1. **Poincaré inequality:** Let \( f : \{0,1\}^n \to \{-1,1\} \). Show that
\[
\text{Var}(f) = 4 \Pr[f(x) = 1] \Pr[f(x) = -1],
\]
where \( \text{Var}(f) \) is as defined in Homework 1. Define the total influence of \( f \) to be
\[
\mathcal{I}(f) = \sum_{\mathcal{X}} \mathbb{E}_{x}[\#\{i \in [n] : f(x) \neq f(x \oplus e_i)\}].
\]
Show that
\[
\mathcal{I}(f) \geq \text{Var}(f).
\]
Interpret this result as a statement about the Hamming graph \( G = (V = \{0,1\}^n, E) \) with edges connecting any two vertices that differ in exactly one coordinate.

2. **Dictatorship test with perfect completeness:** Prove that there is no 3-query dictatorship test that only looks at products \( f(x)f(y)f(z) \) and has perfect completeness (i.e., accepts dictatorships with probability 1) and reject functions \( \epsilon \)-far from dictatorships with some nonzero probability.

3. **Testing resiliency:** Call a function \( f : \{0,1\}^n \to \{-1,1\} \) 1-resilient if \( \hat{f}(S) = 0 \) for all \( |S| \leq 1 \).
   
   (a) Give a combinatorial definition of 1-resiliency.
   
   (b) Give a \( \text{poly}(1/\epsilon) \)-query test that accepts 1-resilient functions with probability at least 2/3, and rejects functions \( \epsilon \)-far from 1-resilient functions with probability at least 2/3. (Notice that this is not quite the same thing as a tester for the property of being 1-resilient.) Do this by amplifying a 2-query test.

4. **Enflo’s distortion lower bound on embedding \( \ell_1 \) into \( \ell_2 \) [2]:** The hypercube \( \{0,1\}^n \) with the Hamming distance is an \( \ell_1 \) metric space (because we can map \( \{0,1\}^n \) to \( \mathbb{R}^n \) in such a way that the Hamming distance is mapped exactly to the \( \ell_1 \) distance). We say that the hypercube can be embedded into \( \ell_2 \) with distortion \( D \) if there exists a mapping \( F : \{0,1\}^n \to \mathbb{R}^m \) for some \( m \) such that for all \( x, y \in \{0,1\}^n \),
\[
\Delta(x,y) \leq \|F(x) - F(y)\|_2 \leq D \cdot \Delta(x,y).
\]
This means that the \( \ell_2 \) distance between \( F(x) \) and \( F(y) \) is the same as the Hamming distance between \( x \) and \( y \) up to a factor of \( D \). It is easy to see that there exists an embedding with distortion \( \sqrt{n} \) (think why). Here we show that this is optimal, and hence this gives an example of an \( \ell_1 \) metric with \( N \) points whose distortion when embedded into \( \ell_2 \) is \( \sqrt{\log N} \). It was recently shown that any \( \ell_1 \) metric with \( N \) points can be embedded into \( \ell_2 \) with distortion \( O(\sqrt{\log N \log \log N}) \) [1] (see also [3]).
   
   (a) Show that for any \( f : \{0,1\}^n \to \mathbb{R} \),
\[
\mathbb{E}_{x}[\sum_i (f(x) - f(x \oplus (1, \ldots, 1)))^2] = 4\|f^{\text{odd}}\|_2^2 \leq 4 \text{Var}[f] \leq 4\mathcal{I}(f) = \sum_{i=1}^{n} \mathbb{E}_{x}[\sum_i (f(x) - f(x \oplus e_i))^2].
\]
(b) Deduce that for any \( F : \{0,1\}^n \rightarrow \mathbb{R}^m \),
\[
\text{Exp}_x[\|F(x) - F(x \oplus (1, \ldots, 1))\|^2_2] \leq \sum_{i=1}^n \text{Exp}_x[\|F(x) - F(x \oplus e_i)\|^2_2].
\]

(c) Use this to conclude that the distortion of any \( F : \{0,1\}^n \rightarrow \mathbb{R}^m \) must be at least \( \sqrt{n} \).

5. **Compactly storing a function:** Let \( f : \{0,1\}^n \rightarrow \mathbb{R} \) be some function, and assume we want to store some information about \( f \) that would allow us to compute \( f(x) \) for any given \( x \in \{0,1\}^n \) to within some accuracy, say, \( \pm 0.01 \). Without any further restrictions on \( f \) we would have to store \( \Omega(2^n) \) bits of information (even for a Boolean \( f \)).

   (a) Show how to reduce the storage to \( \text{poly}(n) \) for functions \( f \) with the property that for all \( S, \hat{f}(S) \geq 0 \) (such functions are called **positive definite**) and moreover, \( \sum_S \hat{f}(S) = 1 \).

   Notice that if \( f = g \circ g \) for some Boolean \( g \) then it satisfies these two requirements.

   (b) Extend this to functions \( f \) satisfying \( \sum_S |\hat{f}(S)| \leq \text{poly}(n) \).

References

