Numerical Methods I: Interpolation (cont'ed)

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Interpolation

Things you should know

- Lagrange vs. Hermite interpolation
- Conditioning of interpolation
- Uniform vs. non-uniform points, Lebesgue constant
- Polynomial bases: Lagrange, Newton, Monomial

The Newton basis $\omega_0, \dots, \omega_n$ is given by

$$\omega_i(t) := \prod_{j=0}^{i-1} (t - t_j) \in \boldsymbol{P}_i.$$

The leading coefficient a_n of the interpolation polynomial of f

$$P(f | t_0, \dots, t_n) = a_n x^n + \dots$$

is called the n*-th divided difference*, $[t_0, \dots, t_n]f := a_n$.

Classical polynomial interpolation

Newton polynomial basis

Theorem: For $f \in C^n$, the interpolation polynomial $P(f|t_0, \ldots, t_n)$ is given by

$$P(t) = \sum_{i=0}^{n} [t_0, \ldots, t_i]f \, \omega_i(t).$$

If $f \in C^{n+1}$, then

$$f(t) = P(t) + [t_0, \ldots, t_n, t]f \; \omega_{n+1}(t).$$

This property allows to estimate the interpolation error.

The divided differences $[t_0, \ldots, t_n]f$ satisfy the following properties:

- $[t_0, \ldots, t_n]P = 0$ for all $P \in \boldsymbol{P}_{n-1}$.

- If $t_0 = \ldots = t_n$:

$$[t_0, \ldots, t_n]f = \frac{f^{(n)}(t_0)}{n!}$$

nodes.

- The following recurrence relation holds for $t_i \neq t_j$ (nodes with a hat are removed):

$$[t_0, \ldots, t_n]f = \frac{\big([t_0, \ldots, \hat{t_i}, \ldots, t_n]f - [t_0, \ldots, \hat{t_j}, \ldots, t_n]f\big)}{t_j - t_i}$$

- If $f \in C^n$ $[t_0, \ldots, t_n]f = \frac{1}{n!} f^{(n)}(\tau)$ with an $a \leq \tau \leq b$, and the divided differences depend continuously on the nodes.

Let us use divided differences to compute the coefficients for the Newton basis for the cubic interpolation polynomial p that satisfies $p(0) = 1$, $p(0.5) = 2$, $p(1) = 0$, $p(2) = 3$.

t_i				
0	$[t_0]f = 1$			
0.5	$[t_1]f = 2$	$[t_0 t_1]f = \frac{[t_1]f - [t_0]f}{t_1 - t_0} = 2$		
1	$[t_2]f = 0$	$[t_1 t_2]f = \frac{[t_2]f - [t_1]f}{t_2 - t_1} = -4$	$[t_0 t_1 t_2]f = -6$	
2	$[t_3]f = 3$	$[t_2 t_3]f = \frac{[t_3]f - [t_2]f}{t_3 - t_2} = 3$	$[t_1 t_2 t_3]f = \frac{14}{3}$	$\frac{16}{3}$

Thus, the interpolating polynomial is

$$p(t) = 1 + 2t + (-6)t(t - 0.5) + \frac{16}{3}t(t - 0.5)(t - 1).$$

Let us now use divided differences to compute the coefficients for the Newton basis for the cubic interpolation polynomial p that satisfies $p(0) = 1$, $p'(0) = 2$, $p''(0) = 1$, $p(1) = 3$.

t_i				
0	$[t_0]f = 1$			
0	$[t_0]f = 1$	$[t_0 t_1]f = p'(0) = 2$		
0	$[t_0]f = 1$	$[t_1 t_2]f = p'(0) = 2$	$[t_0 t_1 t_2]f = \frac{p''(0)}{2!} = \frac{1}{2}$	
1	$[t_3]f = 3$	$[t_2 t_3]f = \frac{[t_3]f - [t_0]f}{t_3 - t_0} = 2$	0	$-\frac{1}{2}$

Thus, the interpolating polynomial is

$$p(t) = 1 + 2t + \frac{1}{2}t^2 + (-\frac{1}{2})t^3$$

Approximation error

If $f \in C^{(n+1)}$, then

$$f(t) - P(f|t_0, \ldots, t_n)(t) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\omega_{n+1}(t) \qquad \left(= \prod_{j=0}^{n}(t-t_j)\right)$$

for an appropriate $\tau = \tau(t)$, $a < \tau < b$.

In particular, the error depends on the choice of the nodes.

$$f(t) - P(f|t_0, \ldots, t_n)(t) = [t_0, t_1, \ldots, t_n, t]f \; \omega_{n+1}(t) = \frac{f^{(n+1)}(\tau)}{(n+1)!}\omega_{n+1}(t), \quad \tau \in (a,b)$$

For Taylor interpolation, i.e., $t_0 = \ldots = t_n$, this results in:

$$f(t) - P(f|t_0, \ldots, t_n)(t) = \frac{f^{(n+1)}(\tau)}{(n+1)!}(t - t_0)^{n+1}$$

Classical polynomial interpolation

Approximation error

Consider functions

$$\{f \in C^{n+1}([a,b]) : \sup_{\tau \in [a,b]} |f^{n+1}(\tau)| \leq M(n+1)!\}$$

for some $M > 0$, then the approximation error depends on $\omega_n(t)$, and thus on $t_0, \ldots, t_n$.

Thus, one can try to minimize

$$\max_{a \leq t \leq b} |\omega_{n+1}(t)|,$$

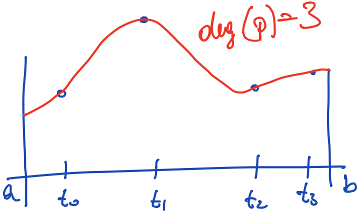
which is achieved by choosing the nodes as the roots of the Chebyshev polynomial of order $(n+1)$.

Summary on pointwise convergence:

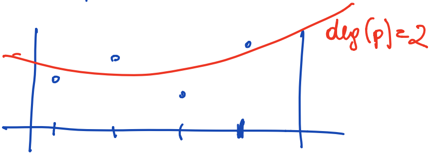
- If an interpolating polynomial is close/converges to the original function depends on the regularity of the function and the choice of interpolation nodes
- For a good choice of interpolation nodes, fast convergence can be obtained for almost all functions

Classical polynomial interpolation

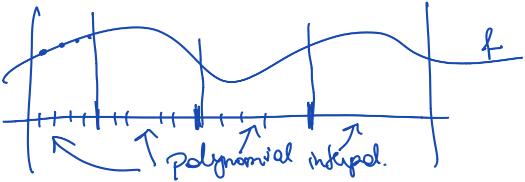
Interpolation/Least square approximation/Splines



- Polynomial interpolation



- Least squares with polynomials
- Splines (i.e., piecewise polynomial interpolation):



Splines

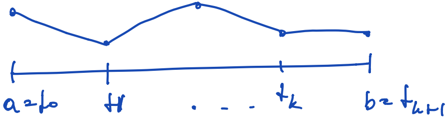
Assume $(l+2)$ pairwise disjoint nodes:

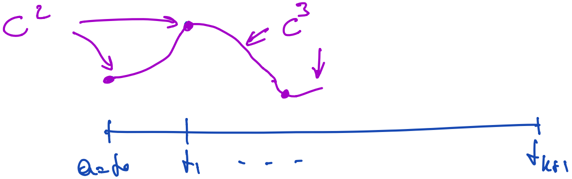
$$a = t_0 < t_1 < \ldots < t_{l+1} = b.$$

A spline of degree $k-1$ (order k) is a function in C^{k-2} which on each interval $[t_i, t_{i+1}]$ coincides with a polynomial in $\boldsymbol{P}_{k-1}$.

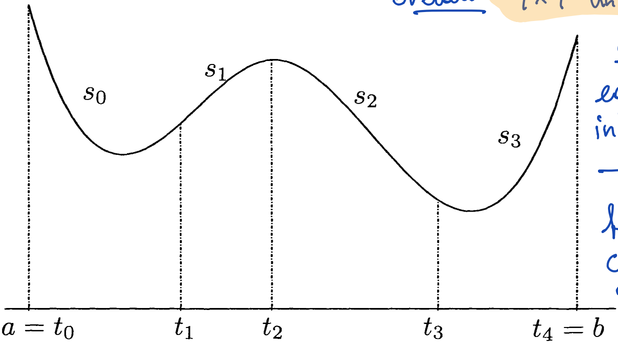
Most important examples:

- linear splines, $k = 2$
- cubic splines, $k = 4$





Cubic splines look smooth:



s_i polynomials of degree 3 (4 degrees of freedom)

overall: 4×4 unknowns

constraints:

each s_i is required to interpolate at 2 points → 4×2 conditions

first & second derivative coincide at intersection points → 3×2 = 6 cond

($s_0'(t_1) = s_1'(t_1)$, $s_0''(t_1) = s_1''(t_1)$, ...)

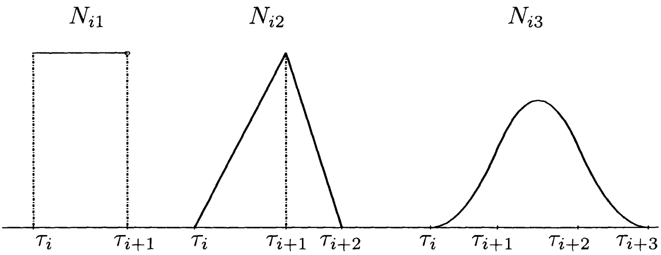
→ 2 free degrees of freedom, e.g. $s_0''(a) = s_3''(b) = 0$

Splines

B-splines

B-splines are a basis in the spline space that:

- has local support
- satisfies a 3-term recursion
- non-negative



- Coefficients for interpolation with the B-spline basis can be computed efficiently using the De Boor algorithm.
- Splines are essential in Computer Aided Design (CAD).
- Also important in CAD: Bezier curves (these do not interpolate points and have useful geometrical properties).

Instead of polynomials, use $\sin(jt), \cos(jt)$ for different $j \in \mathbb{N}$.

For $N \geq 1$, we define the set of complex trigonometric polynomials of degree $\leq N-1$ as

$$\boldsymbol{T}_{N-1} := \left\{ \sum_{j=0}^{N-1} c_j e^{ijt}, c_j \in \mathbb{C} \right\},$$

where $i = \sqrt{-1}$.
Complex interpolation problem: Given pairwise distinct nodes $t_0, \ldots, t_{N-1} \in [0, 2\pi)$ and corresponding nodal values $f_0, \ldots, f_{N-1} \in \mathbb{C}$, find a trigonometric polynomial $p \in \boldsymbol{T}_{N-1}$ such that $p(t_i) = f_i$, for $i = 0, \ldots, N-1$.

Trigonometric Interpolation

- There exists exactly one $p \in \boldsymbol{T}_{N-1}$, which solves this interpolation problem.
- Choose the equidistant nodes $t_k := \frac{2\pi k}{N}$ for $k = 0, \ldots, N-1$. Then, the trigonometric polynomial that satisfies $p(t_i) = f_i$ for $i = 0, \ldots, N-1$ has the coefficients

$$c_j = \frac{1}{N} \sum_{k=0}^{N-1} e^{-\frac{2\pi i j k}{N}} f_k.$$

- For equidistant nodes, the linear map from $\mathbb{C}^N \to \mathbb{C}^N$ defined by $(f_0, \ldots, f_{N-1}) \mapsto (c_0, \ldots, c_{N-1})$ is called the discrete Fourier transformation (DFT).

inverse DFT:

$$(c_0, \ldots, c_{N-1}) \longrightarrow \sum_{j=0}^{N-1} c_j e^{ijt}$$

evaluate at $t_0, t_1, \ldots$ $\longrightarrow (f_0, \ldots, f_{N-1})$

Discrete Fourier transform

The interpolation problem $(f_0, \ldots, f_{N-1}) \mapsto (c_0, \ldots, c_{N-1})$ and its inverse require the multiplication or solution with a dense $n \times n$ system, i.e., at least $O(n^2)$ flops.

However, the special structure of the system matrix allows performing those operations using a much faster algorith, the Fast Fournier Transform (FFT).

Trigonometric Interpolation

The Fast Fourier Transform (FFT) is a (very famous!) algorithm that computes the DFT and its inverse in $O(n)$ flops.

- Note that uniform nodes are used (and even required for the FFT).
- Tensor products on square domains can be used for two dimensional approximations, i.e., $p(x)p(y)$.
- Can be used to approximate and solve differential equations (see Numerical Methods II).