This chapter will mostly be dedicated to the study of the linear Schrödinger equation
\[
\begin{cases}
i\partial_t u - \Delta u = 0, \\
u(t = 0) = u_0
\end{cases}
\]
\[(LS)\]
where \(u(t, x) \in \mathbb{C}\) is a function of \(t \in \mathbb{R}\) and \(x \in \mathbb{R}^d\).

The most important phenomena in this chapter will be dispersion, which can be summarized as follows: different frequencies travel at different speeds. To be more explicit: we will see that, if the data of (LS) is localized in space around 0, and in frequency around \(\xi\), then \(u\) will travel, roughly speaking, at velocity \(\xi\). In other words: \(u\) will be localized in space-time around \(\{x \sim \xi t\}\). Of course, general data can be broken into smaller pieces, each of which with a given frequency localization.

This seemingly simple phenomenon has far reaching, and often delicate to understand, consequences. The aim of the following is to review and quantify them.

When the equation is set on the torus, the picture becomes much more complicated, since geodesics tend to wrap up the torus instead of escaping to infinity. We will say a few words about this case.

1. Plane waves and wave packets

We establish some elementary properties of (LS) in this introductory section; for the moment, \(u_0\) is assumed to be smooth so that it is easy to make sense of all manipulations.

1.1. Plane waves. Plane waves are the simplest solutions of (LS) set on \(\mathbb{R}^d\): they read
\[u(t, x) = e^{i(x\xi + t|\xi|^2)}, \quad \text{for } \xi \in \mathbb{R}^d.\]

By linearity of (LS), any solution of (LS) can be decomposed into plane waves. This is achieved by taking the Fourier transform (in space) of (LS); the equation becomes
\[
\begin{cases}
i\partial_t \hat{u}(t, \xi) + |\xi|^2 \hat{u}(t, \xi) = 0 \\
\hat{u}(t = 0, \xi) = \hat{u}_0(\xi).
\end{cases}
\]

This is simply an ODE! Its solution reads
\[\hat{u}(t, \xi) = e^{it|\xi|^2} \hat{u}_0(\xi).\]

Consistently with our notation for Fourier multipliers, we write equivalently for the above
\[u(t, x) = e^{-it\Delta} u_0.\]

Coming back to \(u\) through the inverse Fourier transform gives
\[
u(t, x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i(x\xi + t|\xi|^2)} \hat{u}_0(\xi) \, d\xi.
\]\[1.1\]
1.2. The fundamental solution. The previous formula gives \( u \) as a superposition of plane waves weighted by \( \hat{u}_0(\xi) \). In order to substitute to this formula a physical space approach, we need to compute the Fourier transform of complex Gaussians.

**Lemma 1.1.** If \( \sigma \in \mathbb{C} \) with \( \Re \sigma > 0 \),

\[
\mathcal{F}(e^{i\sigma x^2}) = \left( \frac{\pi}{2\sigma} \right)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4\sigma}},
\]

where \( \left( \frac{1}{2\sigma} \right)^{\frac{d}{2}} \) is the square root of \( \left( \frac{1}{2\sigma} \right)^d \) with a positive real part.

**Proof.** It suffices to treat the case \( d = 1 \) and compute \( \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi - \sigma x^2} dx \). By first completing the square, and then using contour integration, one obtains that

\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi - \sigma x^2} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{4\sigma}} \int_{\mathbb{R}} e^{-\sigma(x+i\frac{\xi}{2\sigma})^2} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{4\sigma}} \int_{\mathbb{R}} e^{-\sigma x^2} dx.
\]

Using once again contour integration and the well-known Gaussian integral \( \int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi} \),

\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi - \sigma x^2} dx = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{\xi^2}{4\sigma}} \int_{\mathbb{R}} e^{-\sigma x^2} dx = \frac{1}{\sqrt{2\sigma}} e^{-\frac{\xi^2}{4\sigma}}.
\]

\( \square \)

The fundamental solution of the Schrödinger equation corresponds to the case where \( \Re \sigma = 0 \); it can be reached by a limiting argument. Indeed, assuming for simplicity that \( u_0 \in \mathcal{S} \),

\[
u(t, x) = \mathcal{F}^{-1}(e^{i|\xi|^2/2} u_0(\xi)) = \lim_{\epsilon \to 0} \mathcal{F}^{-1}(e^{i(t-\epsilon)\xi^2/2} u_0(\xi))
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{(2\pi)^{d/2}} \mathcal{F}^{-1}(e^{i(t-\epsilon)|\xi|^2}) * \mathcal{F}^{-1}(\hat{u}_0(\xi))
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{(2\pi)^{d/2} (2(\pi t-\epsilon))^{d/2}} \int_{\mathbb{R}^d} e^{-|y|^2/(2(\pi t-\epsilon))} u_0(y) dy
\]

\[
= \frac{1}{(4\pi i)^{d/2} t^{d/2}} \int_{\mathbb{R}^d} e^{-|x-y|^2/t} u_0(y) dy.
\]

In other words, the fundamental solution of (LS) is given by the following kernel

\[
u(t, x) = [K_t \ast u_0](x) \quad \text{with} \quad K_t(x) = \frac{1}{(4\pi i)^{d/2} t^{d/2}} e^{i|\xi|^2/4t}.
\]

1.3. Wave packets. Wave packets are characterized by the fact that they are well localized in space and frequency. In order to understand their behavior as \( t \to \infty \), we will view (2.1) as an oscillatory integral and use the stationary phase lemma.

**Lemma 1.2** (Stationary phase). Assume that \( \phi \) is smooth and \( f \in C_0^\infty \). Assume furthermore that \( \nabla \phi \) only vanishes at \( x_0 \in \mathbb{R}^d \) and that \( \text{Hess} \phi(x_0) \) is non degenerate. Then

\[
I(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda \phi(x)} f(x) dx = \frac{C}{\lambda^{d/2} (\text{det Hess} \phi)^{d/2}} e^{i\lambda \phi(x_0)} f(x_0) + O \left( \frac{1}{\lambda^{(d+1)/2}} \right).
\]

In order to apply this lemma to (2.1), we write it as

\[
u(t, x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i(x\xi + t|\xi|^2)} \hat{u}_0(\xi) d\xi = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\lambda \phi(x)} \hat{u}_0(\xi) d\xi
\]

where we set

\[
X = \frac{x}{t} \quad \text{and} \quad \Phi_X(\xi) = X\xi + |\xi|^2.
\]
It is easy to see that
\[ \nabla \Phi_X = X + 2\xi \quad \text{while} \quad \text{Hess } \Phi_X = 2 \text{Id}. \]
Therefore, \( \nabla \Phi_X \) only vanishes at \( \xi = -\frac{X}{2} \). The stationary phase lemma gives that
\[ u(t,x) \sim \frac{C}{t^{d/2}} e^{-i|\frac{X}{2}|^2} \widehat{u_0} \left( -\frac{X}{2} \right) \]
or in other words that
\[ u(t,x) \sim \frac{C}{t^{d/2}} e^{-i\frac{|x|^2}{4t}} \widehat{u_0} \left( -\frac{x}{2t} \right). \]
(1.2)

This interesting asymptotic formula followed from an application of the stationary phase lemma for \( X \) (hence \( \frac{X}{2} \)) fixed. It can be proved to hold with some uniformity in \( X \), but we are happy to remain at a heuristic level for the moment.

Let us now pick data which are wave packets, which means that they are well localized in space and frequency. More precisely, assume that
\[ \text{Supp } \widehat{u_0} \subset B(\xi_0, r). \]

Then we learn from (1.2) that
- As \( t \to \infty \), the decay rate of \( u \) is \( \sim \frac{1}{t^{d/2}} \).
- As \( t \to \infty \), the solution \( u \) is mostly localized, at time \( t \), in \( B(-2t\xi_0, 2tr) \). In other words, \( u \) is propagating at the group velocity \(-2\xi_0\), and spreading on a scale \( \sim t \).

2. Elementary \( L^2 \) estimates

2.1. Group property on Sobolev spaces. Recall that, for a regularity index \( s \), the inhomogeneous and homogeneous Sobolev spaces, denoted \( H^s(\mathbb{R}^d) \) and \( \dot{H}^s(\mathbb{R}^d) \) respectively, are defined by their norms
\[ \| f \|_{H^s} = \| \langle D \rangle^s f \|_{L^2} \quad \text{and} \quad \| f \|_{\dot{H}^s} = \| \| D \|^{s} f \|_{L^2}. \]
We saw that, for a smooth data \( u_0 \), the solution of (LS) is given by
\[ e^{-it\Delta}u_0(\xi) = e^{it|\xi|^2} \widehat{u_0}(\xi). \]
Since the Fourier transform is an isometry in \( L^2 \), and \( |e^{it|\xi|^2}| = 1 \), we get that, for \( u_0 \) smooth,
\[ \| e^{-it\Delta}u_0 \|_{H^s} = \| u_0 \|_{H^s} \quad \text{and} \quad \| e^{-it\Delta}u_0 \|_{\dot{H}^s} = \| u_0 \|_{\dot{H}^s}. \]
It is now easy to extend \( e^{it\Delta} \) to all of \( H^s \) and \( \dot{H}^s \). This can be done by adopting the Fourier multiplier definition, which makes sense on these Sobolev spaces. Another (equivalent) possibility is to extend the operator \( e^{-it\Delta} \), using the following result: an linear operator from a Banach space \( A \) to a Banach space \( B \), which is defined on a dense subset of \( A \), and bounded, can be extended in a unique way to a bounded operator from \( A \) to \( B \).

This leads to the following lemma.

Lemma 2.1. Let \( s \in \mathbb{R} \). For any \( t \in \mathbb{R} \), the operator \( e^{-it\Delta} \) is an isometry on \( H^s \):
\[ \| e^{-it\Delta}f \|_{H^s} = \| f \|_{H^s}. \]
Furthermore, \( t \mapsto e^{it\Delta} = S(t) \) is a strongly continuous group on \( H^s \), namely
- \( S(0) = \text{Id} \).
- \( S(t)S(t') = S(t + t') \)
- If \( f \in H^s \), \( S(t)f \to S(t_0)f \) in \( H^s \) as \( t \to t_0 \).
2.2. Virial identity. This is the most simple way of quantifying dispersion.

Lemma 2.2. If $f$, $xf$, and $\nabla f$ belong to $L^2$, then

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^d} |x|^2 |e^{-it\Delta} f|^2 \, dx = 8 \int |\nabla f|^2 \, dx.$$ 

Proof. Taking the Fourier transform of $xe^{-it\Delta} f$,

$$\mathcal{F}(xe^{-it\Delta} f)(\xi) = i\nabla \xi e^{it|\xi|^2} \hat{f}(\xi) = e^{it|\xi|^2} (-2it\xi \hat{f}(\xi) + \nabla \xi \hat{f}(\xi)).$$

Since the Fourier transform is an isometry in $L^2$,

$$\|xe^{-it\Delta} f\|_{L^2}^2 = \| -2t\xi \hat{f}(\xi) + \partial_\xi \hat{f}(\xi)\|_{L^2}^2 = 4t^2 \|\hat{f}(\xi)\|_{L^2}^2 + \{\text{constant and linear terms in } t\}.$$ 

Applying $\frac{d^2}{dt^2}$ to the above inequality gives the desired result. \(\square\)

One can think of the identity above as the fact that $\int_{\mathbb{R}^d} |x|^2 |e^{-it\Delta} f|^2 \, dx \sim 4t^2 \int |\nabla f|^2 \, dx$. The growth of this weighted norm is due to dispersion: at time $t$, the typical scale at which the solution is localized is $|x| \sim 2t \int |\nabla f|^2$. If one imagines that $\int |f|^2 = 1$ and $\hat{f}$ is localized around $\xi_0$, this becomes $|x| \sim 2t|\xi_0|$. This is consistent with the value of the group velocity $-2\xi_0$.

2.3. Asymptotic equivalent. We aim at quantifying the asymptotic equivalent (1.2) in $L^2$ topology. Let us first explain how to make sense of $\frac{1}{4\pi^2} e^{-\frac{|y|^2}{4t^2}} \hat{f} \left( -\frac{x}{2t} \right)$ for $f \in L^2$. Observe that, for $f \in C_0^\infty$,

$$\frac{1}{(4\pi i)^{d/2}} e^{-\frac{|y|^2}{4t}} \hat{f} \left( -\frac{x}{2t} \right) = \frac{1}{(4\pi i)^{d/2}} e^{-\frac{|y|^2}{4t}} \int_{\mathbb{R}^d} e^{\frac{iy}{2t} \psi(y)} \, dy 
= \frac{1}{(4\pi i)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{4t^2}} e^{\frac{iy}{2t} f(y)} \, dy 
= e^{it\Delta} \left[ e^{\frac{|y|^2}{4t}} f(y) \right].$$

The key point is that the right-hand side is well defined for $f \in L^2$. For $f \in L^2$, we denote from now on

$$\frac{1}{(4\pi i)^{d/2}} e^{-\frac{|y|^2}{4t}} \hat{f} \left( -\frac{x}{2t} \right) = e^{it\Delta} \left[ e^{\frac{|y|^2}{4t}} f(y) \right] \quad (2.1)$$

Lemma 2.3. If $f \in L^2$,

$$\left\| e^{-it\Delta} f - \frac{1}{(4\pi i)^{d/2}} e^{-\frac{|y|^2}{4t^2}} \hat{f} \left( -\frac{x}{2t} \right) \right\|_{L^2} \to 0 \quad \text{as } t \to \infty.$$ 

Proof. By the definition (2.1),

$$\left\| e^{-it\Delta} f - \frac{1}{(4\pi i)^{d/2}} e^{-\frac{|y|^2}{4t^2}} \hat{f} \left( -\frac{x}{2t} \right) \right\|_{L^2} = \left\| e^{-it\Delta} f - e^{it\Delta} \left[ e^{\frac{|y|^2}{4t}} f(y) \right] \right\|_{L^2} 
= \left\| e^{-it\Delta} \left[ f(y) - e^{\frac{|y|^2}{4t}} f(y) \right] \right\|_{L^2}.$$

Since $e^{it\Delta}$ is an isometry,

$$\cdots \lesssim \left\| f(y) - e^{\frac{|y|^2}{4t}} f(y) \right\|_{L^2} \xrightarrow{t \to \infty} 0$$ 

by dominated convergence. \(\square\)
3. Decay estimates

3.1. Dispersive estimates. Dispersive estimates give decay at a polynomial rate for data in $L^p$ spaces.

**Theorem 3.1.** If $f \in C_0^\infty$, 
\[
\|e^{it\Delta}f\|_{L^p} \lesssim \frac{1}{|t|^\frac{d}{2} - \frac{d}{p}} \|f\|_{L^{p'}} \quad \text{if } 2 \leq p \leq \infty
\]
(where $p'$ is such that $\frac{1}{p} + \frac{1}{p'} = 1$).

In particular, by density of $C_0^\infty$ in $L^{p'}$, $e^{it\Delta}$ can be extended to an operator defined on all of $L^{p'}$ satisfying the above estimate.

**Proof.** By the Riesz-Thorin interpolation theorem, it suffices, in order to prove this theorem, to prove the endpoints $p = 2$ and $p = \infty$. We already proved that $e^{it\Delta}$ is an isometry on $L^2$; this takes care of $p = 2$. In order to treat $p = \infty$, we rely on the fundamental solution of the linear Schrödinger equation and the Young theorem:
\[
\|e^{it\Delta}f\|_{L^\infty} \lesssim \left\| \frac{1}{(4\pi i)^{d/2}} \frac{1}{t^{d/2}} e^{i|x|^2/4t} * f \right\|_{L^\infty} \lesssim \left\| \frac{1}{(4\pi i)^{d/2}} \frac{1}{t^{d/2}} e^{i|x|^2/4t} \right\|_{L^\infty} \|f\|_{L^1}. \]
\]

3.2. Strichartz estimates. These estimates give integrated decay for data in $L^2$ spaces.

**Theorem 3.2.** A pair $(q, r)$ of exponents is called admissible if
\[
\begin{cases}
2 \leq q, r \leq \infty \\
\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \\
(q, r, d) \neq (2, \infty, 2).
\end{cases}
\]

Assuming that $f$ and $F$ are $C_0^\infty$, and that $(q, r)$ and $(Q, R)$ are admissible,

\[
\begin{align*}
& (i) \quad \|e^{it\Delta}f\|_{L_Q^q L_R^r} \lesssim \|f\|_{L^2} \\
& (ii) \quad \left\| \int_\mathbb{R} e^{-is\Delta}F(s) \, ds \right\|_{L^2} \lesssim \|F\|_{L_Q^q L_R^r} \\
& (iii) \quad \left\| \int_0^t e^{i(t-s)\Delta}F(s) \, ds \right\|_{L_Q^q L_R^r} \lesssim \|F\|_{L_Q^q L_R^r}.
\end{align*}
\]

**Proof.** Step 1: (iii) if $q = Q, r = R$ Bound first
\[
\left\| \int_\mathbb{R} e^{i(t-s)\Delta}F(s) \, ds \right\|_{L_Q^q L_R^r} \lesssim \left\| \int_\mathbb{R} e^{i(t-s)\Delta}F(s) \right\|_{L_Q^q} \lesssim \left\| \int_\mathbb{R} \frac{1}{|t-s|^\frac{d}{2} - \frac{d}{r}} \|F(s)\|_{L_R^r} \, ds \right\|_{L_Q^q} \lesssim \|F\|_{L_Q^q L_R^r} (by \text{ the dispersive estimate})
\]

Inequality (iii) if $q = Q, r = R$ corresponds to replacing $\int_\mathbb{R}$ by $\int_0^t$ in the above left-hand side; this is possible thanks to the Christ-Kiselev lemma.
Step 2: (ii) We resort to the $TT^*$ trick:
\[
\left\| \int e^{-is\Delta} F(s) \, ds \right\|_{L^2_x}^2 = \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}} e^{-is\Delta} F(s) \, ds \right] \left[ \int_{\mathbb{R}} e^{-it\Delta} F(t) \, dt \right] \, dx
\]
\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} e^{i(t-s)\Delta} F(s) \, ds \right] F(t) \, dx \, dt \quad \text{(by Fubini’s theorem)}
\]
\[
\lesssim \left\| \int_{\mathbb{R}} e^{i(t-s)\Delta} F(s) \, ds \right\|_{L^q_t L^r_x} \| F \|_{L^q_t' L^r'_x} \quad \text{(by $L^qL^r - L^q'L^r'$ duality)}
\]
\[
\lesssim \| F \|_{L^q_t' L^r'_x}^2 \quad \text{(by the dispersive estimate)}
\]

Step 3: (i) It is actually simply dual to (ii):
\[
\left\| e^{it\Delta} f \right\|_{L^q_t L^r_x} = \sup_{\| F \|_{L^q_t' L^r'_x} \leq 1} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}} e^{it\Delta} F(t) \, dt \right] f \, dx \quad \text{(by $L^qL^r - L^q'L^r$ duality)}
\]
\[
= \sup_{\| F \|_{L^q_t' L^r'_x} \leq 1} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}} e^{-it\Delta} F(t) \, dt \right] f \, dx \quad \text{(by Fubini’s theorem)}
\]
\[
\lesssim \sup_{\| F \|_{L^q_t' L^r'_x} \leq 1} \left\| \int_{\mathbb{R}} e^{-it\Delta} F(t) \, dt \right\|_{L^2_t} \| f \|_{L^2_x} \quad \text{(by the Cauchy-Schwarz inequality)}
\]
\[
\lesssim \| f \|_{L^2_x} \quad \text{(by inequality (ii))}
\]

Step 4: (iii) for $(q,r) \neq (Q,R)$ Bound first
\[
\left\| \int_{\mathbb{R}} e^{i(t-s)\Delta} F(s) \, ds \right\|_{L^q_t L^r_x} = \left\| e^{it\Delta} \int_{\mathbb{R}} e^{-is\Delta} F(s) \, ds \right\|_{L^q_t L^r_x} \quad \text{by (i)}
\]
\[
\lesssim \left\| \int_{\mathbb{R}} e^{-is\Delta} F(s) \, ds \right\|_{L^2_x} \quad \text{by (ii)}
\]
\[
\lesssim \| F \|_{L^Q_t L^R_x} \quad \text{(by impedance (iii))}
\]

Once again, inequality (iii) corresponds to replacing $\int_{\mathbb{R}}$ by $\int_0^t$ in the above left-hand side; this is possible thanks to the Christ-Kiselev lemma. \hfill \Box

4. Smoothing estimate

The following smoothing estimate is due to Kato. It expresses the fact that a solution of the linear Schrödinger equation is, upon localizing in space and averaging in time, more regular than its data $u_0$ (in the $L^2$ Sobolev scale). This can be understood as follows: the more singular, or less smooth parts of $u_0$ correspond to very high frequencies $\xi \to \infty$. But we know that a function localized around $\xi$ in Fourier travels at speed $\sim -2\xi$; so the higher the frequency, the faster the group velocity. Therefore, very high frequency are sent to $\infty$ almost immediately, and the solution becomes locally smoother.

**Theorem 4.1.** If $\epsilon > 0$,
\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{|D|^{1/2} e^{it\Delta} f|^2}{|x|^{1+\epsilon}} \, dx \, dt \lesssim \| f \|_{L^2_x}^2.
\]
Proof. Step 1: switching to Fourier space First denote \( a(x) = \frac{1}{(x+i\epsilon)^2} \). By Plancherel’s theorem, the above left-hand side can be written

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |D|^{1/2} e^{it\Delta} f|^2 a(x) \, dx \, dt = \langle |D|^{1/2} e^{it\Delta} f , a(x)|D|^{1/2} e^{it\Delta} f \rangle_{L^2} \\
= \langle |\xi|^{1/2} e^{it|\xi|^2} \hat{f}(\xi), \left[ |\xi|^{1/2} e^{it|\xi|^2} \hat{a}(\xi) \right] * a(\xi) \rangle_{L^2} \\
= \iiint e^{it(|\xi|^2-|\eta|^2)}|\xi|^{1/2}|\eta|^{1/2}\hat{a}(\xi - \eta)\hat{f}(\xi)\hat{f}(\eta) \, d\eta \, d\xi \, dt.
\]

(4.1)

Step 2: a Fourier analysis formula and its consequence We claim that, if \( F \) is \( C_0^\infty \) and \( \phi \) smooth and non-degenerate,

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} F(x) \, dx \, d\lambda = (2\pi)^d \int_{\phi=0} F(x) \frac{d(x)}{\nabla \phi(x)} \, d\Sigma(x)
\]

(4.2)

(where \( \Sigma \) is the \( d-1 \)-dimensional surface measure on the hypersurface \{\( \phi = 0 \)\} induced by the Lebesgue measure on \( \mathbb{R}^d \). Recalling the coarea formula \( \int g(x)|\nabla u(x)| \, dx = \int_{\mathbb{R}} \int_{u(x)=t} g(x) \, d\Sigma \, dt \), the above left-hand side becomes

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} F(x) \, dx \, d\lambda = \int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{i\lambda\phi} \int_{\phi=0} \frac{d(x)}{\nabla \phi(x)} \, d\Sigma \, dy \, d\lambda
\]

\[
= \iiint e^{i\lambda\phi} F(y) \, dy \, d\lambda = (2\pi)^d \int F(\lambda) \, d\lambda
\]

which is the desired formula (4.2). Setting \( \phi(\xi, \eta) = |\xi|^2 - |\eta|^2 \), the surface \{\( \phi = 0 \)\} becomes \{\( |\xi| = |\eta| \)\}, while \( \nabla \phi = (\xi, \eta)^T \). We can now apply (4.2) to (4.1), so that the quantity to be estimated becomes

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} |D|^{1/2} e^{it\Delta} f|^2 a(x) \, dx \, dt = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \hat{a}(\xi - \eta)\hat{f}(\xi)\hat{f}(\eta) \frac{|\xi|^{1/2} |\eta|^{1/2}}{\sqrt{|\xi|^2 + |\eta|^2}} \, d\Sigma
\]

where \( \Sigma \) is the surface measure on the hypersurface \{\( |\xi| = |\eta| \)\} \( \subset \mathbb{R}^{2d} \). The upshot of this new way of writing the quantity we want to estimate is that no oscillations are present in the above right-hand side; we can take absolute values and estimate directly.

Step 3: the Cauchy-Schwarz inequality Applying it to the above expression, we obtain

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^d} |D|^{1/2} e^{it\Delta} f|^2 a(x) \, dx \, dt \leq \int_{|\xi|=|\eta|} |\hat{a}(\xi - \eta)| \left[ |\hat{f}(\xi)|^2 + |\hat{f}(\eta)|^2 \right] \, d\Sigma
\]

\[
= 2 \int |\hat{f}(\xi)|^2 \int_{|\eta|=|\xi|} |\hat{a}(\xi - \eta)| \, d\Sigma \, d\xi.
\]

In order to obtain the desired bound, it suffices to show that \( K \in L^\infty \).
Step 4: $K \in L^\infty$ Observe first that $\widehat{a}$ satisfies the following estimates

$$|\widehat{a}(\xi)| \lesssim \langle \xi \rangle^{-N}$$

$$|\widehat{a}(\xi)| \lesssim |\xi|^{1+\epsilon-d}.$$  

Furthermore, $K(\xi)$ can be written, after a change of variables

$$K(\xi) = \int_{|\xi| = |\xi|} |\widehat{a}(\xi)| d\Sigma \leq \int_{|\xi| = |\xi|} |\xi|^{1+\epsilon-d} d\Sigma + \int_{|\xi| = |\xi|} |\xi|^{-3d} d\Sigma,$$

which, after a moment of reflection, is seen to be uniformly bounded in $\xi$.  

5. The case of the torus

The question of linear estimates on the torus is very delicate, and has only been settled very recently by Bourgain and Demeter. One obvious difference is that the linear flow on the torus is $4\pi^2$-periodic, so that only local in time estimates are available. We do not attempt to give the global picture, but only treat the most simple example of a Strichartz estimate.

**Lemma 5.1 (Zygmund).** If $f \in L^2(\mathbb{T})$,

$$\|e^{it\Delta} f\|_{L^4([0,4\pi^2] \times \mathbb{T})} \lesssim \|f\|_{L^2(\mathbb{T})}.$$

**Proof.** First decompose $f$ and $e^{it\Delta}$ in Fourier series:

$$f(x) = \sum_{k \in \mathbb{Z}} e^{2\pi i k x} \hat{f}_k \quad \text{and} \quad e^{it\Delta} f(x) = \sum_{k \in \mathbb{Z}} e^{i(2\pi k x + 4\pi^2 k^2 t)} \hat{f}_k.$$

Therefore,

$$|e^{it\Delta} f(x)|^2 = \sum_{k,\ell \in \mathbb{Z}} e^{i(2\pi (k-\ell) x + 4\pi^2 (k^2-\ell^2) t)} \hat{f}_k \hat{f}_\ell$$

$$= \|f\|^2_{L^2} + \sum_{k \neq \ell} e^{i(2\pi (k-\ell) x + 4\pi^2 (k^2-\ell^2) t)} \hat{f}_k \hat{f}_\ell.$$

Simply using that $\|f\|^2_{L^4} = \|f\|^2_{L^2}$,

$$\|e^{it\Delta} f\|^2_{L^4} \leq \|e^{it\Delta} f(x)\|^2_{L^2}$$

$$\leq \|f\|^2_{L^2} + \sum_{k \neq \ell} e^{i(2\pi (k-\ell) x + 4\pi^2 (k^2-\ell^2) t)} \hat{f}_k \hat{f}_\ell$$

$$\lesssim \|f\|^2_{L^2} + \sum_{k \neq \ell} |f_k f_\ell|^2 \lesssim \|f\|^2_{L^2}.$$

In the prior to last inequality, we used Plancherel’s theorem (in space and time), along with the fact that $k - \ell = k' - \ell'$ and $k^2 - \ell^2 = k'^2 - \ell'^2$ imply that $k = k'$ and $\ell = \ell'$.

When treating the nonlinear problem, we will need an estimate for the inhomogeneous problem. In other words, we would like an estimate for the solution $u$ of $i\partial_t u - \Delta u = F$, in terms of $F$. Taking the space time Fourier transform (with dual variables $\tau$ and $k$) gives

$$(\tau + k^2) \tilde{u}(\tau, k) = \tilde{F}(\tau, k),$$

so that (at least formally)

$$\tilde{u}(\tau, k) = \frac{1}{\tau + k^2} \tilde{F}(\tau, k).$$
Therefore, solving the inhomogeneous problems is equivalent to applying to \( \tilde{u} \) the Fourier multiplier \( \frac{1}{\tau + k^2} \).

This explains, at least partly, how the following lemma is related to the solution of the inhomogeneous problem. This will become even more clear when we introduce \( X^{s,b} \) spaces.

**Lemma 5.2** (Bourgain). If \( u \) is a function on \( \mathbb{R} \times T \),

\[
\| u \|_{L^4_{t,x}}^2 \lesssim \sum_k \int \langle \tau + k^2 \rangle^{3/4} |\tilde{u}(\tau, k)|^2 \, d\tau.
\]

**Proof.** First decompose

\[
u = \sum_{j \geq 0} u_j,
\]

where \( \tilde{u}_j \) is supported on \( 2^j < \langle \tau + k^2 \rangle < 2^{j+1} \).

**Step 1: the basic inequality.** We claim that, if \( j, k \in \mathbb{N} \),

\[
\| u_j u_{k+j} \|_{L^2_{t,x}} \lesssim 2^{\frac{1}{2} j} 2^{\frac{3}{4} j} \| u_j \|_{L^2_{t,x}} \| u_{j+k} \|_{L^2_{t,x}},
\]

which, by Plancherel’s theorem, is equivalent to

\[
\left\| \sum_{\ell} \int \tilde{u}_j(\sigma, \ell) \tilde{u}_{k+j}(\tau - \sigma, k - \ell) \, d\sigma \right\|_{L^2_{t,k}(\mathbb{R} \times \mathbb{Z})} \lesssim 2^{\frac{1}{2} j} 2^{\frac{3}{4} j} \| u_j \|_{L^2_{t,x}} \| u_{j+k} \|_{L^2_{t,x}}.
\]

By the Cauchy-Schwarz inequality (in \( \ell \) and \( \sigma \)), followed by Hölder’s inequality (in \( k \) and \( \tau \)), the above left-hand side is bounded by

\[
LHS \lesssim \left\| \left( \sum_{\ell} \int |\tilde{u}_j(\sigma, \ell)|^2 |\tilde{u}_{k+j}(\tau - \sigma, k - \ell)|^2 \, d\sigma \right)^{1/2} \right\|_{L^2_{T,k}}^{1/2} \lesssim 2^{\frac{1}{2} j} 2^{\frac{3}{4} j} \| u_j \|_{L^2_{t,x}} \| u_{j+k} \|_{L^2_{t,x}}.
\]

Therefore, it suffices, in order to prove the claim (5.1), to show that, for any \( \tau \) and \( k \),

\[
\left| \sum_{\ell} \int 1_{(\sigma + \ell^2) \sim 2^{k+1}} 1_{(\tau - \sigma + (k - \ell)^2) \sim 2^{k+j}} \, d\tau \right| \lesssim 2^{\frac{1}{2} j} 2^{\frac{3}{2} j}.
\]

A necessary condition for the integral over \( \tau \) to be nonzero is that \( \tau = -\ell^2 - (k - \ell)^2 + O(2^{k+j}) \), in which case it has size \( O(2^j) \). Therefore, the claim (5.1) would follow if, as soon as \( \tau = -\ell^2 - (k - \ell)^2 + O(2^{k+j}) \),

\[
\left| \sum_{\ell} 1_{(\sigma + \ell^2) \sim 2^{j}} 1_{(\tau - \sigma + (k - \ell)^2) \sim 2^{k+j}} \right| \lesssim 2^{\frac{1}{2} j} 2^{\frac{1}{2} j}.
\]

But this is easily established, giving the claim.
Step 2: proof of the estimate. Start with the trick of writing the $L^4$ norm squared as the $L^2$ norm of the square:

$$\| u \|^2_{L^4_t L^2_x} = \| |u|^2 \|^2_{L^2_t L^2_x} = \left\| \sum_{j,j'} u_j u_{j'} \right\|_{L^4_t L^2_x} \leq \sum_{j,k \geq 0} \| u_j u_{j+k} \|^2_{L^4_t L^2_x}.$$

Resort now to the above claim, followed by the Cauchy Schwarz inequality (in $j$) and the Young inequality (in $k$)

$$\begin{align*}
\sum_{j,k \geq 0} \| u_j u_{j+k} \|_{L^4_t L^2_x} &\lesssim \sum_{j,k \geq 0} 2^{\frac{1}{2}j} 2^{\frac{3}{4}j} \| u_j \|_{L^2_t L^2_x} \| u_j \|_{L^2_t L^2_x} \\
&\leq \sum_{j,k} 2^{\frac{3}{4}j} 2^{\frac{3}{8}(j+k)} 2^{-\frac{1}{8}k} \| u_j \|_{L^2_t L^2_x} \| u_j \|_{L^2_t L^2_x} \\
&\lesssim \left[ \sum_j 2^{\frac{3}{4}j} \| u_j \|^2_{L^2_t L^2_x} \right]^{1/2} \left[ \sum_k \left[ \sum_j 2^{-\frac{1}{8}k} 2^{\frac{3}{8}(j+k)} \| u_j \|_{L^2_t L^2_x} \right]^2 \right]^{1/2} \\
&\lesssim \sum_j 2^{\frac{3}{4}j} \| u_j \|^2_{L^2_t L^2_x} \sim \sum_k \int \langle \tau + k^2 \rangle^{3/4} |\tilde{u}(\tau,k)|^2 \, d\tau.
\end{align*}$$

This is the desired inequality! \[ \square \]

References