Def: $(x,y) \in \mathbb{R}\times\mathbb{R}$ are <u>ρ-correlated normal variables</u> (or, more precisely, (x,y) is distributed like the bivariate normal $N\left(\overset{\text{mean}}{\binom{0}{0}}, \overset{\text{cov. matrix}}{\begin{pmatrix}1&\rho\\ \rho&1\end{pmatrix}}\right)$) if x is chosen according to $N(0,1)$ and then $y = \rho\cdot x + \sqrt{1-\rho^2}\cdot z$ where z is an independent $N(0,1)$ r.v.

Remark: • y's marginal is $N(0,\rho^2) + N(0, 1-\rho^2) = N(0,1)$.

In fact, the definition is symmetric

• $E[xy] = E[\rho x^2] + E[\sqrt{1-\rho^2}\, x\cdot z] = \rho\cdot E[x^2] = \rho$.

Def: $g, h \in \mathbb{R}^n$ are <u>n-dimensional ρ-correlated Gaussians</u> if each coordinate (g_i, h_i) is chosen independently from a ρ-correlated 1-dim Gaussians $N\left(\binom{0}{0}, \begin{pmatrix}1&\rho\\ \rho&1\end{pmatrix}\right)$.

Observation: $E[\langle g,h\rangle] = \sum_{i=1}^{n} E[g_i h_i] = \rho\cdot n$, and in fact $\langle g,h\rangle$ is concentrated around ρn.

Therefore, the angle between g and h is concentrated on $\arccos\rho$.

Noise Stability

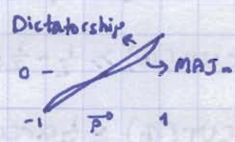
Def: For a function $f: \{0,1\}^n \to \mathbb{R}$ define its noise stability by $S_\rho(f) = \langle f, T_\rho f\rangle = \sum_S \rho^{|S|}\hat{f}(S)^2$.

For ±1 functions, this is $1 - 2\Pr_{x\sim_\rho y}[f(x)\neq f(y)]$ (i.e., $y \sim x + \mu_{\frac{1-\rho}{2}}$).

Remark: $NS_\delta(f) = \frac{1}{2}(1 - S_{1-2\delta}(f))$.

Examples: • Constant function ±1: 1.

• Dictatorship: $S_\rho = \rho$.



Prop: $S_\rho(MAJ_n) = \frac{2}{\pi}\arcsin\rho + O\left(\frac{1}{n}\right)$.

7.4.2008 **Proof sketch:**

$S_\rho(MAJ_n) = 1 - 2\cdot\Pr_{x\sim_\rho y}\left[\mathrm{sign}\left(\frac{\sum(-1)^{x_i}}{\sqrt{n}}\right) \neq \mathrm{sign}\left(\frac{\sum(-1)^{y_i}}{\sqrt{n}}\right)\right]$.

By CLT, $\frac{\sum(-1)^{x_i}}{\sqrt{n}}$ is roughly like $N(0,1)$. (In fact, we need more precise estimate known as the Berry-Esseen theorem). Using a generalization of the CLT to multidimensional distributions, we get that the joint distribution of $\left(\frac{\sum(-1)^{x_i}}{\sqrt{n}}, \frac{\sum(-1)^{y_i}}{\sqrt{n}}\right) \in \mathbb{R}\times\mathbb{R}$ is asymptotically $N\left(\binom{0}{0}, \begin{pmatrix}1&\rho\\ \rho&1\end{pmatrix}\right)$. So, our goal is to compute

$1-2\cdot\Pr[\mathrm{sign}(x)\neq\mathrm{sign}(y)]$ where $(x,y)\sim N\left(\binom{0}{0}, \begin{pmatrix}1&\rho\\ \rho&1\end{pmatrix}\right)$.

This is equal to $1-2\Pr[\mathrm{sign}(x') \neq \mathrm{sign}(\rho x' + \sqrt{1-\rho^2}\,y')]$ where x', y' are $N(0,1)$ independent. Since the angle of (x',y') is uniform on $[0,2\pi)$ we get that the above is $1 - 2\cdot\frac{\arccos\rho}{\pi} = \frac{2}{\pi}\arcsin\rho$. ∎

MAJ IS STABLEST

Thm [Mossel O'Donnell Oleszkiewicz '05]

Fix some $0<\rho<1$. Then if $f:\{0,1\}^n \to [-1,1]$ satisfies $E[f]=0$ and $\forall i.\ \mathrm{Inf}_i(f)\le\varepsilon$ (actually enough if $\mathrm{Inf}_i^{(1-1/\log 1/\varepsilon)}(f)\le\varepsilon$), then $S_\rho(f)\le \frac{2}{\pi}\arcsin\rho + O\left(\frac{\log\log 1/\varepsilon}{\log 1/\varepsilon}\right)$.

Cor ("Reverse" majority is stablest)

Fix $-1<\rho<0$. Let $f:\{0,1\}^n\to[-1,1]$ satisfy $\forall i.\ \mathrm{Inf}_i^{(1-1/\log 1/\varepsilon)}(f)<\varepsilon$, then

$S_\rho(f)\ge \frac{2}{\pi}\arcsin\rho - O\left(\frac{\log\log 1/\varepsilon}{\log 1/\varepsilon}\right)$. Remark: We no longer assume $E[f]=0$.

Proof: Let $f^{odd}(x)=\frac{f(x)-f(x+(1,\dots,1))}{2}$. Then $\widehat{f^{odd}}(S)=\hat f(S)$ if $|S|$ is odd and 0 o.w.

Then, $\mathrm{Inf}_i^{(1-1/\log 1/\varepsilon)}(f^{odd})\le \mathrm{Inf}_i^{(1-\frac{1}{\log 1/\varepsilon})}(f)<\varepsilon$. So, by theorem,

$S_{-\rho}(f^{odd})\le\frac{2}{\pi}\arcsin(-\rho)+O\left(\frac{\log\log 1/\varepsilon}{\log 1/\varepsilon}\right)=-\frac{2}{\pi}\arcsin\rho+O\left(\frac{\log\log 1/\varepsilon}{\log 1/\varepsilon}\right)$.

So, $S_\rho(f)=\sum\rho^{|S|}\hat f(S)^2\ge\sum_{S\ odd}\rho^{|S|}\cdot\hat f(S)^2=-\sum_{S\ odd}(-\rho)^{|S|}\cdot\hat f(S)^2=-S_{-\rho}(f^{odd})$. ∎

Cor: If $f:\{0,1\}^n\to\{-1,1\}$ has no coordinate with large influence, then the probability of "outcome of a 3-candidates vote is rational" is $\frac34-\frac34 S_{-1/3}(f)\le$

$\le\frac34-\frac34\cdot\frac{2}{\pi}\arcsin(-1/3)\approx 0.91226\dots$

Theorem [Khot Kindler Mossel O'Donnell '04]

For all $\eta>0$ and $-1<\rho<0$, it is UG-hard to distinguish given a graph G between

- $\mathrm{MAXCUT}(G)\ge\frac12-\frac12\rho-\eta$, and
- $\mathrm{MAXCUT}(G)\le\frac{1}{\pi}\arccos\rho+\eta$.

In particular, this gives UG-hardness of approximating MAX-CUT to better than $\frac{\frac{1}{\pi}\arccos\rho}{\frac12-\frac12\rho}$, which for $\rho=-0.689$ gives hardness of $0.878567\dots$ matching exactly the Goemans-Williamson algorithm.

Proof: We construct a dictatorship vs. quasirandomness test of "MAXCUT type":

StableTest$_\rho$ (for $-1<\rho<0$)

1. Pick $x\in\{0,1\}^n$ uniformly, and y obtained from x by flipping each bit w.p. $\frac{1-\rho}{2}$.
2. Accept iff $f(x)\ne f(y)$.

Note that $\mathrm{Prob}[\text{test accepts } f]=\frac12-\frac12\langle f,T_\rho f\rangle=\frac12-\frac12 S_\rho(f)$.

Hence, dictators are accepted w.p. $\frac12-\frac12\rho$ and (ε,δ)-quasirandom then

$\Pr[\text{test accepts } f]\le\frac12-\frac12\cdot\frac{2}{\pi}\arcsin\rho+\alpha(\varepsilon,\delta)=\frac{1}{\pi}\arccos\rho+\alpha(\delta,\varepsilon)$, where $\alpha(\varepsilon,\delta)\to0$ as $\varepsilon,\delta\to0$. ∎

Proof sketch of MAJ is Stablest [MOO]

Invariance principle: consider the (multi)linear polynomial $Q_1(x_1,\ldots,x_n) = \frac{x_1+x_2+\ldots+x_n}{\sqrt{n}}$.

If we replace each x_i with an independent $N(0,1)$, we obtain that Q is also $N(0,1)$. If each x_i is ± 1 independently and uniformly, then Q is approximately $N(0,1)$ by the CLT. What about $Q_2(x_1,\ldots,x_n) = x_3$? Here we get a different distribution if we put $N(0,1)$ or ± 1. The "reason" is that x_3 has large influence.

What about higher degree? Take, e.g., $Q_3 = \frac{\sum_{i \neq j} x_i x_j}{n} \approx \left(\frac{\sum x_i}{\sqrt{n}}\right)^2 \leftarrow \chi^2$.

Here we get roughly the same distribution for both $\xrightarrow{N(0,1)}$ $N(0,1)$ and ± 1.

Thm: (invariance principle, roughly)

If $Q(x_1,\ldots,x_n) = \sum_S \alpha_S \cdot \prod_{i \in S} x_i$ is a multilinear polynomial (of low degree, i.e., $\alpha_S = 0$ $\forall S$ of size $> d$, with $\sum \alpha_S^2 = 1$, and $Inf_i(Q) = \sum_{S \ni i} \alpha_S^2$ are all small), then pluggin ± 1 values or $N(0,1)$ values results in similar distributions.

Q: Why multilinear? Otherwise false, e.g., $\sum x_i^4/n$.

Proof: Similar to the Berry-Esseen theorem and uses hypercontractive ineq.

Thm: (MAJ is STABLEST)

Fix $0<\rho<1$. Let $f:\{-1,1\}^n \to [-1,1]$ have $E[f]=0$ and no large influences. Then,

$S_\rho(f) = \langle f, T_\rho f \rangle = \sum \rho^{|S|} \cdot \hat{f}(S)^2 \lesssim \frac{2}{\pi} \arcsin \rho$.

Proof sketch: We can write $f(x_1,\ldots,x_n) = \sum_S \hat{f}(S) \cdot \prod_{i \in S} x_i$.

Define $\tilde{f}:\mathbb{R}^n \to \mathbb{R}$, by $\underset{\mathbb{R}^n}{\tilde{f}(g_1,\ldots,g_n)} = \sum_S \hat{f}(S) \cdot \prod_{i \in S} g_i$.

By the invariance principle, we know that when $g_1,\ldots,g_n$ are chosen from $N(0,1)$ independently then $\tilde{f}(g_1,\ldots,g_n)$ is essentially in $[-1,1]$. Moreover, $E[\tilde{f}(g_1,\ldots,g_n)] = \hat{f}(\phi) = 0$.

Def: For $\tilde{f}:\mathbb{R}^n \to \mathbb{R}$ define $S_\rho(\tilde{f}) = \underset{g \sim_\rho h}{E}[\tilde{f}(g) \cdot \tilde{f}(h)]$, where $g,h \in \mathbb{R}^n$ are ρ-correlated Gaussians. Then, $S_\rho(\tilde{f}) = \sum_S \sum_T \hat{f}(S)\hat{f}(T) \cdot \underset{g \sim_\rho h}{E}[\prod_{i \in S} g_i \cdot \prod_{i \in T} h_i] = \sum_S \rho^{|S|} \hat{f}(S)^2 = S_\rho(f)$.

Thm: [Borell'85] Among all functions $\tilde{f}:\mathbb{R}^n \to [-1,1]$ with $\underset{g}{E}[\tilde{f}] = 0$, the stability $S_\rho(f)$ is maximized by the half space functions $x \mapsto \mathrm{sign}(\langle x,u \rangle)$ for some $u \in \mathbb{R}^n$.

Remark: Because g and h are vectors of length $\sqrt{n}$ and of angle $\sim \arccos \rho$, the stability of the halfspace function is $1 - \frac{2}{\pi}\arccos \rho = \frac{2}{\pi} \arcsin \rho$.

Ex: Prove this formally.

Proof of Borell: Symmetrization.