Strong solutions and weak-strong uniqueness for the nonhomogeneous Navier-Stokes system

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August 24, 2007

Abstract

This article is devoted to the study of the nonhomogeneous incompressible Navier-Stokes system in dimension $d \geq 3$. We use new a priori estimates, that enable us to deal with low-regularity data and vanishing density. In particular, we prove new well-posedness results that improve the results of Danchin [6] by considering a less regular initial density, without a lower bound. Also, we obtain the first uniqueness criterion for weak solutions which is at the scaling of the equation.

1 Introduction

1.1 Presentation of the equation

We shall study in this paper the Cauchy problem for the nonhomogeneous Navier-Stokes system set in the whole space $\mathbb{R}^d$, with $d \geq 3$. It reads

$$(1) \qquad \left\{ \begin{array}{l} \partial_t \rho + u \cdot \nabla \rho = 0 \\ \rho \partial_t u + \rho u \cdot \nabla u - \Delta u = -\nabla p \\ \operatorname{div} u = 0 \\ (\rho, u)_{|t=0} = (\rho_0, u_0) \ . \end{array} \right.$$

It describes the evolution of a fluid of viscosity $\nu = 1$, whose density is not constant (an instance of such a situation is the mixture of two fluids, which do not have the same density, but whose viscosities are equal). The unknowns ρ, u and p stand for the density, velocity, and pressure of the fluid; they are respectively $\mathbb{R}^+$, $\mathbb{R}^d$, and $\mathbb{R}$ valued and they are all functions of the space variable x and of the time variable t. The fluid is furthermore supposed to be incompressible, hence the zero divergence condition on u. Finally, the notation $u \cdot \nabla$ corresponds to the operator $\sum_{i=1}^d u^i \partial_i$.

At least formally, this equation verifies the following conservation law:

$$(2) \qquad \|\sqrt{\rho(t)} u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 \, ds = \|\sqrt{\rho_0} u_0\|_2^2 \ .$$

There is another conserved quantity, namely for any α and β the Lebesgue measure

$$(3) \qquad \mu\{x \in \mathbb{R}^d, \, \alpha \leq \rho(x,t) \leq \beta\} \quad \text{is independent of } t.$$

In particular, the L^∞ norm of the density as well as its lower bound, are independent of t.

Another important feature is the scaling invariance : if (u,ρ) is a solution associated to the initial data (u_0,ρ_0), then

$$(\lambda u(\lambda^2 t,\lambda x),\rho(\lambda^2 t,\lambda x))$$

will be associated to $(\lambda u_0(\lambda x),\rho_0(\lambda x))$.
A functional space for the data (u_0,ρ_0) or for the solution (u,ρ) is said to be at the scaling of the equation if its norm is invariant by the above transformation.

1.2 Weak solutions

Weak solutions have been built up first by the Russian school [1], and more recently in Desjardins [8] and Lions [19]. For instance, it is proved in Lions [19] that weak global solutions of (1) exist provided that, for some $\underline{\rho}>0$ and some $p\in(d/2,\infty)$,

$$\rho_0\in L^\infty \quad,\quad \rho_0 u_0^2\in L^1 \quad \text{and} \quad (\underline{\rho}-\rho_0)^+\in L^p\ .$$

The last condition above is technical, but it is important to notice that it allows the density to vanish on finite-measure sets. These weak solutions naturally belong to the class of finite energy solutions.

Definition 1.1 *A solution (u,ρ) of (1) is said to be a finite energy solution if*

$$\rho\in L^\infty([0,T],L^\infty) \quad,\quad \sqrt{\rho}u\in L^\infty([0,T],L^2) \quad \textit{and} \quad \nabla u\in L^2([0,T],L^2)\ .$$

The solutions of Lions have some additional regularity: they verify (3) and (2) with a "≤" replacing the equality sign. Their uniqueness is an open question in general; Theorem 1.2, below, gives a criterion of uniqueness in the energy class.

1.3 Strong solutions

Regular solutions were first considered by Marsden [20] in the non-viscous case and Ladyzhenskaya [16] in the viscous case. Itoh [11] [12] studied the non-viscous limit for (1).
More accurate results were obtained by Danchin [5] who proved the local well-posedness independently of the viscosity of (1) for initial data with a regularity higher than the scaling.
Danchin [6] also proved the existence of local and unique solutions of (1) for spaces of initial data at the scaling of the equation, but then one has to resort to Besov spaces, ie consider initial data

$$u_0\in \dot{B}^{\frac{d}{2}-1}_{2,1} \quad \text{and} \quad (\rho_0^{-1}-1)\text{ small in }\dot{B}^{\frac{d}{2}}_{2,\infty}\cap L^\infty$$

(see the paper of Danchin for a definition of the Besov spaces $\dot{B}^s_{p,q}$). Notice that in particular ρ_0 has to be bounded from below by a positive constant.
For initial data

$$u_0\in H^2(\mathbb{R}^3) \quad \text{and} \quad \rho_0\in H^1(\mathbb{R}^3)\cap L^{3/2}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)\ , \tag{4}$$

Choe and Kim [2] could remove the assumption of non-vanishing of the viscosity by adding the following compatibility condition:

$$\nu\Delta u_0-\nabla p_0=\rho_0^{1/2}g \tag{5}$$

for some (p_0,g) in $H^1\times L^2$.

1.4 Weak-strong uniqueness

Let us come back for a moment to the setting of the standard Navier-Stokes equations, ie with a constant density. Then there also exists global weak solutions, which were constructed by Leray [18], and their uniqueness is not known in general. However, partial criteria are known under the name of 'weak-strong uniqueness' results. The first result of that kind was proved by Serrin [23]: suppose that u is a weak Leray solution, and that it belongs to the (strong solution) class

$$L^p([0,T],L^q) \quad \text{with} \quad \frac{2}{p}+\frac{d}{q}=1$$

for some $T>0$. Then u is unique on $[0,T]$ in the class of weak solutions. Notice that the space above is at the scaling of the Navier-Stokes equation. More refined weak-strong uniqueness results have been derived since: see Lemarié-Rieusset [17] and Germain [9] for a review and recent results.
For the system (1), uniqueness criteria for weak solutions are given by several authors, see [19] [2] [7]. We would like however to get a result similar to the one of Serrin for the Navier-Stokes system, that is the condition should be at the scaling of the equation and involve only u and its space derivatives (not the time derivative of u). This is done in Theorem 1.2.

Let us also mention here a result related to weak-strong uniqueness. It is proved by Kim [15] (see his paper for more details) that if a solution corresponding to initial data as in (4) (5) leaves the natural solution space corresponding to these data at some time T^*, then necessarily

$$\int_0^{T^*} \|u(s)\|^p_{L^{q,\infty}}\,ds=\infty \qquad \text{for any } p,\,q\ ,\ \ \frac{2}{p}+\frac{3}{q}=1\ ,\ \ 3<q\leq\infty\ ,$$

where $L^{q,\infty}$ denotes the usual weak Lebesgue space.

1.5 Statement of the results

In the following, we will set

$$r=\frac{1}{\rho}.$$

and work with the following formulation of (1)

$$(NNS)\quad \left\{\begin{array}{l} \partial_t r+u\cdot\nabla r=0 \\ \partial_t u+u\cdot\nabla u-r\Delta u=-r\nabla p \\ \operatorname{div} u=0 \\ (r,u)_{|t=0}=(r_0,u_0)\ . \end{array}\right.$$

The definition of finite energy solutions becomes

Definition 1.2 *Data (u_0,r_0) are of finite energy if*

$$\inf r_0>0 \quad \text{and} \quad \frac{u_0}{\sqrt{r_0}}\in L^1\ ;$$

A solution (u,r) is of finite energy if

$$\inf_x r(t,x)\in L^\infty([0,T])\ ,\ \ \frac{u}{\sqrt{r}}\in L^\infty([0,T],L^2)\ \ \text{and}\ \ \nabla u\in L^2([0,T],L^2)\ .$$

The two theorems that we are about to state both rely on the following idea: it is well-known that the oscillations of the density are a major problem for the compressible Navier-Stokes equation, and also for the (NNS) system. So it makes sense to estimate the density in a BMO-type norm, for then the focus is precisely on oscillations. We apply this idea by estimating r in BMO, or in spaces that embed in BMO. Actually, we will use critical homogeneous Sobolev spaces, which are known to embed in BMO, and which are defined as follows.

Definition 1.3 *(The following definition requires further justification ; in order to avoid technicalities for the moment, we delay it until Section 6.1.)*
We define $\dot{W}^{\sigma,d/\sigma}$ *as the homogeneous Triebel-Lizorkin space* $\dot{F}^{\sigma}_{d/\sigma,2}$. *It turns out that this is a Banach space modulo constants; in particular it does not embed in* $\mathcal{S}'$, *but in* $\mathcal{S}'$ *modulo constants.*
We will in the following use an equivalent norm on $\dot{W}^{\sigma,d/\sigma}$. *It is provided by*

$$\|u\|_{\dot{W}^{\sigma,d/\sigma}} = \|\Lambda^{\sigma} u\|_{d/\sigma} \ ,$$

where Λ^{σ} *is the Fourier multiplier given by*

$$\mathcal{F}\left(\Lambda^{\sigma} f\right)(\xi) \overset{def}{=} |\xi|^{\sigma} \widehat{f}(\xi) \ , \tag{6}$$

with $\mathcal{F}$ *or* $\widehat{\ }$ *denoting the Fourier transform operator.*

It might seem problematic to estimate as we will do the reciprocal of the density r in $\dot{W}^{s,d/s}$, since we would not like it to be defined modulo a constant. But this is to be understood only as an indication of its regularity, and r will always be a distribution.

First, we give a theorem on the existence of local strong solutions. The following theorem improves the results of Danchin [6], in that the density is at the scaling of the equation, does not have an inferior bound, and is taken to be less regular.

Theorem 1.1 *Let* $d = 3$, $s \in (2, \frac{5}{2})$, *or* $d \geq 4$, $s \in (\frac{d}{2}, \frac{d}{2}+1)$ *and let* u_0 *and* r_0 *be functions such that*

$$u_0 \in H^s \quad \frac{1}{r_0} \in L^{\infty} \quad \text{and} \quad \Lambda^{s-1} r_0 \in L^{\frac{d}{s-1}} \ .$$

Suppose also that, for a constant C,

$$\left\|\Lambda^{s-1} r_0\right\|_{\frac{d}{s-1}} \leq C \left(\inf r_0\right) \ .$$

Then there exists $T > 0$ *and a local solution* (u, r) *of* (NNS) *defined on* $[0, T]$ *such that*

$$\begin{gathered} u \in L^{\infty}([0,T], H^s) \cap L^2([0,T], H^{s+1}) \\ \frac{1}{r} \in L^{\infty}([0,T], L^{\infty}) \quad \text{and} \quad \Lambda^{s-1} r \in L^{\infty}([0,T], L^{\frac{d}{s-1}}) \\ \nabla p \in L^2([0,T], H^{s-1}) \ . \end{gathered}$$

Some remarks are in order

- No compatibility condition is required, and the density does not have to be bounded from below.

- In the above theorem, the reciprocal of the density is taken in $\dot{W}^{s-1,\frac{d}{s-1}}$, which is a Banach space modulo constants, so it fails in particular to embed in L^∞. There holds nevertheless

  $$\dot{W}^{s-1,\frac{d}{s-1}} \hookrightarrow BMO \ . \tag{7}$$

  This implies that r_0 can go to infinity, but typically in a logarithmic way; in particular, it belongs to any space L^p_{loc}, with $p < \infty$. For the density ρ_0, this translates into a possible vanishing, but at a very slow speed.

  Also notice that the conditions we impose on the density allow for a non-zero limit at (spatial) infinity, which is the physically most relevant case.

- The functional space for the density is at the scaling of the equation. This is not the case for the velocity, but one could achieve a scale invariant condition on the velocity by using methods of paradifferential calculus.

- The existence time in the above theorem would depend on the viscosity, had it not been fixed to one. A priori no inviscid limit results hold for the data of the theorem.

The second theorem gives a criterion of uniqueness of the energy class solutions. Notice that this is the first such criterion at the scaling of the equation ; so it somehow generalizes the Serrin criterion [23] to the nonhomogeneous Navier-Stokes system.

Theorem 1.2 *Let $d \geq 3$, and let (u_0, r_0) be initial data of finite energy. Suppose there exists an associated solution (u, r) such that*

1. *(u, r) is of finite energy.*
2. *(u, r) verifies (C is a constant)*

$$\begin{gathered} u \in L^\infty([0,T], L^d) \\ \nabla r \in L^\infty([0,T], L^d) \quad \text{and} \quad \|r\|_{BMO} \leq C\,(\inf r) \quad \forall t \in [0,T] \\ \nabla u \in L^1([0,T], L^\infty) \quad \text{and} \quad \sqrt{t}\nabla u(t) \in L^2([0,T], L^\infty) \\ \sqrt{t}\Delta u(t) \in L^2([0,T], L^d) \ . \end{gathered}$$

Under these conditions, (u, r) is unique on $[0, T]$ in the class of energy solutions.

Let us mention some remarks

- The above weak-strong uniqueness criterion is at the scaling of the equation.
- A more general version of Theorem 1.2 is given in section 4.
- We could get a much bigger set of indices for the L^pL^q spaces that yield uniqueness of (u, r) in the above theorem, or in its generalized version in section 4. However, we would then have to take $\frac{1}{r}$ in spaces of the type L^pL^q, with $p < \infty$, which would not be very natural, so we chose not to include this possibility in our theorems.

Corollary 1.1 *Assume $d = 3$, $s \in (2, \frac{5}{2})$, or $d \geq 4$, $s \in (\frac{d}{2}, \frac{d}{2} + 1)$. If*

$$u_0 \in H^s \ , \quad \frac{1}{r_0} \in L^\infty \ , \quad \Lambda^{s-1} r_0 \in L^{\frac{d}{s-1}} \ , \quad \text{and} \quad \left\|\Lambda^{s-1} r_0\right\|_{\frac{d}{s-1}} \leq C\,(\inf r_0) \ ,$$

then the solution built up in Theorem 1.1 is unique in the energy class.

PROOF: One verifies indeed easily that this solution satisfies the hypotheses of Theorem 1.2. ■

So Theorem 1.2 gives the local uniqueness of finite energy solutions in the case where the density datum belongs to a functional space at the scaling of the equation, but not the velocity.

What would happen if we would take as data a smooth density, and a velocity belonging to a functional space at the scaling of the equation? Then we expect a regularizing effect on the velocity, because of the parabolic nature of the equation, and it should follow that Theorem 1.2 applies.

The last case is if both the initial density and velocity are at the scaling of the equation; as was pointed out to us by Raphaël Danchin, it is not very likely that the uniqueness theorem applies then. Indeed, the density remains then rough and a priori no regularizing effect is to be expected for the velocity.

Acknowledgements: The author would like to thank Raphaël Danchin for interesting and useful conversations during the writing of this paper, and the referee, whose numerous suggestions helped improve greatly this article.

2 Three lemmas

2.1 An elliptic estimate with BMO coefficients

Let us recall first the Theorem of Coifman Rochberg Weiss [4], which we call in the following CRW theorem.

This theorem was proved by Coifman Rochberg Weiss [4] in the case $r = 2$; for the general case, see for instance Taylor [25].

Theorem 2.1 (Coifman Rochberg Weiss [4]) *If we denote by ϕ the multiplication operator by the function ϕ, and P is a Calderon-Zygmund operator, then the norm of the commutator is bounded as a linear operator on L^r:*

$$\|[\phi, P]\|_{\mathcal{L}(L^r)} \leq C_r \|\phi\|_{BMO}$$

for some constant C_r.

It is well-known that $\mathbb{P}$ and $\mathbb{Q}$, that we define in the next paragraph, are Calderon-Zygmund operators.

Definition 2.1 *Let $\mathbb{P}$ (respectively $\mathbb{Q}$) be the projection operator in L^2 on the divergence free (respectively curl free) vector fields.*

Lemma 2.1 *Let p verify*

$$\mathbb{Q}(\phi \nabla p) = f \ ,$$

where ϕ is a positive function. Then there holds

$$\|\nabla p\|_2 \leq \left(\inf_x \phi(x) \right)^{-1} \|f\|_2 \ ,$$

and for $r \in (1, \infty)$, and for a constant C depending on r,

$$\left(\inf_x \phi(x) - C \|\phi\|_{BMO} \right) \|\nabla p\|_r \leq \|f\|_r \ .$$

Proof: The first point is proved by simply taking the scalar product of the equation with ∇p. For the second point, we simply write

$$\phi\nabla p = [\mathbb{Q}\,,\,\phi]\,\nabla p - \mathbb{Q}(\phi\nabla p) = [\mathbb{Q}\,,\,\phi]\,\nabla p - f\ ,$$

which implies, due to the CRW theorem,

$$\left(\inf_x \phi(x)\right)\|\nabla p\|_r \le \|\phi\nabla p\|_r \le \|f\|_r + \|\,[\mathbb{Q}\,,\,\phi]\,\nabla p\|_r \le \|f\|_r + C\|\phi\|_{BMO}\|\nabla p\|_r\ . \quad \blacksquare$$

2.2 A generalized Gronwall inequality

Lemma 2.2 *Suppose that the following inequality holds*

$$f' + \left(g'\right)^2 \le \alpha f + \beta g' g\ . \tag{8}$$

where f, g', α, and β are positive functions of the real variable such that

$$f \in L^\infty \quad , \quad g(0) = 0 \quad , \quad g' \in L^2 \quad , \quad \alpha \in L^1 \quad \text{and} \quad \sqrt{t}\beta(t) \in L^2\ .$$

Then there holds

$$\left(e^{-\int_0^t \alpha} - \frac{1}{2} - \frac{1}{2}\int_0^t s\beta(s)^2\,ds\right)\int_0^t \left(g'\right)^2 + e^{-\int_0^t \alpha} f(t) \le f(0)\ .$$

Remark 2.1 *Notice that, due to the inequality (9) that follows, the assumptions $\alpha \in L^1$, $\sqrt{s}\beta \in L^2$ are enough for (8) to make sense.*

Proof: Since $g(0) = 0$, Hölder's inequality gives that

$$g(s) \le \sqrt{s}\sqrt{\int_0^s \left(g'\right)^2}\ . \tag{9}$$

With the help of this inequality, we have

$$\begin{aligned}
\int_0^t \left(g'\right)^2 &\le e^{\int_0^t \alpha}\int_0^t e^{-\int_0^s \alpha}\left(g'\right)^2\,ds \\
&\le e^{\int_0^t \alpha}\int_0^t e^{-\int_0^s \alpha} g'(g' - \beta g)\,ds + e^{\int_0^t \alpha}\int_0^t e^{-\int_0^s \alpha}\beta g g'\,ds \\
&\le e^{\int_0^t \alpha}\int_0^t e^{-\int_0^s \alpha} g'(g' - \beta g)\,ds + e^{\int_0^t \alpha}\int_0^t \left|\frac{g}{\sqrt{s}}\sqrt{s}\beta g'\right|\,ds \\
&\le e^{\int_0^t \alpha}\int_0^t e^{-\int_0^s \alpha} g'(g' - \beta g)\,ds + \frac{1}{2}e^{\int_0^t \alpha}\int_0^t \left(g'\right)^2 + \frac{1}{2}e^{\int_0^t \alpha}\left(\int_0^t \sqrt{s}\beta g'\,ds\right)^2 \\
&\le e^{\int_0^t \alpha}\int_0^t e^{-\int_0^s \alpha} g'(g' - \beta g)\,ds + \frac{1}{2}e^{\int_0^t \alpha}\int_0^t \left(g'\right)^2 + \frac{1}{2}e^{\int_0^t \alpha}\int_0^t s\beta^2\,ds\int_0^t \left(g'\right)^2
\end{aligned}$$

which is equivalent to

$$\left(e^{-\int_0^t \alpha} - \frac{1}{2} - \frac{1}{2}\int_0^t s\beta^2\,ds\right)\int_0^t \left(g'\right)^2 \le \int_0^t e^{-\int_0^s \alpha} g'(g' - \beta g)\,sds\ .$$

On the other hand, we have, by the fundamental theorem of calculus,

$$f(t)e^{-\int_0^t \alpha} - f(0) = \int_0^t e^{-\int_0^s \alpha}\left(f' - \alpha f\right)\,ds\ .$$

Adding the two lines above, and using (8), we find the desired inequality

$$\left(e^{-\int_0^t \alpha} - \frac{1}{2} - \frac{1}{2}\int_0^t s\beta^2\,ds\right)\int_0^t \left(g'\right)^2 + f(t)e^{-\int_0^t \alpha} - f(0) \le 0\ . \quad \blacksquare$$

2.3 Some commutator estimates

We denote here Λ^σ for the Fourier multiplier defined in (6), and $[\Lambda^\sigma\, ,\, f]$ for the commutator

$$[\Lambda^\sigma\, ,\, f]\, g = \Lambda^\sigma(fg) - f\Lambda^\sigma g\ .$$

Lemma 2.3 *The following estimates hold true*
For $1 < p < \infty$ *and* $\sigma \geq 0$,

$$\left\|[\Lambda^\sigma\, ,\, f]\, g\right\|_p \leq C\left(\|\nabla f\|_\infty \|\Lambda^{\sigma-1} g\|_p + \|\Lambda^\sigma f\|_p \|g\|_\infty\right)\ .$$

For $1 \leq \sigma < \frac{d}{2}$,

$$\left\|[\Lambda^\sigma\, ,\, f]\, g\right\|_2 \leq C\|\Lambda^\sigma f\|_{d/\sigma}\|\Lambda^\sigma g\|_2\ . \tag{10}$$

For $1 \leq \sigma < \frac{d}{2}$,

$$\left\|[\Lambda^\sigma\, ,\, f]\, g\right\|_{d/\sigma} \leq C\|\Lambda^{\sigma-1} g\|_{d/\sigma}\left(\|\nabla f\|_\infty + \|f\|_{\dot{H}^{\frac{d}{2}+1}}\right)\ . \tag{11}$$

Proof: The first estimate, already classical, is due to Kato and Ponce [13]. They worked in the framework of non-homogeneous derivatives, that is with $\sqrt{1-\Delta}$ instead of Λ ; this makes however only a small difference with the estimates given above.
We refer to the appendix for the proof of the second and third estimates. ■

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, we shall first prove a priori estimates for smooth solutions, and then use a limiting procedure.
Recall that $s \in (\frac{d}{2}, \frac{d}{2}+1)$, and $s \geq 2$.
We denote Λ^s for the Fourier multiplier defined in (6).
We will in the following paragraphs show how one can control the homogeneous norms $\|u\|_{\dot{H}^s}$ and $\|\nabla p\|_{\dot{H}^{s-1}}$. It is the difficult part of the argument; controlling lower order derivatives is then easy, and it yields estimates of the nonhomogeneous norms $\|u\|_{H^s}$ and $\|\nabla p\|_{H^{s-1}}$.

Estimate on the pressure

We have

$$r\Lambda^{s-1}\nabla p = \Lambda^{s-1}\left(r\nabla p\right) + \left[r\, ,\, \Lambda^{s-1}\right]\nabla p\ .$$

Taking the scalar product with $\Lambda^{s-1}\nabla p$ and using Lemma 2.3, we get

$$\left(\inf_{\mathbb{R}^d} r\right)\|\Lambda^{s-1}\nabla p\|_2 \leq \left\|\mathbb{Q}\Lambda^{s-1}\left(r\nabla p\right)\right\|_2 + C\left\|\Lambda^{s-1} r\right\|_{\frac{d}{s-1}}\left\|\Lambda^{s-1}\nabla p\right\|_2\ . \tag{12}$$

Using the momentum conservation equation, we have, due to the zero divergence of u

$$\mathbb{Q}\Lambda^{s-1}\left(r\nabla p\right) = -\mathbb{Q}\Lambda^{s-1}\left(u\cdot\nabla u\right) + \mathbb{Q}\Lambda^{s-1}\left(r\Delta u\right)\ .$$

Since H^s is an algebra for $s > \frac{d}{2}$, we find

$$\left\|\Lambda^{s-1}\left(u\cdot\nabla u\right)\right\|_2 = \left\|\Lambda^{s-1}\nabla\cdot\left(u\times u\right)\right\|_2 \leq C\|u\|_{H^s}^2\ . \tag{13}$$

The CRW theorem, the boundedness of $\mathbb{Q}$ on Lebesgue spaces, another use of Lemma 2.3, and the embedding (7) give

$$\begin{aligned}\left\|\mathbb{Q}\Lambda^{s-1}\left(r\Delta u\right)\right\|_2 &\le \left\|\left[\mathbb{Q}\,,\,r\right]\Lambda^{s-1}\Delta u\right\|_2 + \left\|\mathbb{Q}\left[r\,,\,\Lambda^{s-1}\right]\Delta u\right\|_2\\ &\le C\left\|r\right\|_{BMO}\left\|u\right\|_{H^{s+1}} + C\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\left\|u\right\|_{H^{s+1}}\\ &\le C\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\left\|u\right\|_{H^{s+1}}\ .\end{aligned}\tag{14}$$

So gathering (12), (13), and (14) we get

$$\left((\inf r) - C\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\right)\|\Lambda^{s-1}\nabla p\|_2 \le C\|u\|_{H^s}^2 + C\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\|u\|_{H^{s+1}}\ .\tag{15}$$

Estimate on the velocity

Apply the operator Λ^s to the momentum conservation equation in (NNS), and take the space scalar product with $\Lambda^s u$. The result reads

$$\begin{aligned}\frac{d}{dt}\|\Lambda^s u\|_2^2 &= -\langle\Lambda^s(u\cdot\nabla u)\,,\,\Lambda^s u\rangle + \langle\Lambda^s\left(r\Delta u\right)\,,\,\Lambda^s u\rangle - \langle\Lambda^s\left(r\nabla p\right)\,,\,\Lambda^s u\rangle\\ &= I + II + III\ .\end{aligned}$$

We will estimate one by one I, II and III. Using the identity $\langle u\cdot\nabla v\,,\,v\rangle = 0$ (due to the incompressibility of u) and Lemma 2.3, we get

$$\begin{aligned}|I| &= |\langle\Lambda^s(u\cdot\nabla u)\,,\,\Lambda^s u\rangle|\\ &\le |\langle\Lambda^s(u\cdot\nabla u) - u\cdot\nabla\Lambda^s u\,,\,\Lambda^s u\rangle|\\ &\le C\|\nabla u\|_\infty\|u\|_{H^s}^2 \le C\|u\|_{H^{s+1}}\|u\|_{H^s}^2\ ,\end{aligned}$$

where the Sobolev embedding theorem was used in the last line. Let us now estimate II.

$$\begin{aligned}II &= \langle\Lambda^{s-1}\left(r\Delta u\right)\,,\,\Lambda^{s+1}u\rangle\\ &= -\langle r\Lambda^{s+1}u\,,\,\Lambda^{s+1}u\rangle - \langle\left[r\,,\,\Lambda^{s-1}\right]\Delta u\,,\,\Lambda^{s+1}u\rangle\ .\end{aligned}$$

Making use now of Hölder's inequality and of Lemma 2.3, we find

$$II \le -\left(\inf r\right)\|\Lambda^{s+1}u\|_2^2 + C\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\|\Lambda^{s+1}u\|_2^2\ .$$

Only the term III is left. Using in the last line the incompressibility of u, we can write it as

$$\begin{aligned}III &= \langle\Lambda^{s-1}\left(r\nabla p\right)\,,\,\Lambda^{s+1}u\rangle\\ &= \langle r\Lambda^{s-1}\nabla p\,,\,\Lambda^{s+1}u\rangle + \langle\left[\Lambda^{s-1}\,,\,r\right]\nabla p\,,\,\Lambda^{s+1}u\rangle\\ &= \langle\left[\mathbb{P}\,,\,r\right]\Lambda^{s-1}\nabla p\,,\,\Lambda^{s+1}u\rangle + \langle\left[\Lambda^{s-1}\,,\,r\right]\nabla p\,,\,\Lambda^{s+1}u\rangle\ .\end{aligned}$$

The CRW theorem, Lemma 2.3, and the embedding (7) give finally

$$|III| \le C\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\|\Lambda^{s-1}\nabla p\|_2\|\Lambda^{s+1}u\|_2\ .$$

Gathering the estimates on I, II and III, we have

$$\begin{aligned}\frac{d}{dt}\|\Lambda^s u\|_2^2+&\left((\inf r)-C\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\right)\|\Lambda^{s+1}u\|_2^2\\ &\leq C\|u\|_{H^{s+1}}\|u\|_{H^s}^2+\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\|\Lambda^{s-1}\nabla p\|_2\|\Lambda^{s+1}u\|_2\ .\end{aligned}\tag{16}$$

Estimate on the density

The conservation of mass equation in (NNS) can be rewritten as

$$\partial_t r+u\cdot\nabla r=0\ .$$

Applying Λ^{s-1} to this equation, and taking the scalar product with $\left|\Lambda^{s-1}r\right|^{\frac{d}{s}-2}\Lambda^{s-1}r$, we get, due to the zero divergence of u

$$\frac{d}{dt}\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\leq\left\|\left[\Lambda^{s-1}\,,\,u\right]\cdot\nabla r\right\|_{\frac{d}{s-1}}\ .$$

Now applying Lemma 2.3, and using the Sobolev embedding theorem, we can estimate

$$\begin{aligned}\frac{d}{dt}\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}&\leq C\left\|\Lambda^{s-2}\nabla r\right\|_{\frac{d}{s-1}}\left(\|\Lambda u\|_\infty+\|u\|_{\dot H^{\frac{d}{2}+1}}\right)\\ &\leq C\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\|u\|_{H^{s+1}}\ .\end{aligned}\tag{17}$$

Local control of u, p and r

The estimates above were made for the homogeneous norms $\|u\|_{\dot H^s}$ and $\|\nabla p\|_{\dot H^{s-1}}$; it is then easy to estimate the L^2 norms for lower order derivatives, and thus get similar bounds for the nonhomogeneous norms $\|u\|_{H^s}$ and $\|\nabla p\|_{H^{s-1}}$.

Assume that the initial data satisfy

$$(\inf r_0)-C\left\|\Lambda^{s-1}r_0\right\|_{\frac{d}{s-1}}=2c>0\ .\tag{18}$$

The estimate (17) gives for $t\leq 1$

$$\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\leq 2ce^{C\int_0^t\|u\|^2_{H^{s+1}}}\ ,$$

so for some $\epsilon>0$, as long as $\int_0^t\|u\|^2_{H^{k+1}}\leq\epsilon$, we have

$$(\inf r)-C\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\geq c\quad\text{ and }\quad\left\|\Lambda^{s-1}r\right\|_{\frac{d}{s-1}}\leq C'\ .$$

If these two inequalities hold, we deduce from the a priori estimates (15) (16) that

$$\frac{d}{dt}\|u\|^2_{H^s}+c'\|u\|^2_{H^{s+1}}\leq C\|u\|^4_{H^s}\ ,$$

and this gives a local control of u in $L^\infty_t H^s_x\cap L^2_t H^{s+1}_x$. This in turn implies that for some $t>0$, $\int_0^t\|u\|^2_{H^{k+1}}\leq\epsilon$, closing the argument.

We thus deduce under the hypothesis (18) the existence of $T > 0$ such that on $[0,T]$ an a priori smooth solution $(u, r, \nabla p)$ is bounded in

$$\big(L^\infty([0,T],H^s) \cap L^2([0,T],H^{s+1})\big) \times L^\infty([0,T],\dot{W}^{s-1,d/(s-1)}) \times L^2([0,T],H^{s-1}) \ .$$

The approximation scheme

In order to approximate the initial data, we need the following lemma, whose proof will be given in Section 6.2.

Lemma 3.1 *Let r_0 be a non negative function which satisfies (18). Then there exists a sequence of smooth functions (r_0^n) such that*
(i) $r_0^n \in L^\infty$
(ii) $r_0^n \longrightarrow r_0$ *in* $\dot{W}^{s-1,\frac{d}{s-1}}$ *as* $n \to \infty$
(iii) $r_0^n \longrightarrow r_0$ *in* $\mathcal{S}'$ *and in* L^p_{loc} *(*$\forall p < \infty$*)* *as* $n \to \infty$
(iv) $(\inf r_0^n) \longrightarrow (\inf r_0)$ *as* $n \to \infty$
So in particular (r^n) satisfies (18) for n big enough.

Remark 3.1 *Though the result of the above lemma is intuitively clear, the proof, given in Section 6.2, requires a little work. The lemma becomes obvious if one is willing to lose some generality, for instance if one assumes that r_0 takes a non zero constant value close to infinity.*

The above lemma gives us an approximating sequence for r_0. Now consider an approximation of the identity (ϕ^n), define

$$u_0^n = \phi^n * u_0$$

and denote (u^n, r^n, p^n) for the solution of

$$(19) \qquad \begin{cases} \partial_t r^n + u^n \cdot \nabla r^n = 0 \\ \partial_t u_n + u^n \cdot \nabla u^n - r^n \Delta u^n = r^n \nabla p^n \\ \operatorname{div} u^n = 0 \\ (r^n, u^n)_{|t=0} = (r_0^n, u_0^n) \ . \end{cases}$$

The solution (u^n, r^n, p^n) is defined at least on $[0,T]$ due to the following argument: the data are smooth, with the density bounded away from zero, so by the theory for regular solutions, there exists a local solution. Furthermore, we have an a priori bound on $\big(u^n, \frac{1}{r^n}, \nabla p^n\big)$ which entails in particular the boundedness of ∇u^n in $L^1([0,T], L^\infty)$. This last fact prevents any blow up before T (use for instance the blow up criterion in [7] and Proposition 2.1 in [6]).
So we have a sequence of solutions defined on $[0,T]$,

$$\left(u^n, r^n, \frac{1}{r^n}, \nabla p^n\right) \ ,$$

which is uniformly bounded in

$$\big(L^\infty([0,T],H^s) \cap L^2([0,T],H^{s+1})\big) \times L^\infty([0,T],\dot{W}^{(s-1),d/(s-1)}) \times L^\infty([0,T],L^\infty) \times L^2([0,T],H^{s-1}) \ .$$

Compactness and passage to the limit

The uniform estimates above and the first equation of (19) give a uniform bound for $\|\partial_t r_n\|_{L^\infty L^d}$. By Ascoli's theorem, this implies that there exists R such that $R(0)=0$ and (up to an extraction)

$$R_n \overset{\text{def}}{=} r^n - r_0^n \to R \quad \text{in } \mathcal{C}([0,T],L^d) \ . \tag{20}$$

Therefore, defining $r = R + r_0$, we have

$$r^n = R^n + r_0^n \longrightarrow r \quad \text{in } \mathcal{S}' \ . \tag{21}$$

Since r^n is uniformly bounded in $L^\infty([0,T],\dot{W}^{s-1,d/(s-1)})$, we also obtain (up to a new extraction) that $r \in L^\infty([0,T],\dot{W}^{s-1,d/(s-1)})$.
We also observe that, since u^n is bounded in L^∞, the density propagates at finite speed. Since by lemma 3.1 r_0^n is uniformly bounded (in n) in L^p_{loc} for any finite p, we get that r^n is bounded in $L^\infty([0,T],L^p_{loc})$ for any finite p. This remark and the second equation of (19) give a uniform bound on $\partial_t u^n$ in $L^2([0,T],L^d_{loc})$, and Ascoli's theorem an extracted sequence such that

$$u^n \overset{n\to\infty}{\longrightarrow} u \quad \text{in } \mathcal{C}([0,T],L^d_{loc}) \ . \tag{22}$$

Finally, the uniform bound on ∇p^n in $L^2([0,T],H^{s-1})$ gives, up to an extraction,

$$\nabla p^n \longrightarrow \nabla p \quad \text{as } n \to \infty \quad \text{weakly in } L^2([0,T],H^{s-1}) \ . \tag{23}$$

As is easily verified, the convergences (20) (21) (22) (23) are enough to pass to the limit in the non-linear terms and show that (u,ρ) is a solution of (NNS). ■

4 Proof of Theorem 1.2

We will actually prove a more general version of Theorem 1.2; it is the following

Theorem 4.1 *Let (u_0, r_0) be initial data of finite energy. Suppose there exists an associated solution (u,r) such that*

1. *(u,r) is of finite energy.*
2. *(u,r) verifies* **either** *(C is a small constant)*

$$\nabla r \in L^\infty([0,T],L^d) \quad \textit{and} \quad \|r\|_{BMO} \le (\inf r) \tag{24}$$

$$\nabla u \in L^p([0,T],L^q) \quad \textit{with} \quad \frac{2}{p} + \frac{d}{q} = 2 \ , \ \ 1 < p < \infty \tag{25}$$

$$\sqrt{t}(u\cdot\nabla u)(t) \in L^2([0,T],L^d) \tag{26}$$

$$\sqrt{t}\Delta u(t) \in L^2([0,T],L^d) \ . \tag{27}$$

or

$$\nabla r \in L^\infty([0,T],L^d) \tag{28}$$

$$\nabla u \in L^p([0,T],L^q) \quad \textit{with} \quad \frac{2}{p} + \frac{d}{q} = 2 \ , \ \ 1 < p < \infty \tag{29}$$

$$\sqrt{t}(u\cdot\nabla u)(t) \in L^2([0,T],L^d) \tag{30}$$

$$\sqrt{t}\partial_t u(t) \in L^2([0,T],L^d) \ . \tag{31}$$

Under these conditions, (u, r) is unique on $[0, T]$ in the class of energy solutions.

PROOF: We shall prove the theorem only with the assumptions (24) (25) (26) (27); this is the harder case. We assume in the following that (u, r, p) and $(\bar{u}, \bar{r}, \bar{p})$ are two energy class solutions of (NNS), and that $(\bar{u}, \bar{r})$ satisfies the hypotheses of the theorem.

Estimate on the pressure

We first bound the pressure in L^d. Due to the equation satisfied by $(\bar{u}, \bar{r}, \bar{p})$, we have

$$\mathbb{Q}\left(\bar{r}\nabla\bar{p}\right) = \mathbb{Q}\left(r\Delta\bar{u}\right) + \mathbb{Q}\left(\bar{u}\cdot\nabla\bar{u}\right) \ .$$

The hypothesis (24) and Lemma 2.1 imply that

$$\|\nabla\bar{p}\|_d \leq C\left\|\mathbb{Q}\left(r\Delta\bar{u}\right) + \mathbb{Q}(\bar{u}\cdot\nabla\bar{u})\right\|_d \ .$$

Now using Theorem 2.1, we can bound

$$\begin{aligned}\|\nabla\bar{p}\|_d &\leq C\left(\|\bar{u}\cdot\nabla\bar{u}\|_d + \left\|\left[\mathbb{Q}\,,\,\bar{r}\right]\Delta\bar{u}\right\|_d\right)\\ &\leq C\left(\|\bar{u}\cdot\nabla\bar{u}\|_d + \|\bar{r}\|_{BMO}\,\|\Delta\bar{u}\|_d\right)\end{aligned}$$

Estimate on the conservation of the momentum equation for $u - \bar{u}$

Let us denote

$$v = u - \bar{u} \qquad R = r - \bar{r} \ .$$

Take the difference of the equations satisfied by (u, r) and $(\bar{u}, \bar{r})$, and perform an energy estimate to get (recall $\rho = \frac{1}{r}$)

$$\begin{aligned}\partial_t\|\sqrt{\rho}v\|_2^2 + \|\nabla v\|_2^2 &= -\int_{\mathbb{R}^d}\rho(v\cdot\nabla\bar{u})\cdot v\,dx + \int_{\mathbb{R}^d}R\Delta\bar{u}\cdot\rho v\,dx - \int_{\mathbb{R}^d}R\nabla\bar{p}\cdot\rho v\,dx\\ &\overset{\text{def}}{=} I + II + III \ .\end{aligned}\tag{32}$$

We will now estimate one by one I, II and III. These three terms are estimated with the help of Hölder's inequality and the Sobolev embedding theorem. For the first piece, we get

$$|I| = \left|\int_{\mathbb{R}^d}(\sqrt{\rho}v\cdot\nabla\bar{u})\cdot\sqrt{\rho}v\,dx\right| \leq C\|\nabla v\|_{\frac{d}{q}}\|\rho\|_\infty^{\frac{d}{2q}}\|\sqrt{\rho}v\|_2^{2-\frac{d}{q}}\|\nabla\bar{u}\|_q \ .\tag{33}$$

The second piece is not much harder

$$|II| = \left|\int_{\mathbb{R}^d}R\Delta\bar{u}\cdot\rho v\,dx\right| \leq C\|R\|_2\|\Delta\bar{u}\|_d\|\rho\|_\infty\|\nabla v\|_2 \ .\tag{34}$$

In order to estimate the third piece, we use the bound on the pressure in L^d and get

$$|III| = \left|\int_{\mathbb{R}^d}R\nabla\bar{p}\cdot\rho v\,dx\right| \leq C\|R\|_2\left(\|\Delta\bar{u}\|_d + \|\bar{u}\cdot\nabla\bar{u}\|_d\right)\|\rho\|_\infty\|\nabla v\|_2 \ .\tag{35}$$

Estimate on the conservation of mass equation for $r - \bar{r}$

The equation of conservation of mass becomes

$$\partial_t R + u \cdot \nabla R = -v \cdot \nabla \bar{r} \ .$$

Since u is divergence free, we get

$$\partial_t \|R\|_2 \leq \|\nabla v\|_2 \|\nabla \bar{r}\|_d \ ,$$

which can also be written, since $R(0) = 0$,

$$\|R(t)\|_2 \leq C \int_0^t \|\nabla v\|_2 ds \tag{36}$$

The differential inequality

Gathering the estimates (32) (33) (34) (35) (36), we see that

$$\begin{aligned} \partial_t \|\sqrt{\rho} v\|_2^2 + \|\nabla v\|_2^2 &\leq C \|\nabla v\|_2^{\frac{d}{q}} \|\sqrt{\rho} v\|_2^{2-\frac{d}{q}} \|\nabla \bar{u}\|_q + C \|R\|_2 \left(\|\Delta \bar{u}\|_d + \|\bar{u} \cdot \nabla \bar{u}\|_d\right) \|\nabla v\|_2 \\ &\leq \frac{1}{2} \|\nabla v\|_2^2 + C \|\sqrt{\rho} v\|_2^2 \|\nabla \bar{u}\|_q^p + C \int_0^t \|\nabla v\|_2 \, ds \, \|\nabla v\|_2 \left(\|\Delta \bar{u}\|_d + \|\bar{u} \cdot \nabla \bar{u}\|_d\right) \ . \end{aligned}$$

So denoting

$$f(t) = \|\sqrt{\rho} v(t)\|_2^2 \quad \text{and} \quad g(t) = \int_0^t \|\nabla v(s)\|_2 ds$$

we see that there exists α in L^1 and β verifying $\sqrt{t}\beta(t) \in L^2$ such that

$$f' + \left(g'\right)^2 \leq \alpha f + \beta g' g \ .$$

It now suffices to apply Lemma 2.2 to conclude the proof of the theorem. ■

5 Proof of the commutator estimates

5.1 Coifman and Meyer's generalized product operators and Littlewood-Paley theory

The proof of the commutator estimates will use two tools, which we now present: the boundedness criterion for generalized product operators, and the Littlewood-Paley theory.

The generalized product operators of Coifman and Meyer

Let us first introduce the following notation for the generalized product operators of Coifman and Meyer [3]:

$$T_m(f, g) = \int_{\mathbb{R}^d} e^{ix \cdot (\xi + \eta)} m(\xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) \, d\xi d\eta \ .$$

We now state the fundamental theorem of Coifman and Meyer [21] [14] [10]: if the kernel m satisfies

$$|\partial_\xi^\alpha \partial_\eta^\beta m(\xi, \eta)| \leq \frac{C}{\left(|\xi| + |\eta|\right)^{|\alpha| + |\beta|}} \ , \tag{37}$$

then T_m is bounded from $L^p \times L^q$ to L^r, with $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$, $1 < p, q \leq \infty$, $1 \leq r < \infty$.

Littlewood-Paley theory

We now come to the Littlewood Paley decomposition, first defining it, and then reviewing some classical facts that will be needed in the following; see Lemarié [17] for a more thorough treatment. Consider Ψ a function supported in the annulus centered in 0, of small radius 3/4, and big radius 8/3, such that

$$\sum_{j \in \mathbb{Z}} \Psi\left(\frac{\xi}{2^j}\right) = 1 \qquad \text{for } \xi \neq 0 \ .$$

Then the Fourier multipliers Δ_j and S_j are defined by

$$\Delta_j = \Psi\left(\frac{D}{2^j}\right)$$
$$S_N = 1 - \sum_{j \geq N+1} \Psi\left(\frac{D}{2^j}\right) \ .$$

Thus any distribution can be decomposed (modulo polynomials) into a sum of elementary elements that are localized in frequency:

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f \qquad \text{or} \qquad f = S_N f + \sum_{j \geq N+1} \Delta_j f \ .$$

The L^2-Sobolev norms can be expressed using the Littlewood-Paley decomposition:

$$\|f\|_{H^s}^2 \sim \sum_{j \in \mathbb{Z}} 2^{2js} \|\Delta_j f\|_{L^2}^2 \ .$$

Actually, if $f = \sum_j f_j$, with $\mathrm{Supp}\widehat{f_j} \subset B(0, C2^j)$, and $s > 0$, we even have

$$\|f\|_{H^s}^2 \leq C \sum_{j \in \mathbb{Z}} 2^{2js} \|f_j\|_{L^2}^2 \ . \tag{38}$$

For L^p spaces, we have the following bound (this is a consequence of the Littlewood-Paley theorem):

$$\|f\|_{L^p}^2 \leq C \sum_{j \in \mathbb{Z}} \|\Delta_j f\|_p^2 \qquad \text{if } p \geq 2 \ .$$

We also have an analog of (38): if $f = \sum_j f_j$, with $\mathrm{Supp}\widehat{f_j} \subset B(0, C2^j)$, if $s > 0$ and $p \geq 2$,

$$\|f\|_{W^{s,p}}^2 \leq C \sum_{j \in \mathbb{Z}} 2^{2js} \|f_j\|_{L^p}^2 \ . \tag{39}$$

Finally, we have the Bernstein inequality

$$\|S_j f\|_{L^p} \leq C 2^{jd(\frac{1}{q} - \frac{1}{p})} \|S_j f\|_{L^q} \qquad \text{if } 1 \leq p \leq q \leq \infty \ , \tag{40}$$

and we record the following estimates

$$\begin{aligned} \|\Lambda^s S_j f\|_{L^p} &\leq C 2^{js} \|S_j f\|_p \\ \|\Lambda^s \Delta_j f\|_{L^p} &\leq C 2^{js} \|\Delta_j f\|_p \ . \end{aligned} \tag{41}$$

The paraproduct algorithm

The idea of the paraproduct algorithm is to decompose the product of two functions (or distributions) f and g, depending on the relative sizes of the frequencies of f and g.
We will explain this idea, first in the framework of generalized product operators, and then using Littlewood-Paley theory; these two approaches are of course equivalent, as is easily seen.
In the proofs of the estimate (10) and (11), we will apply the paraproduct algorithm to expressions that are not exactly products (they involve differential operators), but the general philosophy remains exactly the same.

First of all, we can write the product fg as

$$fg = \int_{\mathbb{R}^d} e^{ix\cdot(\xi+\eta)} \widehat{f}(\xi)\widehat{g}(\eta)\, d\xi d\eta \ .$$

Now we split the domain of integration in three regions, $|\xi| << |\eta|$, $|\xi| \sim |\eta|$ and $|\xi| >> |\eta|$
We do this using three functions ϕ_i of a real variable, such that: their sum is identically 1, and their supports are respectively the ball of radius 2/10, the annulus of inner radius 1/10 and outer radius 10, and the complementary of the ball of radius 9. We have then

$$fg = \int_{\mathbb{R}^d} e^{ix\cdot(\xi+\eta)} \left(\phi_1\left(\frac{|\xi|}{|\eta|}\right) + \phi_2\left(\frac{|\xi|}{|\eta|}\right) + \phi_3\left(\frac{|\xi|}{|\eta|}\right)\right) \widehat{f}(\xi)\widehat{g}(\eta)\, d\xi d\eta \ , \tag{42}$$

and the three summands correspond to the regions $|\xi| << |\eta|$, $|\xi| \sim |\eta|$ and $|\xi| >> |\eta|$.

Another approach is to use Littlewood-Paley theory. We write

$$\begin{aligned} fg &= \left(\sum_j \Delta_j f\right)\left(\sum_j \Delta_j g\right) \\ &= \sum_j \Delta_j f S_{j-1} g + \sum_{|j-k|\leq 2} \Delta_j f \Delta_k g + \sum_j \Delta_j g S_{j-1} f \end{aligned} \tag{43}$$

The first (and third) sum above has a very nice property: due to the frequency localization of the Δ_j and S_j operators, each one of the summands $\Delta_j f S_{j-1} g$ is localized in frequency in an annulus centered at 0 of size $\sim 2^j$.
The analogy with the previous formulation of the algorithm is obvious. However, the three sums of (43) do not correspond exactly to the three integrals of (42). Actually, in order to make them equal, we should take the index j to be continuous instead of discrete, and define Δ_j slightly differently. This would change only small details, so for notational

convenience we consider in the following that

$$\int_{\mathbb{R}^d} e^{ix\cdot(\xi+\eta)}\phi_1\left(\frac{|\xi|}{|\eta|}\right)\widehat{f}(\xi)\widehat{g}(\eta)\,d\xi d\eta \sim \sum_j \Delta_j g S_{j-1} f$$

$$\int_{\mathbb{R}^d} e^{ix\cdot(\xi+\eta)}\phi_2\left(\frac{|\xi|}{|\eta|}\right)\widehat{f}(\xi)\widehat{g}(\eta)\,d\xi d\eta \sim \sum_j \Delta_j f \Delta_j g$$

$$\int_{\mathbb{R}^d} e^{ix\cdot(\xi+\eta)}\phi_3\left(\frac{|\xi|}{|\eta|}\right)\widehat{f}(\xi)\widehat{g}(\eta)\,d\xi d\eta \sim \sum_j \Delta_j f S_{j-1} g$$

Notice that in the second line above, we replaced $\sum_{|j-k|\leq 2}\Delta_j f \Delta_k g$ by $\sum_j \Delta_j f \Delta_j g$. Once again, this does not remove a difficulty, but makes the notations lighter.

The idea in the proofs of estimates (10) and (11), which follow, is the following: first, use the paraproduct algorithm to split the commutator into several pieces. For each of this pieces, prove the desired boundedness property by using the theorem of Coifman and Meyer (if the symbol is smooth) or the Littlewood-Paley theory (if it is not).

5.2 Proof of estimate (10)

Our commutator can be rewritten as

$$[\Lambda^\sigma\,,\,f]\,g = \int_{\mathbb{R}^d} e^{ix\cdot(\xi+\eta)}\left(|\xi+\eta|^\sigma - |\eta|^\sigma\right)\widehat{f}(\xi)\widehat{g}(\eta)\,d\xi d\eta = T_m(f,g)$$

by setting

$$m(\xi,\eta) = \left(|\xi+\eta|^\sigma - |\eta|^\sigma\right)\ .$$

We will distinguish between the regions where $|\xi| << |\eta|$, $|\xi| \sim |\eta|$ and $|\xi| >> |\eta|$, and will have to use different techniques for each of those regions. So we consider as explained above three functions ϕ_i of a real variable, whose sum is identically 1, and whose supports are respectively the ball of radius 2/10, the annulus of inner radius 1/10 and outer radius 10, and the complementary of the ball of radius 9

Let us begin with $|\xi| << |\eta|$; this corresponds to the symbol

$$m_1(\xi,\eta) = \phi_1\left(\frac{|\xi|}{|\eta|}\right)m(\xi,\eta)\ .$$

A small computation shows that m_1 can be written as

$$m_1(\xi,\eta) = \sum_{k=1}^{d}\mu(\xi,\eta)\frac{\xi^k + 2\eta^k}{|\eta|}\xi^k|\eta|^{\sigma-1}\ ,$$

where μ is a homogeneous function of order 0, smooth outside the origin, hence satisfying the estimates (37). This is also the case for

$$\widetilde{\mu}(\xi,\eta) = \mu(\xi,\eta)\frac{\xi^k + 2\eta^k}{|\eta|}\ .$$

So we have

$$T_{m_1}(f,g) = T_{\widetilde{\mu}}(\nabla f, \Lambda^{\sigma-1} g)\ ,$$

where $\widetilde{\mu}$ satisfies the estimates (37). The theorem of Coifman and Meyer yields thus

$$\|T_{m_1}(f,g)\|_2 = \left\|T_{\widetilde{\mu}}(\nabla f, \Lambda^{\sigma-1}g)\right\|_2 \leq C\|\nabla f\|_d \|\Lambda^{\sigma-1}g\|_{\frac{2d}{d-2}} \leq C\|\Lambda^\sigma f\|_{\frac{d}{\sigma}} \|\Lambda^\sigma g\|_2 \ . \tag{44}$$

Now we look at the operator with symbol

$$m_2(\xi,\eta) = |\eta|^\sigma \left(\phi_2\left(\frac{|\xi|}{|\eta|}\right) + \phi_3\left(\frac{|\xi|}{|\eta|}\right)\right)$$

which corresponds to the second component of our commutator in the regions $|\xi| \sim |\eta|$ and $|\xi| >> |\eta|$. We notice that $T_{m_2}(f,g)$ is nothing but the paraproduct operator of $\Lambda^\sigma g$ (lower frequencies) by f (higher frequencies), so by the classical theorem (see for instance Stein [24], Chapter IV), we get

$$\|T_{m_2}(f,g)\|_2 \leq C\|f\|_{BMO}\|\Lambda^\sigma g\|_2 \leq C\|\Lambda^\sigma f\|_{d/\sigma}\|\Lambda^\sigma g\|_2 \ ,$$

where we used in the last inequality the embedding (7).

The next operator has symbol

$$m_3(\xi,\eta) = |\xi+\eta|^\sigma \phi_2\left(\frac{|\xi|}{|\eta|}\right) \ .$$

It corresponds to the first component of our commutator in the region $|\xi| \sim |\eta|$. Switching to Littlewood-Paley notation as explained in the previous section, we observe that $T_{m_3}(f,g)$ is of the form

$$\Lambda^\sigma \sum_j \Delta_j f \Delta_j g$$

By the inequality (38), we have

$$\begin{aligned}
\left\|\Lambda^\sigma \sum_j \Delta_j f \Delta_j g\right\|_2^2 &= \left\|\sum_j \Delta_j f \Delta_j g\right\|_{H^\sigma}^2 \\
&\leq C \sum_j 2^{2j\sigma} \|\Delta_j f \Delta_j g\|_2^2 \\
&\leq C \sum_j 2^{2j\sigma} \|\Delta_j f\|_\infty \|\Delta_j g\|_2^2 \\
&\leq C \|\Delta_j f\|_\infty \sum_j 2^{2j\sigma} \|\Delta_j g\|_2^2 \\
&\leq C \|\Lambda^\sigma f\|_{d/\sigma}^2 \|\Lambda^\sigma g\|_2^2 \quad \text{by Bernstein's inequality (40) and (41)} \ .
\end{aligned}$$

Finally, we have to deal with the operator whose symbol reads

$$m_4(\xi,\eta) = |\xi+\eta|^\sigma \phi_3\left(\frac{|\xi|}{|\eta|}\right) \ ;$$

it is the first component of our commutator in the region $|\xi| >> |\eta|$. We can write m_4 as

$$m_4(\xi,\eta) = \phi_3\left(\frac{|\xi|}{|\eta|}\right) \frac{|\xi+\eta|^\sigma}{|\xi^\sigma|} |\xi|^\sigma = \rho(\xi)|\xi|^\sigma \ .$$

Applying the theorem of Coifman and Meyer we get

$$\|T_{m_4}(f,g)\|_2 = \|T_\rho(\Lambda^\sigma f, g)\|_2 \leq C\|\Lambda^\sigma f\|_{d/\sigma}\|g\|_{\frac{2d}{d-2\sigma}} \leq C\|\Lambda^\sigma f\|_{d/\sigma}\|\Lambda^\sigma g\|_2 \ .$$

The last inequality is implied by Sobolev's embedding theorem.

To conclude the proof of (10), it suffices to observe that

$$m = m_1 + m_2 + m_3 + m_4 \ . \quad \blacksquare$$

5.3 Proof of estimate (11)

We follow the same idea and use the same notations as in Section 5.2, so we begin by writing the commutator as

$$[\Lambda^\sigma \, , \, f]\, g = \int_{\mathbb{R}^d} e^{ix\cdot(\xi+\eta)} \left(|\xi+\eta|^\sigma - |\eta|^\sigma\right) \widehat{f}(\xi)\widehat{g}(\eta)\, d\xi d\eta = T_m(f,g) \ .$$

Reasoning as above, we consider first

$$m_1(\xi,\eta) = m(\xi,\eta)\phi_1\left(\frac{|\xi|}{|\eta|}\right) = \widetilde{\mu}(\xi,\eta)\xi^k|\eta|^{\sigma-1} \ ,$$

and use the theorem of Coifman and Meyer to bound

$$\|T_{m_1}(f,g)\|_{d/\sigma} = \|T_{\widetilde{\mu}}(\nabla f, \Lambda^{\sigma-1}g)\|_{d/\sigma} \leq C\|\nabla f\|_\infty \|\Lambda^{\sigma-1}g\|_{d/\sigma} \ .$$

The next step is to consider the symbol

$$n_1(\xi,\eta) = |\xi+\eta|^\sigma\left(\phi_2\left(\frac{|\xi|}{|\eta|}\right) + \phi_3\left(\frac{|\xi|}{|\eta|}\right)\right) \ .$$

It corresponds to frequencies $|\xi| \sim |\eta|$ or $|\xi| >> |\eta|$, that is to say in the Littlewood-Paley framework, we are considering the first and second sums of (42). So if we switch to Littlewood-Paley theory, $T_{n_1}(f,g)$ becomes equivalent to

$$\Lambda^\sigma \sum_j \Delta_j f S_{j+3} g \ ,$$

Using (39),

$$\begin{aligned}
\left\|\Lambda^\sigma \sum_j \Delta_j f S_{j+3}g\right\|_{d/\sigma}^2 &\leq C\sum_j 2^{2j\sigma}\|\Delta_j f S_{j+3}g\|_{d/\sigma}^2 \\
&\leq C\sum_j 2^{2j\sigma}\|\Delta_j f\|_p^2 \|S_{j+3}g\|_d^2 \quad \text{with } \frac{1}{p}+\frac{1}{d}=\frac{\sigma}{d} \\
&\leq C\|g\|_d^2 \sum_j 2^{j\sigma}\|\Delta_j f\|_p^2 \\
&\leq C\|g\|_d^2 \sum_j 2^{j(d+2)}\|\Delta_j f\|_2^2 \quad \text{by Bernstein's inequality (40)} \\
&\leq C\|g\|_d^2 \|\Lambda^{(\frac{d}{2}+1)} f\|_2^2 \ .
\end{aligned}$$

Finally, we have to deal with

$$n_2(\xi,\eta) = |\eta|^\sigma \left(\phi_2\left(\frac{|\xi|}{|\eta|}\right) + \phi_3\left(\frac{|\xi|}{|\eta|}\right)\right) \ .$$

It corresponds to frequencies $|\xi| \sim |\eta|$ or $|\xi| >> |\eta|$, so if we switch to Littlewood-Paley theory, $T_{n_2}(f,g)$ becomes equivalent to

$$\sum_j \Delta_j f \Lambda^\sigma S_{j+3} g \ .$$

By the Sobolev embedding theorem, for $\epsilon > 0$ small, one has

$$\left\|\sum_j \Delta_j f \Lambda^\sigma S_{j+3} g\right\|_{d/\sigma} \leq \left\|\sum_j \Delta_j f \Lambda^\sigma S_{j+3} g\right\|_{W^{\epsilon,q}} \qquad \text{with} \ \ \frac{d}{q} - \epsilon = s \ .$$

Using (39),

$$\begin{aligned}
\left\|\sum_j \Delta_j f \Lambda^\sigma S_{j+3} g\right\|^2_{W^{\epsilon,p}} &\leq C \sum_j 2^{2j\epsilon} \|\Delta_j f S_{j+3} g\|_q^2 \\
&\leq C \sum_j 2^{2j\epsilon} \|\Delta_j f\|_p^2 \|\Lambda^\sigma S_{j+3} g\|_d^2 \quad \text{ with } \ \frac{1}{p} + \frac{1}{d} = \frac{1}{q} \\
&\leq C \|g\|_d^2 \sum_j 2^{2j(\epsilon+\sigma)} \|\Delta_j f\|_p^2 \quad \text{ by (41)} \\
&\leq C \|g\|_d^2 \sum_j 2^{j(d+2)} \|\Delta_j f\|_2^2 \quad \text{ by Bernstein's inequality (40)} \\
&\leq C \|g\|_d^2 \|\Lambda^{(\frac{d}{2}+1)} f\|_2^2 \ .
\end{aligned}$$

To conclude the proof of (11), we just notice that

$$m = m_1 + n_1 + n_2 \ . \quad \blacksquare$$

6 The space $\dot{W}^{\sigma,d/\sigma}$

6.1 Definition

We sketched in Section 1.5 a definition of the space $\dot{W}^{\sigma,d/\sigma}$. We want here to give the rigorous definition.

Definition 6.1 *Let $\sigma \in (0,d)$. A function $f \in \mathcal{S}'$ is said to belong to $\dot{W}^{\sigma,d/\sigma}$ if*

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f \ \ \text{in} \ \ \mathcal{S}' \ \ \text{modulo constants} \ ,$$

and the following norm is finite

$$\|f\|_{\dot{F}^\sigma_{d/\sigma,2}} \overset{def}{=} \left\| \left(\sum_j 2^{2j\sigma} |\Delta_j f|^2 \right)^{1/2} \right\|_{d/\sigma} \ .$$

An equivalent norm is given by

$$\|\Lambda^\sigma f\|_{d/\sigma} \overset{def}{=} \lim_{N\to\infty} \left\| \Lambda^\sigma \sum_{-N}^{N} \Delta_j f \right\|_{d/\sigma} .$$

Finally, if $\sigma \in (0,1)$, a third equivalent norm is provided by

$$\left\| \left(\int_0^\infty t^{-2\sigma} \left(\frac{1}{t^d} \int_{|h|<t} |g(x) - g(x+h)| dh \right)^2 \frac{dt}{t} \right)^{1/2} \right\|_{d/\sigma} \tag{45}$$

The three quantities above are equivalent norms on $\dot{W}^{\sigma,d/\sigma}$, making it a Banach space. But it is only a Banach space modulo constants: it embeds continuously in $\mathcal{S}'/\mathbb{R}$ but not in $\mathcal{S}'$.

Let us now justify carefully the above definition

- In defining $\dot{W}^{\sigma,d/\sigma}$ as the Triebel-Lizorkin space $\dot{F}^{\sigma}_{d/\sigma,2}$, we follow (for instance) Runst and Sickel [22].
- The equivalence of the first and the second norms follows from the Littlewood-Paley theorem.
- A priori, $\dot{F}^{\sigma}_{d/\sigma,2}$ is only defined modulo polynomials. It is proved in Lemarié-Rieusset's book [17], in the very similar case of Besov spaces, that a definition modulo constants is possible.
- The equivalence of the third norm is proved in the book of Runst and Sickel [22], page 41, in the non-homogeneous case; this can be adapted to the homogeneous case.

The definition of $\dot{W}^{\sigma-1,\frac{d}{\sigma}}$ can be given along the same lines, but one does not need to work modulo constants any more. We have the following easy lemma.

Lemma 6.1 *There holds*

$$\|f\|_{\dot{W}^{\sigma,d/\sigma}} \sim \|\nabla f\|_{\dot{W}^{\sigma-1,d/\sigma}}$$

Proof: On the one hand, by the above definitions, and denoting the Riesz transform $\frac{\partial_k}{|D|}$ by R_k,

$$\|f\|_{\dot{W}^{\sigma,d/\sigma}} = \|\Lambda^\sigma f\|_{d/\sigma} \geq C \|R_k \Lambda^\sigma f\|_{d/\sigma} = C \left\| \partial_k \Lambda^{\sigma-1} f \right\|_{d/\sigma} .$$

The other inequality is also easy to prove. ■

6.2 Proof of the approximation lemma 3.1

First, let us define a sequence of functions $(F_n) : \mathbb{R}^+ \to \mathbb{R}^+$ which verify the following properties

- F_n is non decreasing.
- F_n is the identity on $[0,n]$.

- F_n is identically equal to $n+1$ on $[n+1, \infty)$.
- All derivatives of F_n are bounded, uniformally in n.

We also need an approximation of the identity $\phi_n = n^d \phi(n \cdot)$ such that $\mathcal{F}(\phi)$ equals 1 in $B(0,1)$ and zero outside of $B(0,2)$.
Now given r_0 as in Lemma 3.1, we consider the following approximation scheme

$$r_0^n = \phi_n * F_n(r_0)$$

and we claim that it satisfies the points (i) (ii) (iii) and (iv) of Lemma 3.1. This is clear for (i) and (iii).

In order to see that (iv) holds, one notices first that applying F_n to r is harmless as far as the infimum of r is concerned. So we only have to examine the convolution by ϕ_n. But since almost any x is a Lebesgue point for r_0, the infimum of r_0^n converges to the infimum of r_0, that is, (iv) holds.

So we are left with proving (ii), that is

$$r_0^n \longrightarrow r_0 \quad \text{in } \dot{W}^{\sigma,d/\sigma} \quad \text{as} \quad n \to \infty \ .$$

In other words, we will prove the following lemma

Lemma 6.2 *Suppose f is a positive function belonging to $\dot{W}^{\sigma,d/\sigma}$, then*

$$\phi_n * F_n(f) \longrightarrow f \quad \textit{in } \dot{W}^{\sigma,d/\sigma} \quad \textit{as} \quad n \to \infty \ .$$

PROOF : The definition of the norm of $\dot{W}^{\sigma,d/\sigma}$ as a Triebel-Lizorkin space gives at once the convergence of $\phi_n * f$ to f in $\dot{W}^{\sigma,d/\sigma}$. The only remaining thing to check is the convergence of $F_n(f)$.
Our first aim will be to show that $F_n(f)$ is bounded in $\dot{W}^{\sigma,d/\sigma}$. We will use the third definition of $\dot{W}^{\sigma,d/\sigma}$ given in the previous section, which relies on difference estimates.

<u>Boundedness for $0 < \sigma < 1$</u>

Let us suppose first that $\sigma \in (0,1)$. Then, since the derivatives of F_n are bounded,

$$\begin{aligned} \|F_n(f)\|_{\dot{W}^{\sigma,d/\sigma}} &\leq C \left\| \left(\int_0^\infty t^{-2\sigma} \left(\frac{1}{t^d} \int_{|h|<t} |F_n(f)(x) - F_n(f)(x+h)| dh \right)^2 \frac{dt}{t} \right)^{1/2} \right\|_{d/\sigma} \\ &\leq C \left\| \left(\int_0^\infty t^{-2\sigma} \left(\frac{1}{t^d} \int_{|h|<t} |f(x) - f(x+h)| dh \right)^2 \frac{dt}{t} \right)^{1/2} \right\|_{d/\sigma} \\ &\leq C \|f\|_{\dot{W}^{\sigma,d/\sigma}} \ . \end{aligned}$$

<u>Boundedness for $1 < \sigma$</u>

Let us suppose now that $\sigma \in (1,2)$ (larger σ are treated in a very similar way; to avoid technicalities, we shall not give the proof for $\sigma > 2$). Denoting ∂_i for the partial derivative with respect to x_i, we observe that $F_n(f) \in \dot{W}^{\sigma,d/\sigma}$ if and only if $\partial_i F_n(f) = \partial_i f F_n'(f) \in \dot{W}^{\sigma-1,d/\sigma}$. Therefore, using a similar formula as (45), but for $\dot{W}^{\sigma-1,d/\sigma}$, we have

$$\begin{aligned}\|F_n(f)\|_{\dot{W}^{\sigma,d/\sigma}} &\leq C\|\partial_i f F_n'(f)\|_{\dot{W}^{\sigma-1,d/\sigma}} \\ &\leq C\left\|\left(\int_0^\infty t^{2(1-\sigma)}\left(\frac{1}{t^d}\int_{|h|<t}|g(x)-g(x+h)|dh\right)^2\frac{dt}{t}\right)^{1/2}\right\|_{d/\sigma} ,\end{aligned}$$

where we denoted $g = \partial_i f F_n'(f)$. Now

$$\begin{aligned}g(x)-g(x+h) &= \left(\partial_i f(x) - \partial_i f(x+h)\right) F_n'(f)(x+h) + \left(F_n'(f)(x) - F_n'(f)(x+h)\right)\partial_i f(x) \\ &= I + II \ .\end{aligned}$$

Since F_n' is bounded, and $\partial_i f \in \dot{W}^{\sigma-1,d/\sigma}$, it is clear that

$$\left\|\left(\int_0^\infty t^{2(1-\sigma)}\left(\frac{1}{t^d}\int_{|h|<t}|I|dh\right)^2\frac{dt}{t}\right)^{1/2}\right\|_{d/\sigma} \leq C\|\partial_i f\|_{\dot{W}^{\sigma-1,d/\sigma}} \leq C\|f\|_{\dot{W}^{\sigma,d/\sigma}} \ .$$

We are left with the term involving II, which reads

$$\begin{aligned}&\left\|\left(\int_0^\infty t^{2(1-\sigma)}\left(\frac{1}{t^d}\int_{|h|<t}|\left(F_n'(f)(x) - F_n'(f)(x+h)\right)\partial_i f(x)|dh\right)^2\frac{dt}{t}\right)^{1/2}\right\|_{d/\sigma} \\ &= \left\||\partial_i f(x)|\left(\int_0^\infty t^{2(1-\sigma)}\left(\frac{1}{t^d}\int_{|h|<t}|F_n'(f)(x) - F_n'(f)(x+h)|dh\right)^2\frac{dt}{t}\right)^{1/2}\right\|_{d/\sigma} .\end{aligned}$$

By Sobolev embedding, $\partial_i f \in L^d$; also F_n' is Lipschitz, so in order to bound the above term we just have to bound

$$\left\|\left(\int_0^\infty t^{2(1-\sigma)}\left(\frac{1}{t^d}\int_{|h|<t}|f(x)-f(x+h)|\,dh\right)^2\frac{dt}{t}\right)^{1/2}\right\|_{\frac{d}{\sigma-1}} \leq C\|f\|_{\dot{W}^{\sigma-1,\frac{d}{\sigma-1}}} \leq C\|f\|_{\dot{W}^{\sigma,d/\sigma}} \ ,$$

where the last inequality follows from Sobolev embedding.

Convergence

We have showed above that $F_n(f)$ is bounded; now we will see that it converges to f. Let us denote

$$E_n = \{x \in \mathbb{R}^d \, , \, f(x) \leq n\} \ .$$

By definition, $F_n(f) = f$ in E_n. Let us prove the convergence in the case $0 < \sigma < 1$, the other cases being very similar. We want to evaluate the $\dot{W}^{\sigma,d/\sigma}$ norm of $F_n(f) - f$. We have to examine

$$A_n(x,h) = |F_n(f)(x) - f(x) - F_n(f)(x+h) + f(x+h)| \ ,$$

which corresponds to the integrand in the difference formula (45) for the norm of $\dot{W}^{\sigma,d/\sigma}$. We observe that the above quantity is always bounded by $|f(x)-f(x+h)|$, and that it is 0 if both x and $x+h$ belong to E_n. Also, we have

$$\cup_{n\in\mathbb{N}} E_n = \mathbb{R}^d \ .$$

As a result,

$$\|F_n(f)-f\|_{\dot{W}^{\sigma,d/\sigma}} \leq C \left\| \left(\int_0^\infty t^{-2\sigma} \left(\frac{1}{t^d} \int_{|h|<t} |A_n(x,h)| dh \right)^2 \frac{dt}{t} \right)^{1/2} \right\|_{d/\sigma} \overset{n\to\infty}{\longrightarrow} 0 \ ,$$

proving the lemma. ■

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