# Long Monotone Paths in Line Arrangements

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**Abstract**

We show how to construct an arrangement of $n$ lines having a monotone path of length $\Omega(n^{2-(d/\sqrt{\log n})})$, where $d > 0$ is some constant, and thus nearly settle the long standing question on monotone path length in line arrangements.

## 1 Introduction

Let $L = \{\ell_1, \ldots, \ell_n\}$ be a set of $n$ given lines in $\mathbb{R}^2$ . A path in the arrangement $A(L)$ is a simple polygonal chain joining a set of distinct vertices in $V = \{\ell_i \cap \ell_j, i < j\}$ by segments which are on lines in $L$. The length of a path is one plus the number of vertices in $V$ at which the path turns. A path is *monotone in direction* $(a, b)$ if its sequence of vertices is monotone when projected orthogonally along the line with equation $ay - bx = 0$. An interesting open question asks for the value of $\lambda_n$, the maximal monotone path length that can occur in an arrangement of $n$ lines[1]. Clearly $\lambda_n \leq \binom{n}{2} + 1$.

A sequence of results by Sharir (see [2]), Matoušek [3], and Radoičić and Tóth [4] established that $\lambda_n = \Omega(n^{3/2})$, $\lambda_n = \Omega(n^{5/3})$, $\lambda_n = \Omega(n^{7/4})$, respectively. The last paper also showed $\lambda_n \leq 5n^2/12$. Here we give an explicit construction that proves

**Theorem 1** *For any integers $n, m > 0$ such that $m \leq \frac{1}{2}\sqrt{\log n}$, there is an arrangement of at most $2n + 2(30^m)n$ lines in which there is a monotone path of length at least $2^{-m} \cdot n^{2-1/(m+1)}$.*

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[1]Clearly, this is equivalent to the usual definition that considers paths monotone in the direction of the $x$-axis.

Notice that for $m = 3$ this gives the previously best bound $\lambda_n = \Omega(n^{7/4})$.

**Corollary 1** *The maximal monotone path length satisfies*

$$\lambda_n = \Omega(n^{2-\frac{d}{\sqrt{\log n}}})$$

*where $d > 0$ is some constant.*

**Proof:** Let $m$ be $\frac{1}{2}\sqrt{\log n}$. Then Theorem 1 gives a monotone path of length at least $n^{2-(3/(\sqrt{\log n}))}$ using at most $2n + 2(30^{\frac{1}{2}\sqrt{\log n}})n$ lines. A straightforward calculation gives the claimed bound on $\lambda_n$. ■

# 2 The Construction

## 2.1 The Basic Setup

Observe that $k$ parallel horizontal lines and $k$ parallel vertical lines give a path that is monotone in any direction $(a, b)$ with $a, b > 0$, has length $n = 2k$, and uses $n$ lines. We call this path a *"staircase"*.

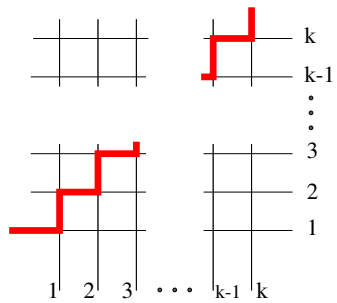


**Figure 1. A "staircase" with $n = 2k$ lines, and having length $n$.**

Given an integer $m > 0$ let $\alpha_k = 1/((k+1)(k+2))$, $0 \le k < m$, and $\alpha_m = 1/(m+1)$. Since $\alpha_0 + \ldots + \alpha_k = (k+1)/(k+2)$,

$$\alpha_0 + \cdots + \alpha_m = \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{m(m+1)} + \frac{1}{m+1} = 1. \tag{1}$$

Let $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$. In the course of the proof we shall set an $\varepsilon > 0$ that will be suitably small. For now we treat $\varepsilon$ as an infinitesimal quantity. We develop a notation to describe points in a hierarchical construction. For $\varepsilon > 0$, the vector-matrix product

$$(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^m) \begin{pmatrix} i_0 & i_1 & \ldots & i_m \\ j_0 & j_1 & \ldots & j_m \end{pmatrix}^T$$

is a point of the plane that we will denote by $\begin{bmatrix} i_0 & i_1 & \ldots & i_m \\ j_0 & j_1 & \ldots & j_m \end{bmatrix}$. The construction uses the set $S$ of points for which $i_0, j_0, \ldots, i_m, j_m$ are integers with

$$\begin{aligned} 0 \le \quad & i_0, j_0 && \le \lfloor n^{\alpha_0} \rfloor - 1 \stackrel{def}{=} D_0 \\ 0 \le \quad & i_1, j_1 && \le \lfloor n^{\alpha_1} \rfloor - 1 \stackrel{def}{=} D_1 \\ & \vdots \\ 0 \le \quad & i_m, j_m && \le \lfloor n^{\alpha_m} \rfloor - 1 \stackrel{def}{=} D_m. \end{aligned}$$

In view of (1), the number of points in $S$ is at most $(n^{\alpha_0})^2(n^{\alpha_1})^2\cdots(n^{\alpha_m})^2 = n^2$.

For $k < m$ write $B_k$ for the subset of $S$ where $i_r = j_r = 0$, $r > k$. That is,

$$B_k = \left\{ P = \begin{bmatrix} i_0 & i_1 & \dots i_{k-1} & i_k & 0 & \dots & 0 \\ j_0 & j_1 & \dots j_{k-1} & j_k & 0 & \dots & 0 \end{bmatrix} \right\}. \tag{2}$$

There are at most $(n^{\alpha_0})^2\cdots(n^{\alpha_k})^2 = n^{2-2/(k+2)}$ such points.

Another way to think about $B_k$ is as follows: let us call the square $[x, x+t) \times [y, y+t) \subseteq \mathbb{R}^2$ the "square of side $t$ at $(x, y)$". The points of $B_0$ are given by the intersection of the integer lattice $\mathbb{Z} \times \mathbb{Z} \subseteq \mathbb{R}^2$ with the square of side $\lfloor \sqrt{n} \rfloor$ at $(0, 0)$. To get the points of $B_1$, the next level of the hierarchy, replace each point $P \in B_0$ by the intersection of the square of side $\varepsilon \lfloor n^{\alpha_1} \rfloor$ at $P$ with the points $P + \varepsilon(\mathbb{Z} \times \mathbb{Z})$. For $1 \le k < m-1$ we construct $B_{k+1}$ by replacing each point $P \in B_k$ by the intersection of the square of side $\varepsilon^{k+1} \lfloor n^{\alpha_{k+1}} \rfloor$ at $P$ and the points $P + \varepsilon^{k+1}(\mathbb{Z} \times \mathbb{Z})$.

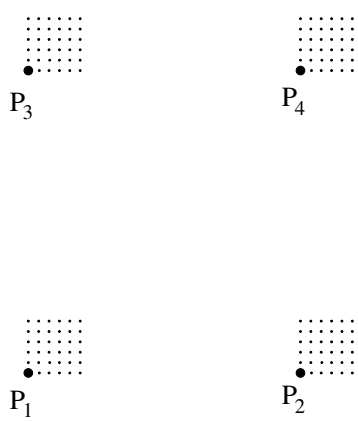


**Figure 2. Some points in $B_{k+1}$**

For example in Figure 2, $P_1, P_2, P_3, P_4$ are neighboring points in $B_k$, each the lower-left corner of a square of side $\varepsilon^{k+1} \lfloor n^{\alpha_{k+1}} \rfloor$ that contains $\lfloor n^{\alpha_{k+1}} \rfloor^2$ grid points. If in Figure 2 $P_1$ has coordinates

$$\begin{bmatrix} i_0 & \dots & i_{k-1} & I & 0 & \dots & 0 \\ j_0 & \dots & j_{k-1} & J & 0 & \dots & 0 \end{bmatrix} \in B_k,$$

then $P_2$ and $P_4$ have $i_k = I + 1$, and $P_3$ and $P_4$ have $j_k = J + 1$.

We now pick a direction in which we want our path to be monotone. Our choice is $\mathbf{w} = (\sqrt{2}, 1)$. Orthogonal to this is the direction $\mathbf{w}' = (-\mathbf{1}, \sqrt{\mathbf{2}})$. A vector is said to *point forward* if it has positive scalar product with $(\sqrt{2}, 1)$. In particular, $\mathbf{u}$ and $\mathbf{v}$ point forward. For $p, q > 0$ the vector $(-q, p)$ points forward iff $\frac{p}{q} > \sqrt{2}$, and $(q, -p)$ points forward iff $\frac{p}{q} < \sqrt{2}$. In the first case we say $p/q$ *approximates* $\sqrt{2}$ *from above*; in the second, $p/q$ *approximates* $\sqrt{2}$ *from below*.

For each point in $S$ consider the horizontal line and the vertical line that go through this point and let $L$ be the union of all these lines. The points of $S$ have at most $n$ distinct $x$ coordinates and at most $n$ distinct $y$ coordinates, so $L$ **has at most** $2n$ **lines**. As we will see later, our monotone path goes through every point in $B_{m-1}$. Whenever it reaches a point

$$\begin{bmatrix} i_0 & \dots & i_{m-1} & 0 \\ j_0 & \dots & j_{m-1} & 0 \end{bmatrix} \in B_{m-1},$$

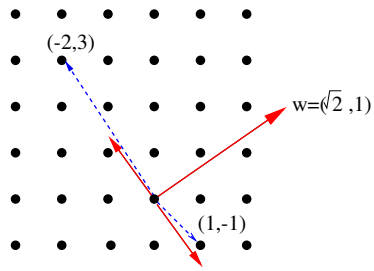


**Figure 3.** $\mathbf{w}$ is the chosen direction of monotonicity. $(-2,3)$ and $(1,-1)$ point forward, since $\frac{3}{2}$ approximates $\sqrt{2}$ from above and $1/1$ from below.

it follows the staircase to

$$\begin{bmatrix} i_0 & \dots & i_{m-1} & D_m \\ j_0 & \dots & j_{m-1} & D_m \end{bmatrix} \in S.$$

This staircase is a monotone path because $\mathbf{u}$ and $\mathbf{v}$ both point forward. We use the following coarse lower bound on the number of staircases (which is good enough for our claim):

$$\lfloor n^{\alpha_0} \rfloor^2 \dots \lfloor n^{\alpha_{m-1}} \rfloor^2 \geq 2^{-m} (n^{\alpha_0})^2 \dots (n^{\alpha_{m-1}})^2 = 2^{-m} n^{2-2/(m+1)}$$

where the first inequality holds since $n^{\alpha_k} \geq n^{2/\log n} = 4$ for all $0 \leq k \leq m-1$. On each of these staircases the path makes $2\lfloor n^{1/(m+1)} \rfloor - 1 \geq n^{1/(m+1)}$ turns, so if we could move from staircase to staircase in a monotone fashion, the resulting path would have length at least $2^{-m} n^{2-1/(m+1)}$, as required.

## 2.2 Helping Lines

In this section we complete the construction by showing how to connect the staircases using at most $2(30^m)n$ extra lines, and moving along each in a direction that points forward with respect to $\mathbf{w}$.

Suppose we project the points of $S$ orthogonally onto the line $\ell$ given by the equation $\sqrt{2}y - x = 0$. The points in $B_0$ project to distinct points on $\ell$ and are ordered by these projections. When each point in $B_0$ is replaced by a square of side $\varepsilon \lfloor n^{\alpha_1} \rfloor$, each square projects to an interval, and if $\varepsilon$ is suitably small, these intervals will be disjoint. This gives an ordering for the points in $B_1$ based first on the ordering for $B_0$, and then on the ordering for points with the same $i_0, j_0$. Inductively, the points in $B_k$ are ordered, and when we replace each by a square of side $\varepsilon^{k+1} \lfloor n^{\alpha_{k+1}} \rfloor$, each square projects to an interval; if $\varepsilon$ is suitably small, these intervals will be disjoint. This gives an ordering for the points in $B_{k+1}$, first based on the ordering of points in $B_k$, and then on the ordering of points with the same values of $i_r, j_r$, $r \leq k$.

To sum up, we obtain a lexicographic ordering of the points in $S$. We define $Q \in S$ to be the *successor* of $P \in S$ if it comes immediately after $P$ in this ordering. These observations imply that the set of staircases can be connected in a monotone manner. We also obtain,

**Lemma 1** *Let*

$$P = \begin{bmatrix} i_0 & \dots & i_{k-1} & i_k & D_{k+1} & \dots & D_m \\ j_0 & \dots & j_{k-1} & j_k & D_{k+1} & \dots & D_m \end{bmatrix}$$

*be a point in $S$ with either $i_k \neq D_k$ or $j_k \neq D_k$, and $k < m$. The successor of $P$ is a point*

$$Q = \begin{bmatrix} i_0 & \dots i_{k-1} & i'_k & 0 & \dots & 0 \\ j_0 & \dots j_{k-1} & j'_k & 0 & \dots & 0 \end{bmatrix}$$

*with either $i'_k \neq i_k$, $j'_k \neq j_k$, or both.*

The point $P$ can be seen as the top of a staircase at level $k$. Let us define this notion more precisely: for $0 \leq k < m$ define $T_k \subseteq S$ as

$$T_k = \left\{ P = \begin{bmatrix} i_0 & \dots & i_k & D_{k+1} & \dots & D_m \\ j_0 & \dots & j_k & D_{k+1} & \dots & D_m \end{bmatrix} \in S : (i_k, j_k) \neq (D_k, D_k) \right\}. \quad (3)$$

These points are the **tops of staircases at level** $k$ of the hierarchy. Let us consider Figure 4 for some fixed $k < m$. All the points in the figure except $P_2$ and $P_5$ are in $B_{k+1}$. Moreover, the points that are at the bottom left of the shaded squares are also in $B_k$. $P_2$ is in $T_k$ and $P_5$ is in $T_{k-1}$. Hence, we can write

$$\begin{aligned} P_1 &= \begin{bmatrix} i_0 & \dots & i_{k-1} & i_k & 0 & \dots & 0 \\ j_0 & \dots & j_{k-1} & j_k & 0 & \dots & 0 \end{bmatrix} \in B_k, \\ P_2 &= \begin{bmatrix} i_0 & \dots & i_{k-1} & i_k & D_{k+1} & \dots & D_m \\ j_0 & \dots & j_{k-1} & j_k & D_{k+1} & \dots & D_m \end{bmatrix} \in T_k, \\ P_4 &= \begin{bmatrix} i_0 & \dots & i_{k-1} & D_k & 0 & \dots & 0 \\ j_0 & \dots & j_{k-1} & D_k & 0 & \dots & 0 \end{bmatrix} \in B_k, \\ P_5 &= \begin{bmatrix} i_0 & \dots & i_{k-1} & D_k & \dots & D_m \\ j_0 & \dots & j_{k-1} & D_k & \dots & D_m \end{bmatrix} \in T_{k-1}. \end{aligned}$$

Finally, notice that $P_3 \in B_k$ is the successor of $P_2 \in T_k$ while the successor of $P_5 \in T_{k-1}$ in some point from $B_{k-1}$ which is not shown.

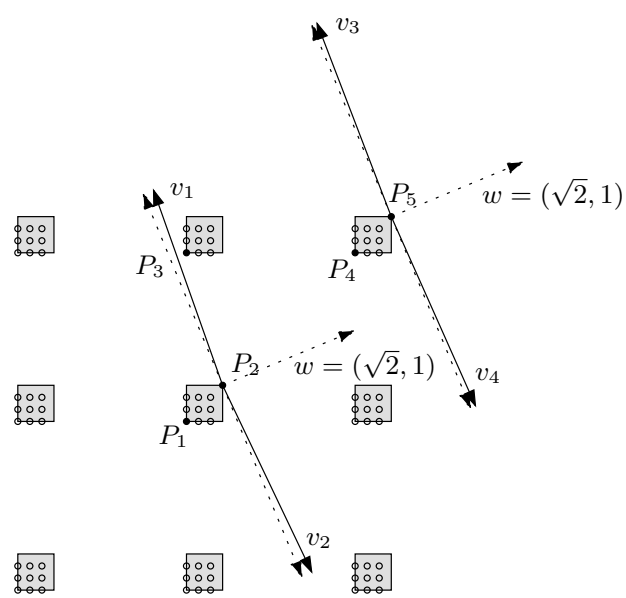


**Figure 4. Successors at level $k$.**

We now discuss the issues concerning the choice of lines used to move from a point to its successor. We call these lines *helping lines*. Let us first use Figure 4 to describe the main ideas. From points in $T_k$ we either follow a line in direction $v_1$ or a line in direction $v_2$. The actual choice is determined by the position of the successor: for example, from $P_2$ we choose the direction $v_1$ because $P_3$ is above $P_2$. In order to be able to move from a point in $T_k$ to its successor in $B_k$, the directions $v_1$ and $v_2$ must be almost orthogonal to $w$. However, as we will explain next, it is crucial that neither $v_1$ nor $v_2$ are completely orthogonal to $w$.

As we said above, we need a helping line for every point in $T_k$. But there are as many as $n^{2-2/(k+2)} \gg 2(30^m)n$ such points! The main idea is to *reuse each helping line, many times*. Hence, even though we define a helping for every point in $T_k$, the number of *distinct* helping lines is actually much lower. The way to reuse a line is the following: when we move to the successor of a point in $T_{k-1}$ we do so on a helping line that is more orthogonal to $w$ than the helping line used for points in $T_k$. For example, in Figure 4, $v_3$ and $v_4$ point less forward than $v_1$ and $v_2$. This essentially allows us to cross $v_1$ and $v_2$ on the way to the successor and then to use them again. Let us now describe the choice of the helping lines more formally.

**Definition 1** *A best upper approximator of $\sqrt{2}$ is a rational number $\frac{p}{q} > \sqrt{2}$ such that no other rational $\frac{p'}{q'}$ with $q' \leq q$ approximates $\sqrt{2}$ better from either above or below. A best lower approximator of $\sqrt{2}$ is a rational $\frac{r}{s} < \sqrt{2}$ such that no other rational $\frac{r'}{s'}$ with $s' \leq s$ approximates $\sqrt{2}$ better from either above or below.*

**Lemma 2** *For every $t \geq 1$ there is a best upper approximator $\frac{p}{q}$ and a best lower approximator $\frac{p'}{q'}$ of $\sqrt{2}$ such that $t < q, q' \leq 10t$.*

**Proof:** The convergents of the simple continued fraction for $\sqrt{2}$ are $1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots$. They can be defined by $r_i/s_i$ where $s_0 = r_0 = 1$, $r_{i+1} = r_i + 2s_i$ and $s_{i+1} = r_i + s_i$. It is easy to see that for $j \geq 0$

$$\frac{r_{2j}}{s_{2j}} < \frac{r_{2j+2}}{s_{2j+2}} < \sqrt{2} < \frac{r_{2j+3}}{s_{2j+3}} < \frac{r_{2j+1}}{s_{2j+1}}.$$

It is also well known (and easy to check) that $r_{2j}/s_{2j}$ is a best lower approximator of $\sqrt{2}$ and $r_{2j+1}/s_{2j+1}$ is a best upper approximator of $\sqrt{2}$. Since $s_{i+1} = r_i + s_i \leq 3s_i$, for every $t \geq 1$ there exists some $i \geq 0$ such that $t < s_i < s_{i+1} \leq 10t$. ■

For $0 \leq k < m$, let $\frac{p_k}{q_k}$ be a best upper approximator of $\sqrt{2}$ such that $n^{\alpha_k} < q_k \leq 10n^{\alpha_k}$ and let $\frac{p'_k}{q'_k}$ be a best lower approximator of $\sqrt{2}$ such that $n^{\alpha_k} < q'_k \leq 10n^{\alpha_k}$. We can now define for every point $P \in T_k$ two lines that are incident with $P$: one in direction $(-q_k, p_k)$ (an upper helping line, like $v_1$ and $v_3$ in Figure 4) and one in direction $(q'_k, -p'_k)$ (a lower helping line, like $v_2$ and $v_4$ in Figure 4). Formally, $L_k^{up}$ denotes the set of lines of slope $\frac{-p_k}{q_k}$ through the points of $T_k$ and $L_k^{down}$, the lines of slope $\frac{-p'_k}{q'_k}$ through these points. As mentioned above, the monotone path will actually follow only one of these lines but for simplicity we define both.

**Lemma 3** *From each point in $P \in T_k$ there is a monotone path to its successor $Q$, that either follows the line in $L_k^{up}$ through $P$ or the line in $L_k^{down}$ through $P$, and then follows a horizontal line to $Q$ (see Figure 5).*

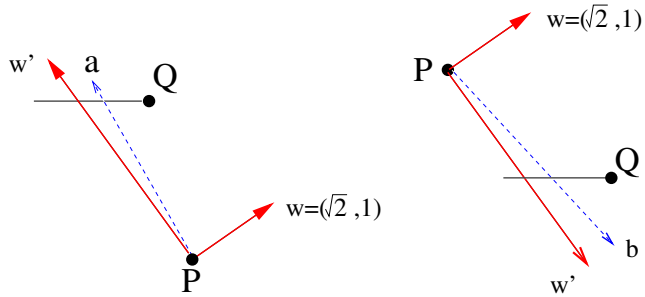


**Figure 5. Helping lines precede successors.**

**Proof:** The choice of $p_k/q_k$ and $p'_k/q'_k$ as best approximators with $q_k, q'_k > n^{\alpha_k}$ guarantee that if $\varepsilon$ is small enough, the successor of $P$ is on a line from $P$ of slope less than $-p_k/q_k$ in the upper case, or greater than $-p'_k/q'_k$ in the lower case. ■

## 2.3 Counting

To complete the proof of the theorem we count the number of distinct helping lines used in the construction.

**Lemma 4** *Let $|L_k^{up}|$ and $|L_k^{down}|$ denote the number of distinct lines in the respective sets, $k < m$. Then the total number of helping lines is*

$$\leq \sum_{k=0}^{m-1} (|L_k^{up}| + |L_k^{down}|) \leq 2(30^m)n. \tag{4}$$

**Proof:** Fix some $k < m$. We just treat $|L_k^{up}|$, the down case being completely analogous. Fix non-negative $I_r, J_r \leq D_r$, $r < k$, and consider the points in

$$A = \{P \in T_k : (i_r, j_r) = (I_r, J_r) \text{ for all } r < k\}.$$

There are at most $N = n^{2\alpha_k}$ such points, one for each possible pair $(i_k, j_k) \neq (D_k, D_k)$, and they require $N$ distinct lines in $L_k^{up}$. Let $R$ be the points in $T_k$ which have the same values of $i_r, j_r$ as do the points in $A$, for all $r < k-1$; i.e.,

$$R = \{P \in T_k : (i_r, j_r) = (I_r, J_r) \text{ for all } r < k-1\}.$$

The $N$ lines just considered (for $A$) will also meet all points in $R$ for which both $i_{k-1} = I_{k-1} - cq_k \geq 0$ and $j_{k-1} = J_{k-1} + cp_k \leq D_{k-1}$ for some integer $c$. For example, in Figure 6, the square $B$ is located $q_k$ squares to the left of $A$ and $p_k$ squares above it and therefore the $N$ lines going through $A$ are the same as the $N$ lines going through $B$. Similarly, $C$ is located $2q_k$ squares to the left and $2p_k$ squares above $A$ and also shares the same $N$ lines.

This indicates that the number of *distinct* lines in $L_k^{up}$ needed for all points in $R$ is less than the trivial bound of $n^{2\alpha_{k-1}} \cdot N$. Indeed, consider the lines of slope $-p_k/q_k$ at those points with $(i_r, j_r) = (I_r, J_r)$, $r < k-1$ and with $i_{k-1} = 0, \ldots, 2\lfloor n^{\alpha_{k-1}} \rfloor$ and

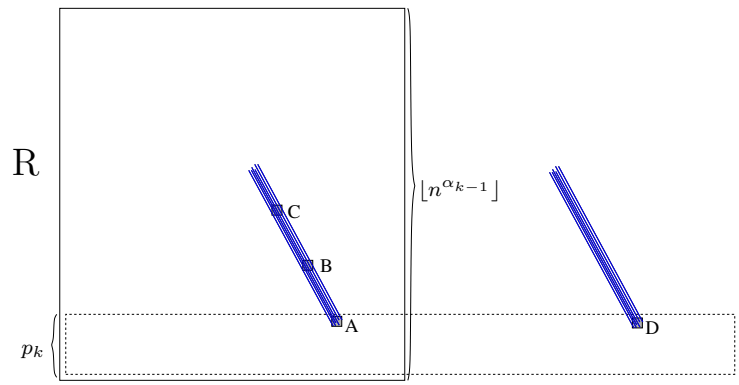


**Figure 6. Lines in $L_k^{up}$ for points with the same $i_r, j_r, r < k-1$**

$j_{k-1} = 0, \ldots, p_k - 1$ (in Fig. 6, these are the lines emanating from the squares inside the dashed rectangle, such as $A$ and $D$). Because $p_k > q_k$ (i.e., the lines form an angle of more than 45° with the $x$-axis), *all* points in $R$ will be covered. Each square uses at most $N$ lines in $L_k^{up}$ and we cover $R$ with at most $2p_k n^{\alpha_{k-1}}$ squares. Hence, the number of distinct lines in $L_k^{up}$ needed for all the points in $R \subseteq T_k$ is at most

$$2p_k \cdot n^{\alpha_{k-1}} N \leq (30n^{\alpha_k}) n^{\alpha_{k-1}} n^{2\alpha_k},$$

where we used the fact that $p_k \leq 1.5q_k$ and $q_k \leq 10n^{\alpha_k}$.

Applying this argument again to points in $T_k$ that have $(i_r, j_r) = (I_r, J_r)$ for $r < k-2$ we deduce that at most

$$(30n^{\alpha_k})^2 n^{\alpha_{k-2}} n^{\alpha_{k-1}} n^{2\alpha_k}$$

lines in $L_k^{up}$ are needed, and continuing inductively, we see that $T_k$ needs at most

$$(30n^{\alpha_k})^k n^{(\alpha_0 + \ldots + \alpha_{k-1})} n^{2\alpha_k} = (30)^k n^{(\alpha_0 + \ldots + \alpha_{k-1} + (k+2)\alpha_k)}$$

lines in $L_k^{up}$. Using the fact that $(k+2)\alpha_k = 1/(k+1)$ and $\alpha_0 + \ldots + \alpha_{k-1} = k/(k+1)$, we obtain

$$|L_k^{up}| \leq (30)^k n.$$

Applying this estimate for each $k$, we establish the bound in (4) and prove the lemma. ■

**Proof of Theorem 1.** We have constructed an arrangement of at most $2n + 2(30^m)n$ lines, at most $n$ horizontal and at most $n$ vertical lines used in the staircases, and the helping lines. Also, as mentioned above, the staircases alone comprise part of a monotone path of length at least $2^{-m} \cdot n^{2-1/(m+1)}$. ■

# 3 Remarks

1. One interesting open question concerns the quantity $\lambda_n(k)$, the length of the longest monotone path in an arrangement of $n$ lines with at most $k$ distinct

slopes. Clearly, $\lambda_n(k)$ increases with $k$ and is at most $\lambda_n$. The construction of Sharir used $k = 4$ different slopes, so $\lambda_n(4) \geq \Omega(n^{3/2})$. Matoušek's construction gives $\lambda_n(5) \geq \Omega(n^{5/3})$. For any constant $m$, our construction uses a set of $O(n)$ lines with $2m + 2$ distinct slopes. Hence, it implies $\lambda_n(2m + 2) \geq \Omega(n^{2-1/(m+1)})$. Recently, Dumitrescu [1] showed that $\lambda_n(k) \leq O(n^{2-1/F_{k-1}})$ where $F_k$ is the $k$'th Fibonacci number ($F_1 = F_2 = 1, F_3 = 2, F_4 = 3$, etc.). In particular, this provides tight upper bounds for $k = 4, 5$.

2. Matoušek [3] also studied arrangements of *pseudo-lines*; i.e., $n$ continuous functions $f_1, \ldots, f_n$ with the same intersection rules as lines. Specifically, for each $i < j$ there is a point $x_{ij}$ (a vertex) such that $(f_i(u) - f_j(u))(f_i(t) - f_j(t)) < 0$ whenever $(u - x_{ij})(t - x_{ij}) < 0$. General position would impose the condition that the vertices be distinct. A "path" moves along a function and may turn at a vertex. Matoušek constructed a pseudoline arrangement with an $x$-monotone path of length $\Omega(n^2/\log n)$. He also had conjectured that $\lambda_n = O(n^{5/3})$, i.e., that his lower bound for monotone path length in line arrangements was optimal. If this were true we would have a neat combinatorial separation of line and pseudoline arrangements based on monotone path length. The result of this paper implies that such a strong separation is impossible. A weaker separation is still possible by showing a $o(n^2/\log n)$ upper bound for $\lambda_n$ (but we don't even know how to show $\lambda_n = o(n^2)$ !).

# References


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