and vertical fluid velocity components in (2.7) is consistent with the equation of mass conservation (and, equivalently, the existence of a stream function). Indeed, we may write $U = \Psi_Y$ and $V = -\Psi_X$ for a stream function $\Psi$, which leads to the transformation introduced in (2.7). The preservation of the irrotational character of the flow in addition to incompressibility in the non-dimensionalisation process requires a regime in which the length scales in the horizontal and vertical directions are of comparable size (that is, $\delta = 1$). The investigations in [9, 10, 12, 15, 20] are performed within the framework of a linear theory in this type of regime. Nevertheless, some simple considerations show that it is possible to dispense of this limitation. To start with, we consider the linear regime $\varepsilon \to 0$ with $\delta$ fixed. A glance at the system (2.10) shows first that $v$ and $p$ are both of order $\varepsilon$, and consequently also $u$. We now perform the scaling

$$p \mapsto \varepsilon p, \quad u \mapsto \varepsilon u, \quad v \mapsto \varepsilon v, \tag{2.11}$$

avoiding the introduction of a new notation. The dimensionless, scaled governing equations are then

$$\begin{cases} \begin{cases} u_x + v_y = 0, \\ u_t + \varepsilon uu_x + \varepsilon vu_y = -p_x, \\ \delta^2\left(v_t + \varepsilon uv_x + \varepsilon vv_y\right) = -p_y, \\ u_y - \delta^2 v_x = 0, \end{cases} \quad \text{in} \quad \varepsilon h(x,t) < y < 1 + \varepsilon f(x,t)\,, \\ p = f \quad \text{on} \quad y = 1 + \varepsilon f(x,t), \\ v = f_t + \varepsilon u f_x \quad \text{on} \quad y = 1 + \varepsilon f(x,t), \\ v = h_t + \varepsilon u h_x \quad \text{on} \quad y = \varepsilon h(x,t)\,. \end{cases} \tag{2.12}$$

We now linearise the system by taking the limit as $\varepsilon \to 0$. The linearised problem in non-dimensional scaled variables is

$$\begin{cases} u_x + v_y = 0 \quad \text{and} \quad u_y - \delta^2 v_x = 0 \quad \text{in} \quad 0 < y < 1, \\ u_t = -p_x \quad \text{and} \quad \delta^2 v_t = -p_y \quad \text{in} \quad 0 < y < 1, \\ p = f \quad \text{and} \quad v = f_t \quad \text{on} \quad y = 1, \\ v = h_t \quad \text{on} \quad y = 0\,. \end{cases} \tag{2.13}$$

While the fluid velocity in the system (2.13) does not originate from a velocity potential if $\delta \neq 1$, there exists a stream function defined up to an additive function of time by

$$u = \psi_y, \quad v = -\psi_x, \qquad 0 < y < 1. \tag{2.14}$$

We can re-formulate (2.13) as

$$\begin{cases} \psi_{yy} + \delta^2\,\psi_{xx} = 0 \quad \text{in} \quad 0<y<1, \\ \psi_{yt} = -p_x \quad \text{and} \quad \delta^2\,\psi_{xt} = -p_y \quad \text{in} \quad 0<y<1, \\ p = f \quad \text{and} \quad \psi_x = -f_t \quad \text{on} \quad y=1, \\ \psi_x = -h_t \quad \text{on} \quad y = 0\,. \end{cases} \tag{2.15}$$

On $y = 1$ we have $f_x = p_x = -\psi_{yt}$ from the second and fourth relation in (2.15), so that $f_{xt} = -\psi_{ytt}$ on $y = 1$. In combination with the fifth relation in (2.15) this yields $\psi_{xx} = \psi_{ytt}$ on $y = 1$. We obtain the linear system

$$\begin{cases} \psi_{yy} + \delta^2\,\psi_{xx} = 0 \quad \text{in} \quad 0<y<1, \\ \psi_{xx} = \psi_{ytt} \quad \text{on} \quad y=1, \\ \psi_x = -h_t \quad \text{on} \quad y = 0\,. \end{cases} \tag{2.16}$$

To find $f$ and $p$, note that the in terms of the solution $\psi$ of (2.16) we have

$$f_t = -\psi_x \quad \text{on} \quad y = 1, \tag{2.17}$$

while $p$ is determined by the second, third, and fourth relation in (2.15).

Before proceeding with the analysis of the system (2.16) in the next section, let us present some considerations related to its derivation. Switching from (2.13) back to the physical variables yields the linear system

$$\begin{cases} U_X + V_Y = 0 \quad \text{and} \quad U_Y - V_X = 0 \quad \text{in} \quad 0<Y<d, \\ U_T = -\dfrac{1}{\rho}\,P_X \quad \text{and} \quad V_T = -\dfrac{1}{\rho}\,P_Y - g \quad \text{in} \quad 0<Y<d, \\ P - P_{atm} = \rho g\,F(X,T) \quad \text{and} \quad V = F_T(X,T) \quad \text{on} \quad Y = d, \\ V = H_T(X,T) \quad \text{on} \quad Y=0\,. \end{cases} \tag{2.18}$$

The first two equations in (2.18) ensure the existence of a velocity potential $\Phi(X,Y,T)$ that is harmonic in the spatial variables $(X,Y)$ throughout the fluid domain and such that $U = \Phi_X$, $V = \Phi_Y$ throughout the domain $0 < Y < d$. We can therefore write (2.18) as

$$\begin{cases} \Delta\Phi = 0 \quad \text{in} \quad 0<Y<d, \\ \Phi_{XT} = -\dfrac{1}{\rho}\,P_X \quad \text{and} \quad \Phi_{YT} = -\dfrac{1}{\rho}\,P_Y - g \quad \text{in} \quad 0<Y<d, \\ P - P_{atm} = \rho g\,F(X,T) \quad \text{and} \quad \Phi_Y = F_T \quad \text{on} \quad Y = d, \\ \Phi_Y = H_T \quad \text{on} \quad Y=0\,. \end{cases} \tag{2.19}$$

The second and third equations in (2.19) are equivalent to the fact that the expression $\Phi_T + \dfrac{1}{\rho}\,(P - P_{atm}) + g(Y-d)$ is, at any fixed instant $T \geq 0$, constant throughout the fluid domain. Absorbing an additive function of time in the definition of $\Phi$, we define $\Phi$ unambiguously by requiring that

$$\Phi_T + \frac{1}{\rho}\,(P - P_{atm}) + g(Y-d) = 0, \qquad 0 \leq Y \leq d,$$

at every fixed $T \geq 0$. This, in combination with the fourth relation in (2.19), yields $\Phi_T + gF(X,T) = 0$ on $Y = d$. Taking into account the fifth relation in (2.19), we obtain $\Phi_{TT} + g\Phi_Y = 0$ on $Y = d$. Thus

$$\begin{cases} \Phi_{XX} + \Phi_{YY} = 0 \quad \text{in} \quad 0 < Y < d, \\ \Phi_{TT} + g\,\Phi_Y = 0 \quad \text{and} \quad \Phi_Y = F_T \quad \text{on} \quad Y = d, \\ \Phi_Y = H_T \quad \text{on} \quad Y = 0\,. \end{cases} \tag{2.20}$$

This is precisely the linearisation of the governing equations obtained in [9] under the assumption that the horizontal and vertical length scales of the motion are comparable. The approach in [9] was different, starting from a direct linearisation of the governing equations, under the assumption that the horizontal and vertical length scales are comparable. For the sake of completeness, we briefly sketch this approach. One starts from the governing equations (2.1)-(2.2)-(2.3)-(2.4)-(2.6) in the physical variables. The first equation in (2.1) and (2.6) ensure the existence of a velocity potential $\Phi(X,Y,T)$ that is harmonic in the spatial variables $(X,Y)$ throughout the fluid domain and such that $U = \Phi_X$, $V = \Phi_Y$. Then the Euler equations in (2.1) simply mean that at each instant $T$ the expression

$$\Phi_T + \frac{\Phi_X^2 + \Phi_Y^2}{2} + g(Y-d) + \frac{1}{\rho}\,(P - P_{atm})$$

is constant throughout the fluid (this is Bernoulli's law). In the absence of waves, that is, for $\Phi \equiv 0$ and for the hydrostatic pressure $P = P_{atm} - \rho g(Y-d)$, this expression is zero. Even in the presence of waves, absorbing the constant in the definition of $\Phi$ we may consider that the expression displayed above is always identically zero in the fluid. Evaluating the expression on the free surface $Y = d + F(X,T)$, where (2.4) holds, yields

$$\Phi_T + \frac{\Phi_X^2 + \Phi_Y^2}{2} + gF(X,T) = 0 \quad \text{on} \quad Y = d + F(X,T).$$

The linearisation of the above relation reads

$$\Phi_T + gF(X,T) = 0 \quad \text{on} \quad Y = d. \tag{2.21}$$

On the other hand, the linearisation of (2.1)-(2.2)-(2.3)-(2.6), obtained by neglecting all nonlinear terms, yields

$$\begin{cases} U_X + V_Y = 0 \quad \text{and} \quad U_Y - V_X = 0 \quad \text{in} \quad 0 < Y < d, \\ U_T = -\frac{1}{\rho}\,P_X \quad \text{and} \quad V_T = -\frac{1}{\rho}\,P_Y - g \quad \text{in} \quad 0 < Y < d, \\ V = F_T(X,T) \quad \text{on} \quad Y = d, \\ V = H_T(X,T) \quad \text{on} \quad Y = 0\,. \end{cases}$$

Since $U = \Phi_X$ and $V = \Phi_Y$, we can write (2.21) and the above relations as the system

$$\begin{cases} \Delta\Phi = 0 \quad \text{in} \quad 0 < Y < d, \\ \Phi_{XT} = -\frac{1}{\rho}\,P_X \quad \text{and} \quad \Phi_{YT} = -\frac{1}{\rho}\,P_Y - g \quad \text{in} \quad 0 < Y < d, \\ \Phi_T + g\,F(X,T) = 0 \quad \text{and} \quad \Phi_Y = F_T \quad \text{on} \quad Y = d, \\ \Phi_Y = H_T \quad \text{on} \quad Y = 0\,. \end{cases}$$

Consequently

$$0 = \Phi_{TT} + g\,F_T(X,T) = \phi_{TT} + g\,\Phi_Y \quad \text{on} \quad Y = d,$$

and we recover the system (2.20).

Rather than working with the linear system (2.20) in the physical variables, we prefer to analyze the system (2.16) in non-dimensional scaled variables. This approach has the advantage that, once we obtain the formula for the solution for a fixed $\delta$, since we are in non-dimensional variables, it is permissible to neglect small terms and we can consider the limits $\delta \to 0$ (shallow water waves) and $\delta \to \infty$ (deep water waves). Another advantage of the formulation (2.13) is that it can be easily adapted to deal with seabed disturbances that do not act on still water but on a water flow that might even be rotational (such flows with vorticity describe non-uniform currents — see [4, 6] for aspects of wave-current interactions related to tsunami). The reason is that, in contrast to (2.20), a velocity potential is not required for (2.13) and stream functions are known to exist for two-dimensional water flows with vorticity (cf. [7, 8]).

## 3. The general solution formula for linear waves

We solve the problem (2.16) by using the space-time Fourier transform. More precisely, we will denote in the following

$$
\begin{aligned}
\hat{\varphi}(\xi, y, t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \varphi(x, y, t)\, dx \\
\tilde{\varphi}(\xi, y, \omega) &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(x\xi + \omega t)} \varphi(x, y, t)\, dx dt.
\end{aligned} \tag{3.1}
$$

This enables us to define on the Schwartz class (functions that are smooth and for which all derivatives decay faster than polynomials at infinity) the Fourier multiplier $m(D)$ by

$$
(\widehat{m(D)\varphi})\,(\xi) = m(\xi)\widehat{\varphi}(\xi), \qquad \xi \in \mathbb{R},
$$

where $D = -i\,\partial_x$. With these notations,

$$
\begin{cases}
\tilde{\psi}_{yy} - \delta^2\, \xi^2\, \tilde{\psi} = 0 \quad \text{in} \quad 0 < y < 1, \\
\xi^2\, \tilde{\psi} = \omega^2\, \tilde{\psi}_y \quad \text{on} \quad y = 1, \\
\xi\, \tilde{\psi} = -\omega\, \tilde{h} \quad \text{on} \quad y = 0\,.
\end{cases} \tag{3.2}
$$

The general solution of the differential equation is of the form

$$
\tilde{\psi}(\xi, y, \omega) = A(\xi, \omega)\, e^{\delta\xi y} + B(\xi, \omega)\, e^{-\delta\xi y}, \qquad 0 \le y \le 1,
$$

and $A$, $B$ are determined from the boundary conditions. Since by (2.17) we have

$$
\omega\, \tilde{f}(\xi, \omega) = -\xi\, \tilde{\psi}(\xi, 1, \omega),
$$

a straightforward calculation yields

$$
\tilde{f}(\xi, \omega) = \frac{\omega^2 \delta^2}{\omega^2 \delta^2\, \cosh(\delta\xi) - \delta\xi\, \sinh(\delta\xi)}\, \tilde{h}(\xi, \omega)\,. \tag{3.3}
$$

Since the function $s \mapsto s\, \tanh(s)$ is even on $\mathbb{R}$ and an increasing diffeomorphism of $[0, \infty)$, for every fixed $\omega > 0$ there are precisely two roots $\xi_\pm(\omega)$ with $\xi_-(\omega) = -\xi_+(\omega)$ of the equation $\omega^2\delta^2\, \cosh(\delta\xi) - \delta\xi\, \sinh(\delta\xi) = 0$. In the limit $\omega \downarrow 0$ the two roots coalesce to $\xi_\pm(0) = 0$, while $\xi_+(\omega) \asymp \omega^2\delta$ for $\omega \to \infty$ since $\tanh(s) \asymp 1$ for $s \to \infty$. It is convenient to apply to (3.3) the inverse Fourier transform, obtaining that

$$
\left[ \partial_t^2 + \frac{D}{\delta}\, \tanh(\delta D) \right]\, f = \frac{\partial_t^2}{\cosh(\delta D)} h. \tag{3.4}
$$

Let us now assume that the up- and downward motions of the sea bottom compensate at each instant:

$$
\int_{\mathbb{R}} h(x, t)\, dx = 0, \qquad t \in \mathbb{R}\,. \tag{3.5}
$$

This will cancel a singularity in the next formula.The considerations in the appendix, cf. (5.7) and (5.8), yield the representation

$$
f(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \int_{\mathbb{R}} \frac{e^{i[(t-s)\sqrt{\tau(\xi)} + x\xi]} - 1}{\sqrt{\tau(\xi)}}\, \frac{\hat{h}_{tt}(\xi, s)}{\cosh(\delta\xi)}\, d\xi ds \tag{3.6}
$$

of the solution, where $\tau(\xi) = \frac{\xi}{\delta}\tanh(\delta\xi)$.

## 4. Asymptotic behaviour of linear waves generated by a moving bed

4.1. **The shallow water regime.** As $\delta \to 0$, equation (3.4) becomes the inhomogeneous linear wave equation

$$\left[\partial_t^2 - \partial_x^2\right] f = \partial_t^2 h.$$

The initial conditions are $f(x,0) = f_t(x,0) = 0$ due to (2.5) and (2.17), so that Duhamel's formula (cf. [19]) yields

$$f(x,t) = \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} h_{tt}(r,s)\,dr\,ds\,, \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{4.1}$$

We will now see that this model gives surprisingly accurate predictions for the propagation of some tsunamis. Let $h(x,t) = a(t)\,b(x)$ with $a \in C^2(\mathbb{R},[0,\infty))$, $b \in C^2(\mathbb{R},\mathbb{R})$ such that $a(t) = 0$ for $t \leq 0$, $a(t) = 1$ for $t > t_0$ (with $t_0 > 0$ representing the duration of the earthquake in the case study of the generation of tsunami waves by submarine earthquakes), and with $b(x) = 0$ for $x \notin (-L, L)$ modelling a localized tsunami source. In this case (4.1) takes on the simpler form

$$f(x,t) = a(t)b(x) + \frac{1}{2}\int_0^t a(s)\,\Big(b'(x+t-s) - b'(x-t+s)\Big)\,ds. \tag{4.2}$$

In the limiting case $t_0 \downarrow 0$, $a(t)$ becomes the Heaviside step function (modelling an instantaneous upward thrust of the seabed near to the earthquake's epicentre) and (4.2) simplifies to

$$f(x,t) = \frac{b(x+t) + b(x-t)}{2}, \qquad x \in \mathbb{R}, \quad t > 0. \tag{4.3}$$

In the special case (4.3) some simple but insightful conclusions can be drawn:

- at each instant $t > 0$ after initiation, the generated wave is localised as $f(x,t) = 0$ for $|x| \geq L + t$;
- the surface wave consists of two travelling waves, one moving to the right and the other moving to the left;
- each of the above travelling waves moves with non-dimensionalised speed 1 (corresponding to the speed $\sqrt{gd}$ in the physical variables) and with a shape that remains unchanged and is precisely that of the bed deformation at half-scale — in particular, the amplitude is half that of the bed deformation.

These conclusions are realistic. Indeed, let us discuss the two largest tsunamis for which records are available — the December 2004 and the May 1960 tsunamis. The December 2004 tsunami was triggered by a submarine earthquake at the interface of the India and Burma plates, off Sumatra. The tsunami initiation region was roughly elliptical, about 100 $km$ wide and more than 1000 $km$ long, with the disturbance subsequently moving eastwards towards Sumatra and Thailand, and westwards towards Sri Lanka and India (cf. [16]). The generating earthquake raised the ocean floor up to 2 $m$ over a distance of about 100 $km$ to the west of the epicenter and lowered it to the east (over 900 $km$). The Indian Ocean/Bay of Bengal as well as the Andaman Basin through which the tsunami propagated are relatively flat, with average depths $d = 4\,km$, respectively $d = 1\,km$. As an indication of the applicability of the above simple considerations, we notice that for India and Sri Lanka, the first tsunami wave reaching the shore was a wave of elevation, while in southern Thailand a wave of depression signaled the arrival of the tsunami. This is consistent with our predictions based on (4.3) since the bed deformation presents a depression eastwards and and elevation westward of the earthquake's epicentre. Also, accurate measurements, provided by a radar altimeter on board a satellite along a track traversing the Indian

Ocean/Bay of Bengal, indicate that out in the open sea the tsunami amplitude was about $1\,m$ (cf. [6]). This is consistent with the halving of the amplitude encompassed in (4.3). Furthermore, the above theory predicts a wave speed of approximately $712\,km/h$, corresponding to $\sqrt{gd}$ with $g = 9.8\,m/s^2$ and $d = 4\,km$. This would mean that in $2\,h\,12\,min$ the tsunami waves would cover a distance of about $1566\,km$, and this is the real time the tsunami needed for the $1550\,km$ across the Indian Ocean/Bay of Bengal, between the epicentre of the earthquake and the firstly affected coast of Sri Lanka (cf. [3]). For the Andaman basin, a predicted speed of $356\,km/h$ would correspond to $\sqrt{gd}$ with $d = 1\,km$. This is also quite accurate since the Thai resort at Hat Ray Leh, about $700\,km$ far from the epicentre of the earthquake, was hit by the tsunami $2\,h$ after initiation (cf. [6]). While the field data for the May 1960 tsunami is scarcer (see, however, the reference list in [5]), it is known that it was caused by the the largest earthquake ever recorded. Several earthquakes in fast succession occurred along $1000\,km$ of fault parallel to the Chilean coastline, with the epicentre within 200 km off the coast of Central Chile, and a submarine lift of $1\,m$ and subsidence of $1.6\,m$ ensued over a stretch of $300\,km$ (cf. [3]). Tsunami waves propagated in the northwest direction across the Pacific Ocean, reaching Hawaii, at about $10500\,km$ from the epicentre of the earthquake, after $14\,h\,48\,min$. The ocean floor of the Central Pacific Basin between Chile and Hawaii is relatively uniform, with a mean depth of about $d = 4\,km$. The above simple theory would therefore predict a wave speed $c = \sqrt{gd} \approx 712\,km/h$, so that the waves would travel $10535\,km$ in $14\,h\,48\,min$, which is about right.

4.2. **The regime of finite, non vanishing $\delta$.** The asymptotic formula (5.13) for the solution to the equation (3.4) is derived in the appendix. It gives

$$f(x,t) \sim -\sqrt{\frac{2\pi}{t}}\frac{\sqrt{\tau(\xi_0)}}{\sqrt{\left|\left[\sqrt{\tau}\right]''(\xi_0)\right|}}\frac{1}{\cosh(\delta\xi_0)}\mathfrak{Im}\left[e^{it(\sqrt{\tau}(\xi_0)+\mathcal{X}\xi_0-\frac{\pi}{4})}\hat{h}(\xi_0,\sqrt{\tau(\xi_0)})\right]$$

where

$$\mathcal{X} = \frac{x}{t}, \quad \tau(\xi) = \frac{\xi}{\delta}\tanh(\delta\xi) \quad \text{and} \quad \left[\sqrt{\tau}\right]'(\xi_0) + \frac{x}{t} = 0.$$

For tsunamis propagating in open sea, typical values of the physical parameters introduced above are (see for instance [3])

$$a = 1\,m, \quad h = 4\,km, \quad \lambda = 200 km.$$

This leads to $\varepsilon = 0.00025$ and $\lambda = 0.02$, which points toward the linear wave approximation. The stationary phase analysis conducted above is justified if

$$t\sqrt{\tau}''(\xi) >> 1. \tag{4.4}$$

Using the discussion in the appendix, $\partial_\xi\sqrt{\tau(\xi)}$ can be written as $F(\delta\xi)$, for an odd function $F$, decreasing on $(0,\infty)$, with $F(0+) = 1$ and $F(\infty) = 0$. It makes sense to restrict $\mathcal{X}$ away from 0: very low frequencies travel very slowly, and correspond thus to waves which will barely reach the coast. With this restriction (say $0.2 < \mathcal{X} < 1$), the stationary phase point $\xi > 0$ where $\partial_\xi\sqrt{\tau(\xi)} + \mathcal{X} = 0$ is of order $\frac{1}{\delta}$. Now, for $\delta\xi = O(1)$, (5.12) yields $|\partial_\xi^2\sqrt{\tau(\xi)}| = O(\delta)$ so that from (4.4) we get the criterion

$$t >> \frac{1}{\delta^2}.$$

Coming back to physical variables, this corresponds to

$$T >> \frac{\lambda}{\sqrt{gd}\delta^2} \sim 2\cdot 10^5 s \sim 50h.$$

Thus, with the choice of the above parameters, stationary phase analysis does not seem to be justified. However, under different physical conditions, stationary phase analysis could be the right choice.

4.3. **The deep water regime.** As $\delta \to \infty$, the equation (3.4) becomes equivalent (to first order in $\delta$) to

$$\left[\partial_t^2 + \frac{|D|}{\delta}\right]\eta = 2\,e^{-\delta|D|}\partial_t^2 h.$$

This corresponds to a dispersion relation $\tau(\xi) = \frac{|\xi|}{\delta}$. One finds $\xi_0 = \frac{t^2}{4\delta x^2}$ and formula (5.3) gives that for $t$ large

$$f(x,t) \sim \sqrt{\frac{\pi}{t}}\left(\frac{t}{\sqrt{\delta x}}\right)^{5/2} e^{-\frac{t^2}{4x^2}}\,\mathfrak{Im}\left[e^{i\left(\frac{t^2}{4\delta x}-\frac{\pi}{4}\right)}\tilde{h}\left(\frac{t}{2\delta x}, -\frac{t^2}{4\delta x^2}\right)\right].$$

This regime is not adequate to model the 2004 Boxing Day or the 1960 Chile tsunami.

## 5. Appendix: asymptotic formulas

5.1. **A general approach.** We want to derive an asymptotic formula, as $t \to \infty$, for the solution $f(x,t)$ of

$$\begin{cases} \left[\partial_t^2 + \tau(D)\right]\, f = \theta, \\ \lim\limits_{t\downarrow -\infty} f = 0, \end{cases} \tag{5.1}$$

where $\tau$ is a given dispersion relation, such that $\tau(\xi) > 0$ for any $\xi \neq 0$, and $\theta$ is a given forcing term. We assume that $\theta$ and $f$ are both of Schwartz class $\mathcal{S}(\mathbb{R}^2,\mathbb{R})$, the limit in (5.1) being also interpreted in the Schwartz class. If $f \in \mathcal{S}(\mathbb{R}^2,\mathbb{R})$ solves (5.1), then the function $v = \partial_t f + i\sqrt{\tau(D)}\,f$ solves $\left[\partial_t - i\sqrt{\tau(D)}\right]v = \theta$. Filtering $v$ by the group $e^{it\sqrt{\tau(D)}}$ gives that $u = e^{-it\sqrt{\tau(D)}}\,v$ solves $\partial_t u = e^{-it\sqrt{\tau(D)}}\,\theta$. This last equation can be solved by integrating in time; coming back to $f$, this means

$$\left(\sqrt{\tau(D)}\,f\right)(x,t) = \mathfrak{Im}\left\{\int_{-\infty}^{t} e^{i(t-s)\sqrt{\tau(D)}}\theta(x,s)\,ds\right\}. \tag{5.2}$$

Spelling out the Fourier multipliers in the above formula, and writing the formula in order to highlight the oscillatory nature of the integral, we see that with $\mathcal{X} = \frac{x}{t}$,

$$f(x,t) = \frac{1}{\sqrt{2\pi}}\,\mathfrak{Im}\left\{\int_{-\infty}^{t}\int_{\mathbb{R}}\frac{1}{\sqrt{\tau(\xi)}}e^{it(\sqrt{\tau(\xi)}+\mathcal{X}\xi)}e^{-is\sqrt{\tau(\xi)}}\hat{\theta}(s,\xi)\,d\xi\,ds\right\}.$$

The stationary phase principle gives for $t$ large,

$$f(x,t) \sim \frac{1}{\sqrt{t\,\tau(\xi_0)\left|[\sqrt{\tau}]''(\xi_0)\right|}}\,\mathfrak{Im}\left\{\int_{-\infty}^{t} e^{it\left[(\sqrt{\tau(\xi_0)}+\mathcal{X}\xi_0)+\sigma\frac{\pi}{4}\right]}e^{-is\sqrt{\tau(\xi_0)}}\hat{f}(s,\xi_0)\,ds\right\}$$

for $t$ large, where $\xi_0$ is the function of $\mathcal{X}$ such that $[\sqrt{\tau}]'(\xi_0) + \mathcal{X} = 0$, and $\sigma$ is the sign of $\sqrt{\tau}''(\xi_0)$. Performing the integration over $s \in \mathbb{R}$ in the above integral gives the asymptotics: for $t$ large,

$$f(x,t) \sim \frac{1}{\sqrt{t\tau(\xi_0)\left|[\sqrt{\tau}]''\right|(\xi_0)}}\,\mathfrak{Im}\left\{e^{it\left[(\sqrt{\tau(\xi_0)}+\mathcal{X}\xi_0)+\sigma\frac{\pi}{4}\right]}\tilde{\theta}(\sqrt{\tau(\xi_0)},\xi_0)\right\}. \tag{5.3}$$

The above derivation has been rigorous, except for two points: we assumed implicitly the uniqueness of $\xi_0$; and we divided by $\sqrt{\tau}$, which is problematic where this function vanishes. We will address these two issues in the next paragraph, where we specialize the above analysis to the case of linear water waves.

5.2. **The case of linear water waves.** We apply here the analysis of the previous paragraph to the equation

$$\left[\partial_t^2 + \frac{D}{\delta}\tanh(\delta D)\right] f = \frac{\partial_t^2}{\cosh(\delta D)} h \tag{5.4}$$

where $h \in \mathcal{D}(\mathbb{R}^2, \mathbb{R})$ is a test function (smooth and with compact support). The setting of the previous paragraph corresponds to the notations

$$\theta(x,t) = \Big(\frac{\partial_t^2 h}{\cosh(\delta D)}\Big)(x,t) \in \mathcal{S}(\mathbb{R}^2, \mathbb{R}), \tag{5.5}$$

and

$$\tau(\xi) = \frac{\xi}{\delta}\tanh(\delta\xi), \qquad \xi \in \mathbb{R}. \tag{5.6}$$

It is easily seen that

- $\tau : \mathbb{R} \to \mathbb{R}$ is an even function, $\tau(-\xi) = \tau(\xi)$;
- $\tau(\xi) > 0$ for $\xi \neq 0$;
- $\limsup_{\xi\to 0} \frac{\tau(\xi)}{\xi^2} \leq 1$;
- $\tau$ has linear growth at infinity .

Since the first property ensures that $\tau(D)\,\varphi$ is real-valued if $\varphi \in \mathcal{S}(\mathbb{R}, \mathbb{R})$, (5.2) yields

$$\Big(\sqrt{\tau(D)}\, f\Big)(x,t) = \mathfrak{Im}\Big\{\int_{-\infty}^t e^{i(t-s)\sqrt{\tau(D)}}\,\theta(x,s)\,ds\Big\},$$

so that

$$f(x,t) = \frac{1}{\sqrt{2\pi}}\,\mathfrak{Im}\Big\{\int_{-\infty}^t \int_{\mathbb{R}} \frac{e^{i(t-s)\sqrt{\tau(\xi)}}}{\sqrt{\tau(\xi)}}\, e^{ix\xi}\hat{\theta}(\xi,s)\,d\xi\,ds\Big\}. \tag{5.7}$$

Let us now assume that $h \in \mathcal{D}(\mathbb{R}^2, \mathbb{R})$ is such that (3.5) holds — this hypothesis expresses that at each instant the upward motion of the sea bed is compensated by the downward motion. This hypothesis cancels the singularity at $\xi = 0$ which is present in (5.7). Indeed, since (3.5) ensures $\hat{h}(0,s) = 0$ for all $s \in \mathbb{R}$ and

$$\hat{\theta}(\xi,s) = \frac{\hat{h}_{tt}(\xi,s)}{\cosh(\delta\xi)}, \tag{5.8}$$

we get $\hat{\theta}(0,s) = 0$ for all $s \in \mathbb{R}$. To make the cancellation of the singularity more obvious, we can rewrite (5.7) as

$$f(x,t) = \frac{1}{\sqrt{2\pi}}\,\mathfrak{Im}\Big\{\int_{-\infty}^t \int_{\mathbb{R}} \frac{e^{i(t-s)\sqrt{\tau(\xi)}}e^{ix\xi} - 1}{\sqrt{\tau(\xi)}}\,\hat{\theta}(\xi,s)\,d\xi\,ds\Big\}. \tag{5.9}$$

As the Fourier transform acts on the Schwartz class as an isomorphism, the absence of singularities on the right of the above ensures $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{R})$. For the evaluation of the asymptotic behaviour of the solution (5.9) by means of the stationary phase principle, let us write (5.9) in the equivalent form

$$f(x,t) = \frac{1}{\sqrt{2\pi}}\,\mathfrak{Im}\Big\{\int_{-\infty}^t \int_{\mathbb{R}} \frac{e^{it(\sqrt{\tau(\xi)}+\mathcal{X}\xi)-is\sqrt{\tau(\xi)}} - 1}{\sqrt{\tau(\xi)}}\,\hat{\theta}(\xi,s)\,d\xi\,ds\Big\} \tag{5.10}$$

where $\mathcal{X} = \frac{x}{t}$. A computation gives

$$\partial_\xi\sqrt{\tau(\xi)} = \frac{\sinh(\delta\xi)\cosh(\delta\xi) + \delta\xi}{2\delta\,\cosh^2(\delta\xi)}\,\sqrt{\frac{\delta\cosh(\delta\xi)}{\xi\sinh(\delta\xi)}} \tag{5.11}$$

and

$$\partial_\xi^2 \sqrt{\tau(\xi)} = -\frac{1}{\tau^{3/2}(\xi)}\,\frac{[\sinh(\delta\xi)\cosh(\delta\xi)-\delta\xi]^2 + 4\,\delta^2\xi^2\sinh^2(\delta\xi)}{4\,\delta^2\cosh^4(\delta\xi)} > 0\,. \tag{5.12}$$

Consequently, $\xi \mapsto \partial_\xi\sqrt{\tau(\xi)}$ is strictly decreasing on $(0,\infty)$ from the asymptotic value 1 towards the asymptotic value 0, and on $(-\infty,0)$ from the asymptotic value 0 towards the asymptotic value $-1$. Thus stationary points that possibly contribute towards the leading order exist only for the range $\mathcal{X} \in (-1,0)\cup(0,1)$, and in this range each fixed $\mathcal{X}$ produces precisely one stationary point $\xi_0 = \xi_0(\mathcal{X})$. Since

$$\tilde{\theta}(\xi_0, \sqrt{\tau(\xi_0)}) = -\frac{\tau(\xi_0)\,\tilde{h}(\xi_0,\sqrt{\tau(\xi_0)})}{\cosh(\delta\xi_0)},$$

the formula (5.3) becomes

$$f(x,t) \sim -\frac{\sqrt{2\pi\tau(\xi_0)}}{\cosh(\delta\xi_0)\sqrt{t\,\left|[\sqrt{\tau}]''(\xi_0)\right|}}\,\mathfrak{Im}\left\{e^{it\left[(\sqrt{\tau(\xi_0)}+\mathcal{X}\xi_0)+\frac{\pi}{4}\right]}\,\tilde{h}(\xi_0,\sqrt{\tau(\xi_0)})\right\}. \tag{5.13}$$

(notice that the singular factor $\frac{1}{\sqrt{\tau}}$ present in (5.3) has been canceled). For $\xi > 0$ fixed, by expanding in (5.11) in power series (in terms of $\delta\xi$), we see that $\delta \mapsto \partial_\xi\sqrt{\tau(\xi)}$ is a smooth ($C^\infty$) and even function of $\delta$. Since $\lim_{\delta\to 0}\partial_\xi\sqrt{\tau(\xi)} = 1$, we must have $\partial_\xi\sqrt{\tau(\xi)} = 1 + \alpha(\xi)\,\delta^2 + O(\delta^3)$ for some function $\alpha(\xi)$. To determine $\alpha(\xi)$, notice that $2\alpha(\xi)$ is the coefficient of $\delta^2$ in the expansion of $[\partial_\xi\sqrt{\tau(\xi)}]^2$ with respect to the parameter $\delta$. As

$$[\partial_\xi\sqrt{\tau(\xi)}]^2 = \frac{\sinh(\delta\xi)}{4\delta\xi\,\cosh(\delta\xi)} + \frac{1}{2\,\cosh^2(\delta\xi)} + \frac{\delta\xi}{4\,\cosh^3(\delta\xi)\,\sinh(\delta\xi)}\,,$$

expanding in power series with respect to $\delta\xi$ yields $[\partial_\xi\sqrt{\tau(\xi)}]^2 = 1 - \delta^2\xi^2 + O(\delta^3)$. Therefore $\partial_\xi\sqrt{\tau(\xi)} = 1 - \frac{1}{2}\,\delta^2\xi^2 + O(\delta^3)$. For $\mathcal{X} \in (-1,0)$ we deduce that the stationary phase point $\xi_0 > 0$ where $\partial_\xi\sqrt{\tau(\xi)} + \mathcal{X} = 0$ is of order $\dfrac{1}{\delta}$. Now, for $\delta\xi_0 = O(1)$, note that (5.12) yields $|\partial_\xi^2\sqrt{\tau(\xi_0)}| = O(\delta)$.

Acknowledgement. The support by the ERC Advanced Grant "Nonlinear studies of water flows with vorticity" (NWFV) is gratefully acknowledged.

## References


[1] B. Alvarez-Samaniego and D. Lannes, Large time existence for 3D water waves and asymptotics, *Invent. Math.* **171** (2008), 485–541.

[2] T. P. Barnett and K. E. Kenyon, Recent advances in the study of wind waves, *Rep. Prog. Phys.* **38** (1975), 667–729.

[3] A. Constantin, On the relevance of soliton theory to tsunami modelling, *Wave Motion* **46** (2009), 420–426.

[4] A. Constantin, A dynamical systems approach towards isolated vorticity regions for tsunami background states, *Arch. Rat. Mech. Anal.* **200** (2011), 239–253.

[5] A. Constantin and D. Henry, Solitons and tsunamis, *Z. Naturforsch.* **64a** (2009), 65–68.

[6] A. Constantin and R. S. Johnson, Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis, *Fluid Dyn. Res.* **40** (2008), 175–211.

[7] A. Constantin and W. Strauss, Exact steady periodic water waves with vorticity, *Comm. Pure Appl. Math.* **57** (2004), 481–527.

[8] D. Coutand and S. Shkoller, Well-posedness of the free-surface incompressible Euler equations with or without surface tension, *J. Amer. Math. Soc.* **20** (2007), 829–930.

[9] D. Dutykh and F. Dias, Water waves generated by a moving bottom, in *Tsunami and nonlinear waves*, Springer (Ed. A. Kundu), pp. 65–95 (2007).

[10] D. Dutykh, F. Dias and Y. Kervella, Linear theory of wave generation by a moving bottom, *C.R. Acad. Sci. Paris* **343** (2006), 499–504.

[11] P. Germain, N. Masmoudi and J. Shatah, Global solutions for the gravity water waves equation in dimension 3, *Ann. Math* (to appear).

[12] A. Hayir, Ocean depth effects on tsunami amplitudes used in source models in linearized shallow-water wave theory, *Ocean Eng.* **31** (2004), 353-361.

[13] R. S. Johnson, *A modern introduction to the mathematical theory of water waves*, Cambridge University Press, Cambridge, 1997.

[14] J. Lighthill, *Waves in fluids*, Cambridge University Press, Cambridge, 1978.

[15] K. T. Ramadan, H. S. Hassan and S. N. Hanna, Modeling of tsunami generation and propagation by a spreading curvilinear seismic faulting in linearized shallow-water wave theory, *Appl. Math. Model.* **35** (2011), 61–79.

[16] H. Segur, Waves in shallow water, with emphasis on the tsunami of 2004, in "Tsunami and Nonlinear Waves" (ed. A. Kundu), Springer, pp. 3–29 (2007).

[17] R. Stuhlmeier, KdV theory and the Chilean tsunami of 1960, *Discrete Contin. Dyn. Syst. Ser. B* **12** (2009), 623–632.

[18] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton, NJ, 1993.

[19] W. A. Strauss, *Partial differential equations. An introduction*, J. Wiley & Sons, Ltd., Chichester, 2008.

[20] M. I. Todorovska, A. Hayir and M. D. Trifunac, A note on tsunami amplitudes above submarine slides and slumps, *Soil Dyn. Earthq. Eng.* **22** (2002), 129–141.

[21] S. Wu, Global well-posedness of the 3-D full water wave problem, *Invent. Math.* (to appear).

Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, 1090 Vienna, Austria

*E-mail address*: adrian.constantin@univie.ac.at

Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, N.Y. 10012, USA

*E-mail address*: pgermain@cims.nyu.edu