Instructions as before.

1. **Talagrand’s lemma:** Let \( f : \{0, 1\}^n \to [-1, 1] \) and assume \( p = \mathbb{E}[|f|] \ll 1 \). Show that \( W_1(f) = \sum_{|S|=1} \hat{f}(S)^2 \leq O(p^2 \log(1/p)) \).

2. **Generalized Chernoff bound:** Let \( p(x_1, \ldots, x_n) \) be a multilinear polynomial over the reals of degree at most \( d \), and assume that \( \mathbb{E}[p(x_1, \ldots, x_n)^2] = 1 \) where the \( x_i \) are chosen independently from \( \{-1, 1\} \) (equivalently, this says that the sum of squares of \( p \)'s coefficients is 1). Then for any large enough \( t \),
   \[
   \Pr[|p(x_1, \ldots, x_n)| \geq t] \leq \exp(-\Omega(t^2/d)),
   \]
   where the \( x_i \) are chosen as before. The case \( d = 1 \) is a version of the Chernoff bound. Hint: use Markov’s inequality and a corollary of the hypercontractive inequality that we saw in class.

3. **Logarithmic Sobolev inequality:**
   (a) Using the hypercontractive inequality, show that for any \( f : \{0, 1\}^n \to \mathbb{R} \) and \( 0 \leq \varepsilon \leq \frac{1}{2} \),
   \[
   \|T_{\sqrt{1-2\varepsilon}} f\|_2^2 \leq \|f\|_{\frac{2}{2-2\varepsilon}}^2.
   \]
   (b) Notice that we have equality at \( \varepsilon = 0 \) and use this to deduce
   \[
   \frac{d}{d\varepsilon} \|T_{\sqrt{1-2\varepsilon}} f\|_2^2 \bigg|_{\varepsilon=0} \leq \frac{d}{d\varepsilon} \|f\|_{\frac{2}{2-2\varepsilon}}^2 \bigg|_{\varepsilon=0}.
   \]
   (c) Show that the left hand side is \(-2\mathbb{I}(f)\).
   (d) Show that the right hand side is \(-\text{Ent}[f^2] \) where \( \text{Ent}[g] \) is defined for non-negative \( g \) as \( \mathbb{E}[g \ln g] - \mathbb{E}[g] \ln \mathbb{E}[g] \) (with \( 0 \ln 0 \) defined as 0). No need to be 100% rigorous.

   This establishes the **logarithmic Sobolev inequality**, saying that for any \( f : \{0, 1\}^n \to \mathbb{R} \),
   \[
   \text{Ent}[f^2] \leq 2\mathbb{I}(f).
   \]
   (e) Show that if \( f : \{0, 1\}^n \to \{-1, 1\} \) has \( p = \Pr[f = -1] \leq \frac{1}{2} \) then
   \[
   \mathbb{I}(f) \geq 2p \ln(1/p).
   \]

   For small value of \( p \), this significantly improves the Poincaré inequality \( \mathbb{I}(f) \geq 4p(1 - p) \) from Homework 1.

4. **Open question:** Fix some \( 0 < \rho < 1 \). Let \( f : \{0, 1\}^n \to [0, 1] \) and let \( \mu = \mathbb{E}[f] \). Note that \( \mathbb{E}[T_\rho f] = \mu \) as well. Clearly, Markov’s inequality implies that \( \Pr[(T_\rho f)(x) \geq \rho \mu] \leq \frac{1}{7} \).
   Can you improve this upper bound to \( o\left(\frac{1}{\rho}\right) \)? perhaps \( O(1/(t\sqrt{\log t})) \)? Intuitively, since \( T_\rho \) smooths \( f \), one would expect the peaks to shrink.