2.1 Limits of sequences

2.1.5: Pick any $\epsilon > 0$. Since $(b_n)$ converges to 0, there exists $N \in \mathbb{N}$ such that: \( \forall n \geq N, \ b_n = |b_n - 0| < \epsilon' = \epsilon/C. \) Then \( \forall n \geq N, \ |x_n - a| \leq Cb_n < C \cdot (\epsilon/C) = \epsilon, \)
in other words, \((x_n)\) converges to \(a\).

2.1.7: a) Let us denote by \(l\) the common limit of \((x_n)\) and \((y_n)\), and pick any $\epsilon > 0$. Since \((x_n)\) converges to \(l\), it is possible to find \(N_1\) such that: \( \forall n \geq N_1, \ |x_n - l| < \epsilon' = \epsilon/2. \) Similarly, since \((y_n)\) converges to \(l\), there exists \(N_2\) for which: \( \forall n \geq N_2, \ |y_n - l| < \epsilon' = \epsilon/2. \) Then: \( \forall n \geq N = \max(N_1, N_2), \)
\[ |x_n - y_n| = |(x_n - l) + (l - y_n)| \leq |x_n - l| + |l - y_n| < (\epsilon/2) + (\epsilon/2) = \epsilon \]
by using the triangular inequality. Hence, \(x_n - y_n\) converges to 0.

For b), it is enough to prove that \(x_n = n\) is not bounded (prove it!), since a convergent sequence is always bounded (property proved in class). Or one can argue by contradiction: if \(x_n = n\) was converging to some real number \(l\), then we would have \(x_n \leq l + 1\) for \(n\) large enough, so that \(n \leq l + 1\), which is a contradiction for \(n\) large.

For c), one can take \(x_n = n\) and \(y_n = n + 1/n\).

2.1.8: This is a much simpler result that the corollary of Bolzano-Weierstrass seen in class, for which we only assumed that all converging subsequences (and not all subsequences) converged to \(a\). If \((x_n)\) converges to \(a\), then we have proved in class that every subsequence \((x_{\phi(n)})\) also converges to \(a\). Conversely, if you know that every subsequence converges to \(a\), then \((x_n)\) being in particular a subsequence of itself (obtained with \(\phi(n) = n\)), it converges to \(a\).

2.2 Limit theorems

2.2.0: a) False: take \(x_n = n + 1\) and \(y_n = -n\) (actually, any real number, as well as +\(\infty\) and -\(\infty\) can be obtained as limits!).

b) This is a consequence of (iv) in the "algebraic rules for computing limits": \(y_n/x_n \rightarrow 0\) if \(y_n\) is bounded (here \(y_n = 1\) and so is bounded), and \(x_n \rightarrow -\infty\) (assume that for each \(n, x_n \neq 0\), which has to be true for \(n\) large enough at least).

c) False, \(1/x_n \rightarrow -\infty\) is possible too (take \(x_n = -1/n\)), or it can oscillate (take \(x_n = 1/((-1)^n)\)). However, the statement is true if \(x_n\) tends to 0 "from above" (staying positive).

d) True: Pick any $\epsilon > 0, M$ such that $(1/2)^M < \epsilon$, and then (using that $x_n \rightarrow +\infty$) $N$ such that for all $n \geq N, x_n \geq M$, so that
\[ (1/2)^{x_n} \leq (1/2)^M < \epsilon. \]

2.2.3: a) Divide both the numerator and the denominator by the corresponding leading powers of \(n, n^2\) for both:
\[ \frac{2 + 3n - 4n^2}{1 - 2n + 3n^2} = \frac{2/n^2 + 3/n - 4}{1/n^2 - 2/n + 3}. \]
Then \(2/n^2 + 3/n - 4 \to 0 + 0 - 4 = -4\) and \(1/n^2 - 2/n + 3 \to 0 - 0 + 3 = 3\) (using rules (i) and (ii)), so that the quotient tends to \((-4)/3 = -4/3\) (rule (iv)).

b) Here, one has to divide the numerator and the denominator by \(n^3\); we get \(1/2\) as a limit.

c) Write \(\sqrt{3n+2} - \sqrt{n} \geq \sqrt{3n} - \sqrt{n} = (\sqrt{3} - 1)/\sqrt{n} \to +\infty\), and the squeeze theorem implies that \(\sqrt{3n+2} - \sqrt{n} \to +\infty\) (you could also multiply and divide by \(\sqrt{3n+2} + \sqrt{n}\)).

d) We have

\[
\frac{\sqrt{4n+1} - \sqrt{n-1}}{\sqrt{9n+1} - \sqrt{n+2}} = \frac{\sqrt{4} - \sqrt{1}}{\sqrt{9} - \sqrt{1}} = \frac{1}{2},
\]

by dividing above and below by \(\sqrt{n}\).

### 2.4 Cauchy sequences

- **2.4.2**: Since \((x_n)\) is Cauchy, there exists \(N\) such that: \(\forall m, n \geq N, |x_m - x_n| < 1/2\). Hence, \((x_m - x_n)\) is an integer in \((-1/2, 1/2)\), it has to be 0! So \(x_m - x_n = 0\): \(\forall m, n \geq N, x_m = x_n\). This implies that \(\forall n \geq N, x_n = x_N\), so \(x_n = a\) if we call \(a = x_N \in \mathbb{Z}\).

- **2.4.4**: The assumption exactly means that the sequence of partial sums

\[
s_n = \sum_{k=1}^{n} x_k
\]

is a Cauchy sequence: indeed, for \(m > n\), \(s_m - s_n = \sum_{k=n+1}^{m} x_k\). So \((s_n)\) converges to a finite limit, from the property that Cauchy sequences in \(\mathbb{R}\) are converging.

- **2.4.5**: Check the criterion of 2.4.4 with \(x_k = (-1)^k/k\): for \(m > n \geq N\),

\[
|\sum_{k=n}^{m} x_k| \leq 1/n \leq 1/N,
\]

so it is sufficient to take \(N\) large enough so that \(1/N < \epsilon\). To prove the inequality, check that if \(n\) is even, the sum oscillates between 0 and \(1/n\), and if \(n\) is odd, it oscillates between \(-1/n\) and 0 (group terms in the sum two by two).

- **2.4.6**: We proved the same result in class in the particular case \(a = 1/2\): \((x_n)\) is automatically a Cauchy sequence, so it has to converge (see also example 2.30 in the textbook).