This chapter will be dedicated to the perturbative approach to the nonlinear Schrödinger equation

\[
\begin{aligned}
&i \partial_t u - \Delta u = \mu |u|^2 u, \\
&u(t = 0) = u_0
\end{aligned}
\]  

(NLS)

where \( u(t, x) \in \mathbb{C} \) is a function of \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^d \) or \( T^d \), and \( \mu = \pm 1 \).

What is meant by \textit{perturbative} is that we will focus on regimes where the linear part of the equation is dominant. To be more explicit, write the equation in Duhamel form

\[
 u(t) = e^{it\Delta} u_0 + \int_0^t e^{i(t-s)\Delta} |u|^2 u(s) \, ds.
\]

Proceeding heuristically, let \( \epsilon \) denote the size of the data \( u_0 \) and of \( u \), which we can assume to be comparable; and let \( T \) denote the time over which we want to solve the above equation. Then

- The linear term \( e^{it\Delta} u_0 \) has size \( \sim \epsilon \).
- The nonlinear term \( \int_0^t e^{i(t-s)\Delta} |u|^2 u(s) \, ds \) has size \( \sim \int_0^T \epsilon^3 \, ds \sim \epsilon^3 T \).

In this naive line of reasoning, we already see that for the nonlinear term to be treated as a perturbation, we need

\[
\epsilon^3 T \ll \epsilon \iff \epsilon^2 T \ll 1.
\]

More generally, we will see that the perturbative regime can be reached by assuming smallness of the data, or of the time interval.

It will also become clear that the perturbative approach, when it applies, does so for \( \mu = \pm 1 \), and gives similar results in both cases. This should not really come as a surprise: the sign of \( \mu \) becomes significant for strongly nonlinear effects (observed for large data, large time, or both); they will be studied in the following chapters.

1. Data in \( H^s \), \( s > \frac{d}{2} \)

The following theorem gives local well-posedness in any \( H^s \), \( s > \frac{d}{2} \), independently of the domain (\( T^d \) or \( \mathbb{R}^d \)), and of course, of the sign of \( \mu \). Such a level of generality makes \( H^s \), with \( s > \frac{d}{2} \) a great framework... but it also means that we are missing most of the structure of the equation, and hence capturing only a small part of the dynamics.

**Theorem 1.1.** Consider (NLS) on \( \mathbb{R}^d \) or \( T^d \), with \( \mu = \pm 1 \).

Fix \( s > \frac{d}{2} \); then (NLS) is locally well-posed in \( H^s \) (in the sense of mild solutions):

- **Existence:** For any \( u_0 \in H^s \), there exists a (mild) solution \( u \) defined on \([0, T_0]\), where \( T_0 = c_s \|u_0\|_{H^s}^{-2} \), for a constant \( c_s > 0 \). It is such that \( \|u\|_{C_{T_0}H^s} \leq 2\|u_0\|_{H^s} \).
- **Uniqueness:** it is unique in \( L^\infty_{T_0} H^s \).
- **Dependence on the data:** the map \( u_0 \mapsto u \) is analytic from \( H^s \) to \( C_{T_0}H^s \) (more precisely: the time of existence \( T_0 \) can be chosen locally uniformly, and around any \( u_0 \) the solution operator can be expanded in series of multilinear operators).
**Proof.** Fix \( s > \frac{d}{2} \). We will not distinguish between the cases of \( \mathbb{R}^d \) and \( \mathbb{T}^d \), and simply denote \( H^s \) for the corresponding Sobolev spaces.

Step 1: a priori estimate. An a priori estimate gives a *quantitative* control on the solution of a PDE, under a *qualitative* assumption (such as: the solution is smooth). It is a very fundamental idea in nonlinear PDE that a priori estimates are the cornerstone of most proofs. To illustrate this idea, we will see first the a priori estimate, and discuss then how it can be converted into a completely rigorous proof. Start with the Duhamel formulation

\[
    u(t) = e^{it\Delta}u_0 + \mu \int_0^t e^{i(t-s)\Delta} |u|^2 u(s) \, ds.
\]  

(1.1)

Take the \( H^s \) norm of both sides. This gives immediately

\[
    \|u(t)\|_{H^s} \leq \|e^{it\Delta}u_0\|_{H^s} + \int_0^t \left\| e^{i(t-s)\Delta} |u|^2 u(s) \right\|_{H^s} \, ds
\]

\[
    = \|u_0\|_{H^s} + \int_0^t \|u(s)\|^2_{H^s} \, ds \quad \text{(since } e^{is\Delta} \text{ is an isometry on } H^s)\]

\[
    \leq \|u_0\|_{H^s} + CT \int_0^t \|u(s)\|^3_{H^s} \, ds \quad \text{(since } H^s \text{ is an algebra)}
\]

Taking the \( L_T^\infty \) bound leads to

\[
    \|u\|_{L_T^\infty H^s} \leq \|u_0\|_{H^s} + CT \|u\|_{L_T^\infty H^s}^3,
\]

which can also be written \( F(\|u\|_{L_T^\infty H^s}) \leq 0 \), with \( F(x) = x - \|u_0\|_{H^s} - CT x^3 \). Choosing \( T = T_0 = c_s \|u_0\|_{H^s}^{-2} \), we see that \( F^{-1}(-\infty, 0) \cap (0, \infty) \) has two separate components, the first one being of the form \( [0, C_0 \|u_0\|_{H^s}] \), for a constant \( C_0 \). Observe that \( F(\|u\|_{L_T^\infty H^s}) \leq 0 \) for any \( \tau \leq T \), and that \( \|u\|_{L_T^\infty H^s} \to 0 \) as \( \tau \to 0 \). By a continuity argument, we therefore obtain that

\[
    \|u\|_{L_T^\infty H^s} \lesssim \|u_0\|_{H^s}.
\]

Step 2: fixed point argument. To turn the above a priori estimate into a rigorous existence proof, define the map \( \Phi \) through

\[
    \Phi : u \mapsto e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta} |u|^2 u(s) \, ds.
\]

This map enables us to reformulate the equation (NLS) as the fixed point problem

\[
    \Phi(u) = u.
\]

It is a consequence of Step 1 that \( \Phi \) maps \( L_T^\infty H^s \) to itself. We will now show that \( \Phi \) is a contraction from the ball \( \mathcal{B} = B_{L_T^\infty H^s}(0, 2\|u_0\|_{H^s}) \) to itself. By Picard’s fixed point theorem, this will imply that \( \Phi \) has a unique fixed point, proving the existence part of the theorem.

- Let us first check that \( \Phi \) maps \( \mathcal{B} \) to itself. This follows essentially the argument in Step 1: if \( u \in \mathcal{B} \), and provided that \( c_s \) is chosen sufficiently small (recall that \( T_0 = c_s \|u_0\|_{H^s}^{-3} \)),

\[
    \|\Phi(u)\|_{L_T^\infty H^s} \leq \|u_0\|_{H^s} + CT_0 \|u\|^3_{L_T^\infty H^s} \leq \|u_0\|_{H^s} + CT_0 \|u_0\|^3_{H^s} \leq 2\|u_0\|_{H^s},
\]

which is the desired result.
Let us now check that $\Phi$ acts as a contraction on $\mathcal{B}$: if $u, v \in \mathcal{B}$, and provided that $c_s$ is chosen sufficiently small,

$$
\|\Phi(u) - \Phi(v)\|_{L^\infty_t H^s} \leq C T_0 \|u^2 u - |v|^2 v\|_{L^\infty_t H^s} \\
\leq C T_0 \left( \|u\|_{L^\infty_t H^s}^2 + \|v\|_{L^\infty_t H^s}^2 \right) \|u - v\|_{L^\infty_t H^s} \\
\leq C T_0 \|u_0\|_{H^s}^2 \|u - v\|_{L^\infty H^s} \leq \frac{1}{2} \|u - v\|_{L^\infty H^s}.
$$

Picard’s fixed point theorem now gives a solution $u \in L^\infty_{T_0} H^s$. It is now easy, thanks to the Duhamel formula (1.1).

**Step 3:** uniqueness in $L^\infty_t H^s$. The uniqueness result in Picard’s fixed point theorem only holds in $\mathcal{B}$; and we are seeking uniqueness in the whole space $L^\infty_t H^s$. In order to prove that it holds, consider data $u_0 \in H^s$, and assume that two solutions of the problem are given: $u$ and $v$, both in $L^\infty_t H^s$, for some $T$. As we saw earlier, they also belong to $C_T H^s$. Let now

$$
t_0 = \max\{t > 0 \text{ such that } u = v \text{ on } [0,T] \}.
$$

Let us argue by contradiction, and assume that $t_0 < T$. Consider then the Cauchy problem, set at the initial time $t_0$, with data $u(t_0) = v(t_0)$. We view it as a fixed point problem on $[t_0, t_1]$, where $t_1$ remains to be determined, and define $\Phi$ similarly as in Step 2. Let $R$ be a majorant of the norm of $u$ and $v$ in $L^\infty_{t_0} H^s$. Then, for $t_1$ sufficiently close to $t_0$, we can show as above that $\Phi$ acts as a contraction on $B_{L^\infty_{[t_0,t_1]} H^s}(0, 2R)$. Therefore, $u = v$ on $[t_0, t_1]$. ... but it contradicts the fact that $t_0 < T$!

**Step 4:** analytic dependence.

\[ \Box \]

**2. Data in the scale-invariant space $\dot{H}^{d-2/2}$ for (NLS) on $\mathbb{R}^d$, $d \geq 2$**

When it is possible, applying a fixed point argument to data in a scale invariant space has several consequences, the most important one being the possibility to construct global solutions by a simple fixed point argument.

Now is therefore a good time to introduce some vocabulary related to large time behavior of solutions of (NLS) on $\mathbb{R}^d$.

- The expectation for small data, or in the defocusing case, is that solutions will be asymptotically linear, namely that there exists $\psi_+$ such that $u(t) \sim e^{it\Delta} \psi_+$ as $t \to \infty$. If this is the case, we will say that $u$ scatters. The (expected) mechanism is the following: as $t \to \infty$, the waves spread so that $u$ converges locally to 0; but then the nonlinear term, which is cubic, becomes irrelevant compared to the linear term.
- One can try and go the other way around, and, for a given $\psi$, try and find $u_0$ such that the associated solution of (NLS) satisfies $u(t) \sim e^{it\Delta} \psi_+$ as $t \to \infty$. The operator $W : \psi \mapsto u_0$ is called wave operator (if it exists!).

Recall that the scale invariant space is $\dot{H}^{d-2/2}$, so that in dimension 2 it is simply $L^2$; this turns out to reduce greatly the technicality of the argument, and we focus for this reason on this case. But a very similar approach would work for any $d \geq 2$.

**Theorem 2.1.** Consider (NLS) set on $\mathbb{R}^2$.

(i) **(Large data local well-posedness)** It is locally well-posed in $L^2$: for any $u_0$ in $L^2$, there exists a time $T > 0$ and a unique solution $u \in L^1_{[0,T]} L^1$, which furthermore depends continuously on $u_0$. 

(ii) (Small data global well-posedness and scattering) There exists \( \epsilon > 0 \) such that: if \( \|u_0\|_{L^2} < \epsilon \), then \( T \) can be chosen = \( \infty \), so that \( u \) is a global solution. It furthermore scatters: there exists \( \psi \in L^2 \) such that
\[
\|u(t) - e^{it\Delta} \psi\|_{L^2} \to 0 \quad \text{as} \ t \to +\infty.
\]

(iii) (Existence of the wave operator for small data at \( \infty \)) There exists \( \epsilon' > 0 \) such that: if \( \|\psi\|_{L^2} < \epsilon' \), then there exists \( u_0 \in L^2 \) such that \( \|u_0\|_{L^2} < \epsilon \), and the associated global solution is such that
\[
\|u(t) - e^{it\Delta} \psi\|_{L^2} \to 0 \quad \text{as} \ t \to +\infty.
\]

Proof. Step 1: the fixed point argument. It is very similar to the case of data in \( H^s \), \( s > \frac{d}{2} \), so that we will give fewer details here. Define as above
\[
\Phi : u_0 \mapsto e^{it\Delta} u_0 + \int_0^t e^{i(t-s)\Delta} |u|^2 u(s) \, ds.
\]

We claim that it is a contraction on \( B = B_{L^4_{[0,T]}L^4}(0,R) \), where the constants \( R \) and \( T \) will be defined soon. We will actually only check that \( \Phi \) maps \( B \) to itself, since checking that it acts as a contraction follows by a simple modification of this argument. Choose \( u \in B \). Then
\[
\Phi(u) \|_{L^4_{[0,T]}L^4} \leq \|e^{it\Delta} u_0\|_{L^4_{[0,T]}L^4} + \left\| \int_0^t e^{i(t-s)\Delta} |u|^2 u(s) \, ds \right\|_{L^4_{[0,T]}L^4}
\]
\[
\leq \|e^{it\Delta} u_0\|_{L^4_{[0,T]}L^4} + C_0 \|u\|_{L^4_{[0,T]}L^4}^3 \quad \text{by Strichartz’ inequality}
\]
\[
= \|e^{it\Delta} u_0\|_{L^4_{[0,T]}L^4} + C_0 \|u\|_{L^4_{[0,T]}L^4}^3
\]

(notice that in the above \( C_0 \) is the constant in the Strichartz estimate \( L^4_{t,x} \to L^4_{t,x} \)). We can now distinguish two cases:

- If \( u_0 \) is not restricted in size, we choose \( T \) such that \( \|e^{it\Delta} u_0\|_{L^4_{[0,T]}L^4} \leq \rho \), where \( \rho \) is chosen such that \( \rho + C_0(2\rho)^3 \leq 2\rho \). Such a \( T \) exists by elementary properties of the Lebesgue integral since \( \|e^{it\Delta} u_0\|_{L^4_{t,x}} < \infty \). This choice of \( T \) ensures that \( \Phi \) is a contraction of \( B \) if \( R = 2\rho \).
- On the other hand, we can set in the above \( T = \infty \) and apply Strichartz’ inequality once again to get
\[
\Phi(u) \|_{L^4_{[0,\infty]}L^4} \leq C_1 \|u_0\|_{L^2} + C_0 \|u\|_{L^4_{[0,\infty]}L^4}^2
\]

(where \( C_1 \) is the constant in the Strichartz estimate \( L^4_{t,x} \to L^4_{t,x} \)). Let \( \epsilon \) be such that \( C_1\epsilon + C_2(2C_1\epsilon)^3 < 2C_1\epsilon \). Then we get that \( \Phi \) maps \( B \) to itself if \( \|u_0\|_{L^2} < \epsilon \) and \( R = 2C_1\epsilon \).

We can now run the fixed point argument as in the case of data in \( H^s \), \( s > \frac{d}{2} \). This gives local well posedness for arbitrary \( u_0 \), and global well-posedness if \( u_0 \) is sufficiently small in \( L^2 \).

Step 2: Scattering for small data. Assuming that \( \|u_0\|_{L^2} < \epsilon \), we obtain by the previous point a global solution \( u \) such that \( \|u\|_{L^4_{[0,\infty]}L^4} < \infty \). In order to show that it scatters, the first step is to write Duhamel’s formula as
\[
u(t) = e^{it\Delta} \left[ u_0 + \int_0^T e^{-is\Delta} |u|^2 u(s) \, ds \right].
If suffices to show that \( \psi \) has a limit in \( L^2 \). But it is a simple consequence of Strichartz’ estimate: indeed, if \( 0 < t_1 < t_2 \),
\[
\| \psi(t_2) - \psi(t_1) \|_{L^2} = \left\| \int_{t_1}^{t_2} e^{-it\Delta} |u|^2 u(s) \, ds \right\|_{L^2} \\
\leq \| u \|^2_{L^4} |t_{3/2}^{1/2} \|_{L^4} = \| u \|^4_{L^4} \longrightarrow 0 \quad \text{as } t_1, t_2 \rightarrow \infty.
\]

Step 3: Existence of wave operators. The Duhamel formulation of this problem is as follows: given \( \psi \in L^2 \), find \( u \in L^4_{[0, \infty)} L^2 \) such that
\[
u(t) = e^{it\Delta} \psi - \frac{1}{\sqrt{2}} e^{i(t-s)\Delta} |u|^2 u(s) \, ds.
\]
The existence (and uniqueness) of \( u \) can be established through a fixed point argument very similar to the above, and which will therefore be skipped. \( \square \)

3. Modified scattering for (NLS) on \( \mathbb{R} \)

3.1. Heuristics. The above approach turns out not too work in dimension 1, and there is a good reason for that: the solution does not scatter. As we will see, one needs to add a logarithmic correction to describe the asymptotic behavior of a generic (small) solution.

An indication that the fixed point argument will not work in dimension 1 can be obtained as follows: assume that \( u \) decays as fast as allowed by the linear estimate, namely \( \| u(t) \|_{L^\infty} \lesssim \frac{1}{|t|^{d/2}} \) (only large times will matter, so it is not a problem to smoothen \( \| u(t) \|_{L^\infty} \) for \( t \) small). Write Duhamel’s formula
\[
u(t) = e^{it\Delta} u_0 + \int_0^t e^{i(t-s)\Delta} |u|^2 u(s) \, ds,
\]
and attempt a direct \( L^2 \) estimate, using that \( \| u(t) \|_{L^\infty} \lesssim \frac{1}{(t)^{d/2}} \) and \( \| u(t) \|_{L^2} = \| u_0 \|_{L^2} \)
\[
\| u(t) \|_{L^2} \leq \| u_0 \|_{L^2} + \int_0^t \| u(s) \|^2_{L^2} \, ds \lesssim \| u_0 \|_{L^2} + \int_0^t \| u_0 \|_{L^2} \frac{ds}{s^{d/2}}.
\]
This gives a uniform bound in \( t \), except if \( d = 1 \).

To understand better what happens if \( d = 1 \), we will study the new dependent variable
\[
u(t) = e^{-it\Delta} u(t) \quad \text{or} \quad \tilde{f}(\xi) = e^{it|\xi|^2} \tilde{u}(t, \xi).
\]
This change of dependent variables is relevant in the weakly nonlinear regime: indeed, \( f \) is constant if \( u \) is linear, and \( u \) scatters if \( f \) converges. We will see that the modified scattering for (NLS) will be characterized by a (logarithmically) slow evolution for \( f \). A small computation shows that \( u \) solves (NLS) if and only if \( \tilde{f} \) is a solution of
\[
i \partial_t \tilde{f}(t, \xi) = \frac{1}{\sqrt{2\pi}} \int e^{it\eta} \tilde{f}(t, \xi - \eta) \bar{f}(t, \xi - \sigma - \eta) \, d\eta \, d\sigma.
\]
We want to view the above as an oscillatory integral, with phase function \( \phi(\eta, \sigma) = \eta(\xi - \sigma) \), so that
\[
\nabla_{\eta, \sigma} \phi = \left( \begin{array}{c} \sigma \\ \eta \end{array} \right) \quad \text{and} \quad \text{Hess } \phi = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right).
\]
Assuming that \( \tilde{f} \) is sufficiently smooth and decaying, we can apply the stationary phase lemma to the above left-hand side to get, as \( t \rightarrow \infty \),
\[
i \partial_t \tilde{f}(t, \xi) \sim -\frac{1}{t} |\tilde{f}(t, \xi)|^2 \tilde{f}(t, \xi).
\]
Integrating this ODE suggests the following modified scattering behavior: there exists a function $W(\xi)$ such that
\[ |\hat{f}(t, \xi)| \sim W(\xi)e^{i\log t|W(\xi)|^2} \quad \text{as } t \to \infty. \]

3.2. The modified scattering result.

**Theorem 3.1.** Consider the (NLS) set on $\mathbb{R}$. For data $u_0$ such that $\|xu_0\|_{L^2} + \|\nabla u_0\|_{L^2} = \epsilon$ is sufficiently small, there exists a global solution $u$ such that
\[ \sup_t \|u(t)\|_{H^1} \lesssim \epsilon, \quad \text{and} \quad \|u(t)\|_{L^\infty} \lesssim \frac{\epsilon}{(\sqrt{t})}. \]

Furthermore, $f = e^{-it\Delta}u$ has the following asymptotic behavior: there exists $W(\xi) \in L^\infty$ such that
\[ \left\| \hat{f}(t, \xi) - W(\xi)e^{i\log t|W(\xi)|^2} \right\|_{L^\infty} \to 0 \quad \text{as } t \to \infty. \]

**Proof.** Step 1: the bootstrap argument. Define the norm
\[ \|u\|_T = \|\sqrt{t}u\|_{L^\infty_{[0,T]}L^\infty} + \|t^{1/10}\nabla u\|_{L^\infty_{[0,T]}L^2} + \|t^{1/10}xu\|_{L^\infty_{[0,T]}L^2} + \|\hat{f}\|_{L^\infty_{[0,T]}L^\infty} + \|u\|_{L^\infty_{[0,T]}L^2}. \]

We will show that, for any $T > 0$, if $\|u\|_T < \epsilon$, then
\[ \|u\|_T \lesssim \epsilon + \|u\|_T^3. \]

By a continuous induction argument, this will show that $\|u\|_\infty < \epsilon$.

Step 2: control of the $L^2$ and $H^1$ norm of $u$. First, $\|u\|_{L^2}$ is conserved by the flow of (NLS); as for the control of $\|\nabla u\|_{L^2}$, it is easily obtained by differentiating NLS
\[ i\partial_t \partial_x u - \Delta \partial_x u = \partial_x |u|^2 u \]
and performing an $L^2$ estimate:
\[ \|\partial_x u\|_{L^2} \lesssim \|\partial_x u_0\|_{L^2} + \int_0^t \|\partial_x |u|^2 u\| \, ds \leq \int_0^t \|u\|_{L^\infty}^2 \|\partial_x u\|_{L^2} \, ds \lesssim \|u\|_T^3 \int_0^t s^{-1/10} \, ds \lesssim t^{1/10}. \]

Step 3: control of the $L^2$ norm of $xf$. Let us argue first in Fourier space, where $xf$ becomes $\partial_x \hat{f}$. Differentiating in $\xi$ the equation (3.1) satisfied by $\hat{f}$ gives
\[ i\partial_t \partial_\xi \hat{f}(t, \xi) = \frac{1}{\sqrt{2\pi}} \int e^{it\eta} \partial_\xi \hat{f}(t, \xi - \eta) \hat{f}(t, \xi - \sigma - \eta) \, d\eta \, d\sigma + \{\text{similar terms}\} \]

Coming back to physical space, an $L^2$ estimate gives then
\[ \partial_t \|xf\|_{L^2} = \left\| \int e^{-it\Delta} \left[ e^{it\Delta} x f e^{it\Delta} \hat{f} e^{it\Delta} \right] \, ds \right\|_{L^2} \lesssim \|e^{it\Delta} xf\|_{L^2} \|u\|_{L^\infty}^2 \lesssim \frac{c^2}{t} \|xf\|_{L^2}. \]

For $\epsilon$ sufficiently small, the integration of this differential inequality leads to the desired estimate.

Step 5: decay of $u$ in $L^\infty$. We use that
\[ e^{it\Delta} g = \frac{1}{(it)^{d/2}} \hat{g} e^{\frac{|x|^2}{4t}} \left( \frac{x}{t} \right) + \frac{1}{t^{1+\frac{d}{2}}} \|xg\|_{L^2}, \]
(quantitative stationary phase) which yields the result at once. \qed
Step 6: control and asymptotic behavior of $\hat{f}$. Starting from (3.1), use Plancherel’s theorem to obtain that
\[
 i\partial_t \hat{f}(t, \xi) = \frac{1}{\sqrt{2\pi}} \int \mathcal{F}^{-1}_{\eta, \sigma} \left[ e^{i\xi \sigma} \mathcal{F}^{-1}_{\eta, \sigma} \left[ \hat{f}(t, \xi - \eta)\hat{f}(t, \xi - \sigma - \eta)\hat{f}(t, \xi - \sigma) \right] \right] dy dz.
\]
Split then the right-hand side into
\[
 i\partial_t \hat{f}(t, \xi) = \frac{1}{\sqrt{2\pi}} \int \mathcal{F}^{-1}_{\eta, \sigma} \left[ \hat{f}(t, \xi - \eta)\hat{f}(t, \xi - \sigma - \eta)\hat{f}(t, \xi - \sigma) \right] d\eta d\sigma
\]
\[
 + \frac{1}{\sqrt{2\pi}} \int \left[ 1 - e^{i\xi \sigma} \right] e^{i\xi(y+z)} \int e^{-ix\xi} f(x-y)\overline{f(x)f(x-z)} \, dx \, dy \, dz
\]
which can also be written
\[
 i\partial_t \hat{f}(t, \xi) = \frac{1}{\sqrt{2\pi t}} |\hat{f}(t, \xi)|^2 \hat{f}(t, \xi) + R(t, \xi).
\]
To conclude, it suffices to show that $R$ decays sufficiently fast. But a direct estimate gives
\[
 |R(s, \xi)| \lesssim \frac{1}{t} \int \int \frac{|y|^{1/8} |z|^{1/8}}{|t|^{1/8}} \left| f(x-y)\overline{f(x)f(x-z)} \right| \, dx \, dy \, dz
\]
\[
 \lesssim \frac{1}{t^{9/8}} \left\| \langle x \rangle^{1/4} f \right\|_2^3 \lesssim \frac{1}{t^{9/8}} \|u\|_T^3.
\]
Using this integrable decay rate, it is now not hard to obtain that $\hat{f}$ is bounded, along with its asymptotic behavior.

**References**