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On the suppression of thermal conduction by magnetic fields

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ABSTRACT

It is commonly assumed that 'tangled' magnetic fields can sufficiently suppress thermal transport to allow one to ignore thermal conduction in various astrophysical systems. In order to assess the validity of this assumption, we study the physics of thermal conduction in the presence of magnetic fields. Following previous work, we relate the effective suppression of thermal conductivity to the average length of magnetic field lines, and use geometric arguments to constrain the length of field lines by a normalized rms magnetic field strength. Numerical simulations are then used to examine the effects of fluid turbulence on field line stretching. Finally, as an illustration, our results are applied to cooling flow models, demonstrating that the magnetic suppression of thermal conductivity is not sufficiently effective in typical cluster haloes.

Key words: conduction – magnetic fields – MHD – turbulence – cooling flows.

1 INTRODUCTION

The suppression of thermal transport by magnetic fields has been widely invoked to justify the neglect of thermal conduction in various astrophysical models. In the presence of magnetic fields, the electron heat flux along the mean magnetic field is reduced because of random magnetic field fluctuations. The stochastic effects of fluctuations in magnetic field lines were first studied in the context of the propagation of cosmic rays in the solar wind (see Jokipii 1966; Jokipii & Parker 1969; Jokipii 1971, and references therein); these studies involved a statistical analysis of the scattering of cosmic rays, and related a diffusion tensor to the power spectrum of the magnetic field. Hollweg & Jokipii (1972) pointed out the relevance of these studies to the problem of heat conduction in the solar wind. Typically, the Spitzer value of the perpendicular thermal conductivity is many orders of magnitude smaller than the corresponding parallel conductivity, because the gyroradius of electrons is much shorter than their mean free path. Under these conditions, the dominant effect of magnetic fields on thermal conduction is to decrease the path-length over which heat-flux-carrying electrons travel (cf. Rosner & Tucker 1989; Tribble 1989; Soker & Sarazin 1990).

These considerations are relevant to astrophysical plasmas in general, because magnetic fields are ubiquitous. In the context of galactic cluster haloes, thermal conduction in the presence of magnetic fields may be relevant to both the steady-state models of the gas (the so-called 'cooling flow' models; for a recent review, see Fabian 1994), as well as to the dynamics of this gas (via the effects of magnetohydrodynamic thermal instabilities; see, for instance, Bodo et al. 1991). The characteristic conduction time for thermal transport proceeding at the unreduced Spitzer rate is about 10^8 yr (cf. Sarazin 1988; Chun & Rosner 1993), which is much less than the halo lifetime (on the order of 10^{10} yr). Since the conduction time-scale depends linearly on the value of the thermal conductivity, the effective thermal conductivity must be suppressed by at least three orders of magnitude for the observed temperature gradients to survive. It is usually assumed that 'tangled' magnetic fields can effectively achieve this suppression of thermal transport (e.g. Binney & Cowie 1981; Stewart et al. 1984; Sarazin 1988; Fabian 1994).

The goal of our work is to study the evolution of magnetic fields in turbulent fluids in order to determine whether magnetic field lines are stretched sufficiently to suppress thermal conduction, and, if this is the case, to what extent. We will show that within our idealized model of turbulent magnetohydrodynamics, we can quantify and bound the effective increase in thermal electron path-length by applying existing theoretical results from fluid turbulence and geometric measure theory.

The structure of this paper is as follows. In Section 2, we discuss the effects of magnetic fields on thermal conduction in typical astrophysical settings. After some geometrical and phenomenological considerations in Section 3, we present numerical evidence in support of our theoretical results in Section 4. In Section 5, we discuss the implications of our result for cooling flow models.

2 THE EFFECT OF MAGNETIC FIELD LINES ON THERMAL CONDUCTION

Many authors have claimed that tangled magnetic fields can significantly reduce the effective thermal conductivity; a quantitative assessment of the physical validity of this assumption is, however, lacking in the literature. Before we present our results, we first examine the effects of magnetic fields on thermal transport. For the purposes of our discussion, we assume that the classical linear approximation to heat conduction holds, i.e., the heat flux is proportional to the local temperature gradient (cf. Spitzer & Härm 1953)

$$\boldsymbol{Q} = -\kappa \nabla T. \tag{1}$$

This formulation of thermal conduction applies where the temperature gradients are not too steep, and becomes invalid where the mean free path of heat-flux-carrying electrons is longer than ~2 per cent of the typical thermal length-scale (Gray & Kilkenny 1980). In the presence of steep temperature gradients, flux-limiting and non-local effects appear; we refer the readers to Chun & Rosner (1993) for a detailed discussion of these effects.

In the limit where the gyroradius of electrons is much less than the characteristic magnetic length-scale, the primary effect of magnetic fields is to increase the effective path-length of conduction electrons. We may evaluate the consequences of this by calculating the effective mean free path for thermal electrons in the presence of tangled magnetic field lines. This is a direct evaluation of the effective thermal conductivity along the mean direction of the heat flux (which, for convenience, we will also take to be the direction of the mean magnetic field). In this case, we can explicitly calculate the effective scattering distance of heat-flux-carrying electrons along the direction of the mean temperature gradient:

$$\lambda_{\rm eff} = \lambda_{\rm e} \left\langle \frac{\Delta z}{\Delta s} \right\rangle, \tag{2}$$

where Δz is the projected distance between successive scatterings along the mean magnetic field, and Δs is the actual path-length traversed by a heat-flux-carrying electron between scattering events (Hollweg & Jokipii 1972). In other words, $\langle \Delta s/\Delta z \rangle$ is the effective increase in path-length. In the rest of the paper, we also use $\langle \Delta s/\Delta z \rangle$ to denote the average length of magnetic field lines of our model system.

To evaluate the effective thermal conductivity along the local magnetic field, we write the local heat flux carried by individual electrons as

$$\langle Q \rangle_B = -\kappa_0 \left\langle \frac{\Delta z}{\Delta s} \right\rangle \nabla T, \tag{3}$$

where κ_0 is the Spitzer value of the thermal conductivity. When we consider electrons passing through a plane perpendicular to the mean large-scale magnetic field, the heat flux carried by these electrons will be reduced by another factor of $\langle \Delta z/\Delta s \rangle$:

$$\kappa_{\rm eff} = \kappa_0 \left\langle \frac{\Delta z}{\Delta s} \right\rangle^2. \tag{4}$$

This is because on average the local magnetic field makes an angle $\delta\theta$ with the plane under consideration, and the average cosine of $\delta\theta$ is given exactly by $\langle \cos(\delta\theta) \rangle = \langle \Delta z/\Delta s \rangle$.[1]

It is now straightforward to quantify the effective field line stretching by the (possibly) turbulent motions in the intracluster medium. We propose the following experiment. Consider a conducting fluid, which is assumed to be in a turbulent state. We wish to study the initial value problem of introducing into this fluid an ordered magnetic field, which will be stretched by the turbulent motions. In particular, we wish to determine whether there exists an asymptotic state for the stretched magnetic field. In the following, we show that such an asymptotic state indeed exists, and apply results from turbulence and geometric measure theory to estimate the effectiveness of turbulent line stretching.[2]

3 PHYSICAL CONSIDERATIONS

To estimate how long we may stretch magnetic field lines, consider an initially straight flux tube of length L, cross-section A, and uniform field strength B. The total magnetic flux in this tube is conserved, and the only way we may stretch the tube is by decreasing its cross-section:

$$\frac{L'}{L} = \frac{A}{A'} = \frac{B'}{B}, \tag{6}$$

where the primed quantities denote the final values.

It is easy to show that when there are tubes of different cross-sections and field strengths, the average length of field

[1]This statement is exact in flux-conserving systems, and we restrict our attention to these systems in this paper. This is precisely because flux conservation leads to the perfect cancellation of oppositely directed field lines as they are stretched by the turbulent motions. (This perfect cancellation of field lines is also the reason why dynamos cannot be generated by steady two-dimensional motions.) However, when we consider fields in systems with non-zero magnetic diffusivity stretched by (steady or unsteady) three-dimensional motions (or time-dependent two-dimensional motions), dynamo action is possible. When one does have such dynamo action, then the implication is that the cancellation is not perfect, and therefore flux may be generated (see Du & Ott 1993 for a discussion of this effect). Hence, in the general case, the effective thermal conductivity is suppressed by a factor of $\langle \Delta z/\Delta s \rangle$ to a power lying between 1 and 2, depending on the statistics of the turbulence. This effect will be reconsidered in a forthcoming paper. For the purposes of our discussion here, we note that equation (4) gives a lower bound to the effective thermal conductivity, and hence we have obtained an upper bound to the suppression of thermal conduction.

[2]We note that our calculations only yield results for the average of the effective stretching of field lines in our model system, i.e., $\langle \Delta s/\Delta z \rangle$, and not $\langle \Delta z/\Delta s \rangle$. However, algebraically one can show that

$$\left\langle \frac{\Delta z}{\Delta s} \right\rangle \geq \frac{1}{\langle \Delta s/\Delta z \rangle}, \tag{5}$$

where the equality is only attained when the individual fractions are identically equal. (To prove this, we can write the averages explicitly for a system of N field lines, and apply twice the arithmetic mean–geometric mean inequality.) In other words, we obtain a lower bound for $\kappa_{\rm eff}$, which is the desired result.

lines is bounded, from the above, by

$$\left\langle\frac{\Delta s}{\Delta z}\right\rangle \leq \left[\frac{\langle B^2\rangle}{|\langle B\rangle|^2}\right]^{1/2}, \tag{7}$$

where the angular brackets denote volume averaging, and we use $\Delta s/\Delta z$ to denote the increase in the length of an initially straight field line. We can make these physical ideas mathematically rigorous for continuum fields by considering the geometry of magnetic field lines in detail. The formal proof utilizes the co-area formula from geometric measure theory and is rather technical; we defer its discussion to the appendix.

Given that the average length of magnetic field lines is bounded from above by the total magnetic field fluctuations, we now estimate the amount of magnetic fluctuations generated by turbulent motions. In the following, we discuss the two-dimensional and three-dimensional cases separately. The difference between magnetized fluid dynamics in two and in three dimensions will become apparent.

Previous estimates of the importance of magnetic fields assume that the non-linear Lorentz forces of the large-scale field become important when the field strength is close to *equipartition*, that is, when the strength of the large-scale magnetic field is comparable to the strength of the large-scale velocity field. Cattaneo & Vainshtein (1991) contended that, because of the process of turbulent cascade, field stretching is much more effective on small scales. In determining whether or not large-scale magnetic fields are slaved to the kinetic motions, it is crucial to take the small-scale components of the field into account. Cattaneo & Vainshtein (1991) found the following criterion for the upper bound of the large-scale average of a passive (kinematic) field

$$|\langle \boldsymbol{B}\rangle_{\rm crit}|^2 = R_{\rm M}^{-1} u^2, \tag{8}$$

where the angular brackets denote volume averaging over the characteristic length-scale of the magnetic field, say L, and u is the rms velocity. The magnetic Reynolds number $R_{\rm M}$ is defined as usual as $R_{\rm M} = u\ell/\eta$, where ℓ is the characteristic velocity length-scale and η is the magnetic diffusivity. They obtained this result from estimating the magnetic fluctuations arising from the dynamics of the two-dimensional induction equation (see equation 9 below). Here we repeat their argument.[3]

In two dimensions, the magnetic induction equation may be written for a scalar potential (the magnetic flux function) as

$$\frac{\partial A}{\partial t} + \boldsymbol{u}\cdot\nabla A = \eta\nabla^2 A, \tag{9}$$

where A is the magnetic flux function (related to the two-dimensional magnetic field by $\boldsymbol{B} = \nabla \times A\hat{\boldsymbol{z}}$). Equation (9) is an advection–diffusion equation for the scaler field, A. Hence we may use a well-known result regarding scalar advection to estimate the magnetic fluctuations.

Consider the following exact equation, obtained by multiplying equation (9) by the magnetic flux function, A, and volume-averaging:

$$\frac{\partial\langle A^2\rangle}{\partial t} = -2\eta\langle|\nabla A|^2\rangle = -2\eta\langle \boldsymbol{B}^2\rangle. \tag{10}$$

This equation is an exact equation when there are no external magnetic sources (Vainshtein & Rosner 1991). Assuming that the fluid is turbulent, i.e., that the mean squared flux function dissipates on the dynamical time-scale (given by the velocity turnover time, $\tau_{\rm T} \approx L/u$), as opposed to the diffusive time-scale, $\tau_{\rm d} \approx L^2/\eta$, the right-hand side of equation (10) must be large. Since η is tiny in most physical and astrophysical circumstances, the magnetic fluctuations, $\langle \boldsymbol{B}^2\rangle$, must be enormous. Physically, the dissipative process is confined to very small scales. This generation of small-scale fluctuations leads to the growth of scalar gradients (here, the magnetic flux function is the advected scalar), and consequently to the growth of the magnetic field energy density via the stretching of field lines. Estimating the magnetic fluctuations from equation (10), we have

$$\frac{u}{L}|\langle \boldsymbol{B}\rangle|^2 L^2 \approx \eta\langle \boldsymbol{B}^2\rangle,$$

or, in terms of the magnetic Reynolds number

$$\langle \boldsymbol{B}^2\rangle \approx R_{\rm M}|\langle \boldsymbol{B}\rangle|^2. \tag{11}$$

We see that the fluctuations are indeed enormous: astrophysical plasmas have $R_{\rm M}$s ranging from 10^7 in the solar convection zone to 10^{14} in the Galaxy.

The advection of the magnetic field remains passive as long as the magnetic fluctuations are much smaller than the kinetic energy density:

$$\langle \boldsymbol{B}^2\rangle \leq u^2, \tag{12}$$

or, in terms of the large-scale magnetic field,

$$|\langle \boldsymbol{B}\rangle|^2 \leq R_{\rm M}^{-1} u^2, \tag{13}$$

which is exactly equation (8), and implies that in typical astrophysical cases even extremely weak magnetic fields can affect turbulent transport.

For fields that are stronger than the critical value, we can estimate the magnetic fluctuations to be comparable to the kinetic energy density. The ratio between the large-scale field and the fluctuations is then given by

$$\langle \boldsymbol{B}^2\rangle \approx u^2 \equiv M^2|\langle \boldsymbol{B}\rangle|^2, \tag{14}$$

where M is the Alfvénic Mach number and is given by the ratio of the rms velocity to the Alfvén wave speed (as determined by the strength of the mean magnetic field).

Based on equations (11) and (14), and motivated by the turbulent diffusion time formula of Cattaneo & Vainshtein (1991; see their equation 12), we suggest the following formula to interpolate between the passive and active magnetic field regimes:

$$\langle \boldsymbol{B}^2\rangle = \left(\frac{1}{R_{\rm M}} + \frac{1}{M^2}\right)^{-1}|\langle \boldsymbol{B}\rangle|^2. \tag{15}$$

When $M^2 \gg R_{\rm M}$, i.e., when the field is very weak, we recover the kinematic estimate, equation (11); when $R_{\rm M} \gg M^2$, i.e., when the field is dynamically limited by equipartition, we

[3]We must add that this argument originated with Zel'dovich (1957), but was made quantitatively more precise by Cattaneo & Vainshtein (1991) and Vainshtein & Rosner (1991).

recover equation (14). An obvious question is whether this interpolation formula is borne out by simulations, i.e., by the statistically stationary solutions to the fully non-linear equations (equation 9, and equations 18 and 19 below); in Section 4, we report on this comparison.

Unfortunately, in three dimensions, the analogy to scalar transport no longer exists: the magnetic field is *not* the gradient of a scalar (cf. Rosner et al. 1989; Rosner, Tao & Howard 1995, in preparation). Instead, the possibility of dynamos makes any simple relation like equation (10) unlikely in the kinematic (passive) case: there is no end to the exponential growth of magnetic energy in the case of a dynamo. Consequently, estimates of magnetic fluctuations (cf. equation 15) become more speculative and non-generic. However, in the dynamical case, where the Lorentz forces of the magnetic field make the very fields resist being stretched and bunched, it is possible that the magnetic energy is bounded from above. In no case is the magnetic energy greater than equipartition, i.e., the magnetic energy grows until it reaches approximately the value of the kinetic energy of the flow field. Even in the case of an α-effect dynamo, the statistically stationary state has approximately equal values in kinetic and magnetic energy densities (see Tao, Cattaneo & Vainshtein 1993 and Cattaneo & Tao 1995, in preparation, where the authors explicitly constructed α-effect dynamos and found that the effective growth rate is suppressed by surprisingly weak large-scale dynamical fields).

Vainshtein & Rosner (1991) suggested the following analogue of equation (11) for the magnetic fluctuations generated by three-dimensional turbulent motions:

$$\langle \boldsymbol{B}^2 \rangle \approx R_{\mathrm{M}}^{m} |\langle \boldsymbol{B} \rangle|^2, \tag{16}$$

where m is a flow-dependent exponent (see, for instance, Cattaneo et al. 1995). Vainshtein et al. (1993) proposed that $m=2$ when the magnetic field is compressed into space-filling tubes, and that $1<m<2$ in general. The lower limit is given by field stretching by strictly two-dimensional motions. The corresponding increase in fluctuations, i.e., the formula analogous to equation (15), is

$$\langle \boldsymbol{B}^2 \rangle = \left(\frac{1}{R_{\mathrm{M}}^{m}} + \frac{1}{M^2} \right)^{-1} |\langle \boldsymbol{B} \rangle|^2. \tag{17}$$

We must emphasize that equation (17) is flow-dependent. The important dynamical features of three-dimensional magnetic field range from magnetic ropes (in the $m=2$ case) to magnetic sheets (in the $m=1$ case). Perhaps, ultimately we may construct models of astrophysical systems including magnetic fields to agree with available observations of the magnetic structures.

Before we turn to the analysis of our numerical data, we conclude this section with a discussion of the level of rigour and the range of applicability of our bounds and estimates in this section. An important issue that has arisen in our study is the use of an incompressible velocity field in our analysis: what are the effects of fluid compressibility?

Our principal result is to demonstrate that the stretching of magnetic field lines is directly linked to the generation of small-scale magnetic fields. The geometric bound (equation 7) regarding the average length of field lines in our model system is independent of the compressibility of our model flow. Equation (7) can be derived directly from the representations of the magnetic field without any considerations for the flow field (see the appendix).

On the other hand, our estimates of the magnetic fluctuations generated by turbulence (cf. equations 11 and 16, and our interpolation formulae, equations 15 and 17) are somewhat dependent on flow-incompressibility. In two dimensions, equation (10), the equation for the generation of magnetic fluctuations by turbulent motions is an exact equation only for incompressible flows; however, the consequences we derived from it (cf. equation 11) are also valid for compressible flows (cf. Batchelor 1959). In three dimensions, magnetic fluctuations are no longer generic, i.e., the fluctuations are flow-dependent. Thus equations (16) and (17) are speculative, but are consistent with the typical dynamical structures observed in numerical simulations (e.g. Meneguzzi, Frisch & Pouquet 1981; Proctor & Weiss 1982; Brandenburg 1990, and references therein).

However, we note that in the case of dynamically important fields (i.e., $\langle \boldsymbol{B} \rangle \geq \langle \boldsymbol{B} \rangle_{\mathrm{crit}}$, cf. equation 8), magnetic fluctuations are bounded from above by the equipartition of kinetic and magnetic energy densities. Thus the magnetic fluctuations generated from turbulence dominated by dynamical magnetic fields are well described by the appropriate limiting case of equation (15) and of equation (17).

4 NUMERICAL RESULTS

To assess quantitatively the phenomenological ideas considered in the previous section, we consider the following idealized model. The evolution equations for the velocity $\boldsymbol{u}$ and the magnetic field $\boldsymbol{B}$ in an incompressible fluid of unit density can be written as

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} = -\nabla p + \frac{1}{M^2} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{F}, \tag{18}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}, \tag{19}$$

with

$$\nabla \cdot \boldsymbol{u} = 0, \qquad \nabla \cdot \boldsymbol{B} = 0, \tag{20}$$

where p is the pressure, ν and η are the viscosity and magnetic diffusivity, respectively (assumed constant in space), $\boldsymbol{F}$ is a forcing term, and M is the Alfvénic Mach number.

We restrict ourselves to periodic boxes (cubes in three dimensions and squares in two dimensions). Initially, the magnetic field is uniform and along the x-direction, i.e., $\boldsymbol{B}(t=0) = (B_0, 0, 0)$. Stokes's theorem tells us that this uniform component does not change even though $\boldsymbol{B}$ changes due to the growth of fluctuations. We choose the strength of the uniform component as the unit of magnetic field intensity, so that M^2 in equation (18) gives the ratio between the kinetic energy and the magnetic energy of the imposed uniform field. As discussed previously, the incompressibility of the flow field is a benign assumption because our results do not significantly depend on it.

To mimic the effects of a turbulent velocity field, we randomly stir the fluid using small-scale forcing functions, represented by $\boldsymbol{F}$ in equation (18). For our two-dimensional

Table 1. Summary of numerical simulations.

Parameters	2D		3D
R_e	100	150	130
R_M	200	400	130
M^2	4^2, 8^2, 16^2, 32^2, 64^2, 128^2	4^2, 8^2, 16^2, 32^2, 64^2	10, 30, 100, 250, 1000, 10000
F(k)	$k \approx 5$	$k \approx 5$	$k \approx 2-3$
Resolution	128^2	256^2	64^3

computations, we choose an $\boldsymbol{F}$ consisting of wave vectors $\boldsymbol{k}$ with wave numbers $k \approx 5$; for our three-dimensional computations, $\boldsymbol{F}$ has $k \approx 2$. With these choices, we have set up a separation of scales between the 'large-scale' magnetic field and the 'small-scale' turbulent motions. The amplitude of each component of the forcing function is chosen randomly with unit correlation time such that the resulting velocity field has an amplitude of order unity. With this normalization condition, ν and η in equations (18) and (19) are of the same order as the inverse of the kinetic and magnetic Reynolds numbers respectively. These Reynolds numbers are defined as $R_e = u\ell/\nu$ (viscous) and $R_M = u\ell/\eta$ (magnetic), where u is the rms velocity field and ℓ is the characteristic length-scale of the velocity field. For a summary of the actual parameter values and the resolution of our simulations, see Table 1.

Equations (18)–(20) are solved numerically on a periodic cube of side length 2π using standard pseudo-spectral techniques: non-linear terms are evaluated in configuration space and derivatives in wave number space. We used different time-advanced schemes: explicit second-order Adams–Bashforth for the non-linear terms and implicit Crank–Nicholson for the linear terms. More details regarding our numerical codes can be found in Cattaneo & Vainshtein (1992), Tao et al. (1993) and Cattaneo & Tao (1995, in preparation). For the three-dimensional system, we do not explicitly compute the pressure term in equation (18); we ensure solenoidality of both the velocity and magnetic fields (equation 20) by projecting the fields on to the Craya decomposition (cf. Tao et al. 1993).

Our objective is to quantify and to bound the effective field line stretching and to see how it varies with the strength of the imposed large-scale magnetic field (as measured by M). In our numerical experiments, we identify the $k=0$ component with the large-scale field, and compute how smaller scale ($k>0$) random velocity field fluctuations act on this large-scale field. Using the result of the previous section, we estimate the effectiveness of the turbulent stretching of magnetic field lines by computing the magnetic fluctuations.

Our first result is that statistically stationary states do indeed exist. In Fig. 1(a), we show a case for which the magnetic fluctuations are limited by diffusion. After an initial transient period, during which the magnetic energy is (exponentially) amplified, the magnetic energy density becomes

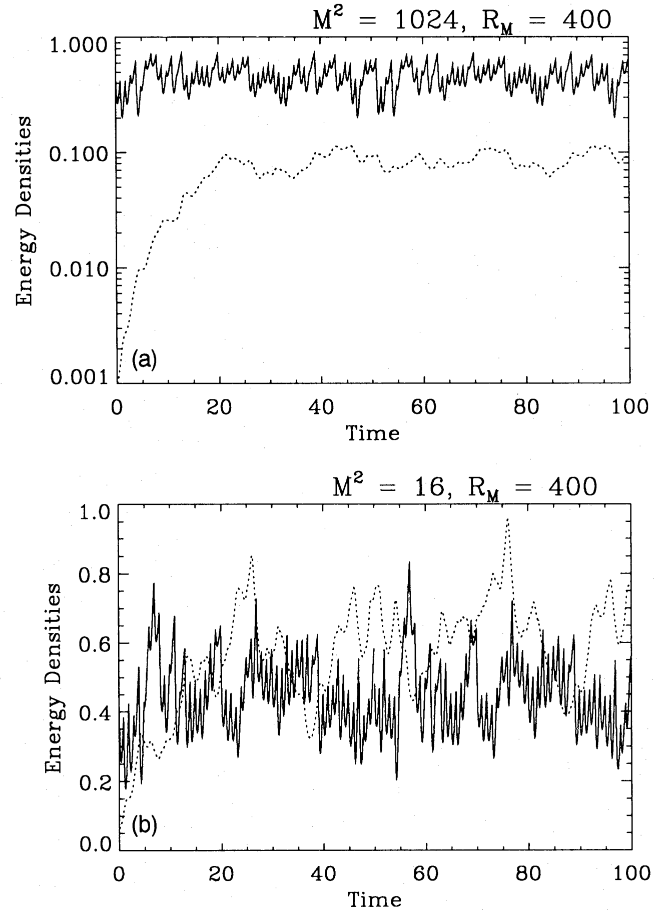

Figure 1. Time history of energy densities ($m=1$ in two dimensions). (a) Case limited by diffusion: $M^2 = 1024 > R_M = 400$. Solid line $\langle u^2 \rangle$; dotted line $\langle B^2 \rangle / |\langle \boldsymbol{B} \rangle|^2$. (b) Case limited dynamically: $M^2 = 16 < R_M = 400$. Solid line $\langle u^2 \rangle$; dotted line $\langle B^2 \rangle / |\langle \boldsymbol{B} \rangle|^2$.

stationary and fluctuates about a mean value (as determined by diffusion).

In Fig. 1(b), we show the time history of energy densities for a case for which the magnetic fluctuations are dynamically limited. We see that just as the case illustrated in Fig.

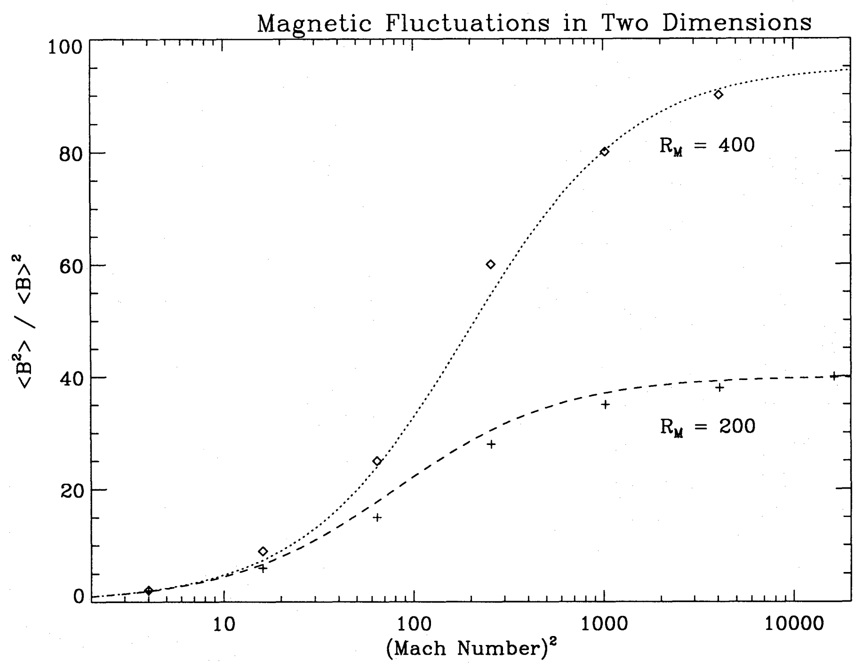

Figure 2. Magnetic fluctuations as functions of the magnetic Mach number. Squares: $R_M=400$; dotted line is the curve $[(4.2/R_M)+(2.0/M^2)]^{-1}$. Crosses: $R_M=200$; dashed line is the curve $[(5.0/R_M)+(2.0/M^2)]^{-1}$.

1(a), there is a transient period of exponential growth during which the magnetic field is amplified by the fluid turbulence. However, in this case, the stationary values of the energy densities are fixed by equipartition: the magnetic energy density increases until it is comparable to the kinetic energy density. As expected, in all cases for which $\boldsymbol{B}(t=0)\geq\langle\boldsymbol{B}\rangle_{\rm crit}$ (cf. equation 8), equipartition between kinetic and magnetic energies is observed.

We also found evidence in support of equations (15) and (17), our interpolation formulae for magnetic fluctuations. The dependence of the small-scale magnetic fluctuations as a function of the large-scale magnetic Mach number is illustrated in Fig. 2.[4] For comparison to our theory, we also plot smooth curves which have the functional form of equation (15) with constants of order one. In Fig. 3, we show the corresponding curve for the three-dimensional magnetic fluctuations. We see that there is very good agreement between the numerical computations and our theoretical predictions: even relatively weak fields can suppress field line stretching. The difference between the order-one constants in the two- and the three-dimensional simulations arises from the difference in the large-scale kinetic forcing. For the two-dimensional computations, the constants are similar between simulations with identical forcing but with different values of R_M.

The physical reason for the suppression of field line stretching seen in these calculations is associated with the dynamics of the small-scale magnetic field fluctuations (see discussions in Cattaneo & Vainshtein 1991, Tao et al. 1993 and Cattaneo & Tao 1995, in preparation, for the identical effect regarding the suppression of turbulent magnetic transport coefficients). As the strength of the imposed field increases (corresponding to decreasing M^2), the magnetic fluctuations and, by implication, the effectiveness of turbulent line stretching, become smaller. If the strength of the imposed field is stronger than the critical value (cf. equation 8), the field is sufficiently strong to alter fluid turbulence on scales larger than the magnetic diffusive scales. Consequently, diffusive-scale magnetic structures are prevented from ever forming and line stretching is not effective. In Fig. 4, we show the time history of the Taylor microscale of the magnetic field, $\lambda=\langle B^2\rangle/\langle|\nabla\times\boldsymbol{B}|^2\rangle$, for two Alfvénic Mach numbers, $M^2=30$ and 1000, corresponding to the dynamical and kinematical regimes, respectively. In the kinematic case, we see that the fluctuations are formed on the small diffusive scales right away. However, in the dynamical field case, the magnetic Taylor microscale is bounded away from the diffusive cut-off, and the diffusive-scale fluctuations are never formed. The small scales that are necessary for effective stretching of magnetic field lines are not formed. The magnetic fluctuations on these scales resist turbulent stretching as equipartition fields are formed on scales larger than the diffusive cut-off.

[4]We note that the magnetic energy densities fluctuate with time; therefore the actual values plotted are averages over suitably many velocity turnover times after the initial transient amplification periods (see Fig. 1).

5 ASTROPHYSICAL IMPLICATIONS

In this paper, we have focused on the effectiveness of magnetic field line stretching by two- and three-dimensional turbulent motions. We have concentrated on this problem because of the common assumption that magnetic field lines can be stretched essentially indefinitely to suppress thermal conduction in various astrophysical systems. We have demonstrated how the extent of field line stretching can be constrained by the physical and measurable properties of the

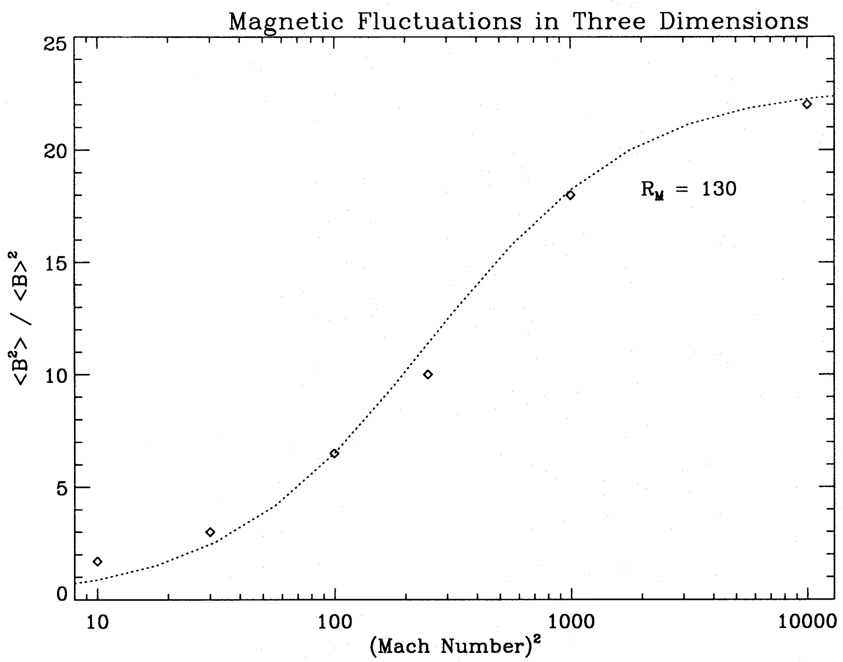

Figure 3. Magnetic fluctuations in three dimensions as a function of the magnetic Mach number. Squares: $R_{\mathrm{M}}=130$; dotted line is the curve $[(5.7/R_{\mathrm{M}})+(10/M^2)]^{-1}$.

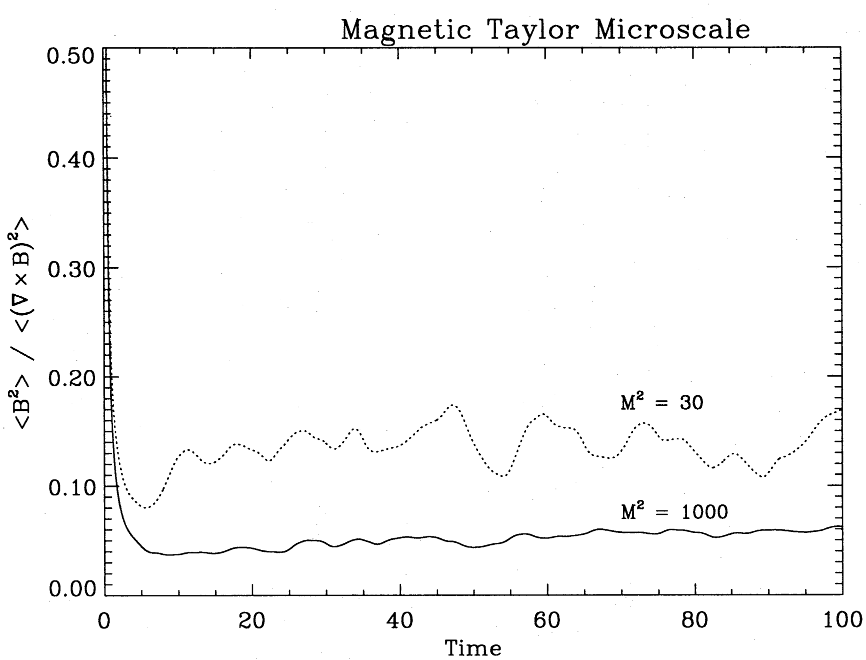

Figure 4. Time history of magnetic field Taylor microscale in three dimensions. Solid line: Case limited by diffusion, $M^2=1000>R_{\mathrm{M}}=130$. Dotted line: Case limited dynamically, $M^2=30<R_{\mathrm{M}}=130$.

magnetic field, which we assume to be embedded in a turbulent flow. In particular, we have shown that the stretching of magnetic field lines can be bounded from above by a normalized square root of the magnetic energy density. Thus we have obtained, for the first time, quantitative results for the increase in the path-length over which thermal electrons in haloes must travel, and hence the factor by which magnetic fields can suppress thermal conduction.

Phenomenologically, magnetized fluid dynamics exhibit two distinct behaviours: a kinematic regime where the magnetic fluctuations are limited by diffusion, and a dynamical regime where the Lorentz back-reaction on the turbulence becomes important, and the generation of small-scale magnetic fluctuations are constrained by the equipartition of kinetic and magnetic energies. For comparison to our theoretical estimates of the magnetic fluctuations, we have also constructed an idealized model in which these fluctuations can be explicitly calculated using numerical simulations. Having studied these issues, we conclude this paper by discussing the implications of our results for the standard 'cooling flow' models of galactic cluster haloes.

Observations of clusters of galaxies show the presence of diffuse intracluster gas in the central region of these clusters. X-ray spectra reveal that the gas is optically thin, and that the radiative emission from this gas is primarily due to bound–bound and free–bound transitions in 'metals' and thermal bremsstrahlung. These observations imply that the gas has electron number densities ranging from 10^{-2} to 10^{-4} cm^{-3} and temperatures of 10^7–10^8 K.[5] Proponents of the standard cooling flow models have interpreted the X-ray data to infer that radiative cooling is driving mass accretion on to the central core galaxy at enormous rates (cf. Binney & Cowie 1981; Stewart et al. 1984, among others); these accretion rates range from 3 $M_\odot$ yr^{-1} for M87 to over 300 $M_\odot$ yr^{-1} for NGC 1275 (cf. Arnaud 1988; Edge, Stewart & Fabian 1992; Fabian 1994).

A peculiar aspect of these cooling flow models is that they do not include the effects of thermal conduction. It is normally argued that thermal conduction in haloes can be neglected because of 'tangled' magnetic field lines. In other studies where thermal conduction is consistently taken into account, a number of authors (Tucker & Rosner 1983; Bertschinger & Meiksin 1986; Meiksin 1988; Rosner & Tucker 1989) have shown that together with internal heating the mass accretion rate can be suppressed to very low values (as small as 10^{-2} $M_\odot$ yr^{-1}). In order to distinguish between models of cluster haloes (with or without thermal conduction), we now examine quantitatively how the stretching of magnetic field lines suppress the effective thermal conductivity in the conditions found in clusters of galaxies.

Take M87 as our typical case: estimates of the magnetic field strengths in the Virgo cluster range from 10^{-7} to 10^{-6} G (Kim, Tribble & Kronberg 1991), while the density is approximately $N \approx 10^{-3}$ cm^{-3} near the centre of the cluster (cf. Sarazin 1988). We estimate the typical kinetic energy density from the temperature near M87, $T \approx 2 \times 10^7$ K (cf. Sarazin 1988), to be around $NkT \approx 3 \times 10^{-12}$ erg cm^{-3}, compared with the magnetic energy density, $B^2/8\pi \approx 4 \times 10^{-14}$ erg cm^{-3}. Thus M^2 is of order 100, which is much smaller than the magnetic Reynolds number, $R_M \gg 10^7$. Hence the magnetic fields in the halo of M87 are not passive (cf. equation 8).

We can immediately place bounds on the effectiveness of magnetic fields in suppressing thermal transport by combining equations (7) and (17) to obtain

$$\left\langle \frac{\Delta s}{\Delta z} \right\rangle \le \left[\frac{\langle B^2 \rangle}{|\langle \boldsymbol{B} \rangle|^2} \right]^{1/2} \approx \left(\frac{1}{R_M^m} + \frac{1}{M^2} \right)^{-1/2}. \tag{21}$$

In the present dynamical field case, we have $R_M \gg M^2$, so that

$$\left\langle \frac{\Delta s}{\Delta z} \right\rangle \lesssim M, \qquad \text{and} \qquad \kappa_{\rm eff} \gtrsim \frac{\kappa_0}{M^2}, \tag{22}$$

i.e., M itself gives an upper bound to the field line stretching, and M^2 gives a lower bound to the effective thermal conductivity. In this case, we estimate therefore that a rough upper limit on the average line stretching between any two points on the same field line within the halo of Virgo is $\lesssim 10$ times the distance between these points, and that, as a consequence, on average the thermal conductivity can be accordingly suppressed at most by a factor of 100 (cf. equation 4).[6] We note that this estimate is a generous upper bound on the effective turbulent line stretching: measurements of the magnetic field strength in the Virgo cluster are of the large-scale field but, in principle, the field could be stronger in the cooling flow region of the cluster.[7] In such an event, the upper limit on field line stretching is even smaller.

We remind the readers that, in the core of cluster haloes, the thermal conduction time-scale (with thermal conduction proceeding at the unsuppressed Spitzer rate) is of the order of 10^8 yr. Therefore, unless turbulent magnetic fields suppress the effective thermal conductivity by at least a factor of 10^3, it is hard to justify the neglect of thermal conductive effects in cluster haloes. We conclude that magnetic field line stretching is not an effective mechanism for the suppression of thermal conduction in galactic cluster haloes. we must emphasize that we have based our discussion on the fundamental fluid equations, while earlier discussions were based on phenomenological models of heat transport (e.g. Rosner & Tucker 1989; Tribble 1989).

To conclude, we have studied the problem of magnetic field line stretching in turbulent fluids, and have shown that one cannot appeal to 'tangled' magnetic fields to suppress thermal transport in cluster haloes. We have argued that line stretching is not sufficiently effective to suppress thermal conduction in the typical conditions of the galactic cluster haloes, thus casting serious doubt on the physical consistency of the standard 'cooling flow' models, which ignore the effects of classical thermal conduction.

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[5]For reviews of these observations, see Fabian, Nulsen & Canizares (1984), Sarazin (1988), Fabian (1992) and Fabian (1994, and references therein).

[6]In the halo of M87, this also implies that the effects of such a field on thermal instabilities is negligible. The critical wavelength of isobaric overstabilities, given by equations (7) and (8) of Malagoli, Rosner & Bodo (1987) at a thermal conduction suppression factor, $\alpha \gtrsim 0.01$, is still comparable to the radius of the halo itself. In fact, from equation (8) of Malagoli et al. (1987), we see that we need to stretch a field by a factor of at least a few times 10^3 for thermal instabilities to take place on the astrophysically interesting spatial scales (i.e., $\lesssim 100$ kpc). In this case, an important additional effect of magnetic fields on thermal conduction is to limit the scales on which thermal perturbation may become unstable.

[7]The author is indebted to Dr Avery Meiksin for this remark.

REFERENCES

Arnaud K., 1988, in Fabian A. C., ed., Cooling Flows in Clusters and Galaxies. Kluwer, Dordrecht, 31
Arnold V. I., 1986, Sel. Math. Sov., 5, 327 (Original in Russian, in Proc. Summer School in Diff. Eqs., Erevan, Armenia SSR Acad. Sci.)
Batchelor G. K., 1959, J. Fluid Mech., 5, 113
Berger M. A., Field G. B., 1985, J. Fluid Mech., 147, 133
Bertschinger E., Meiksin A., 1986, ApJ, 306, L1
Binney J., Cowie L. L., 1981, ApJ, 247, 464
Bodo G., Massaglia S., Rosner R., Ferrari A., 1991, ApJ, 370, 398
Brandenburg A., 1990, PhD dissertation, Univ. Helsinki
Cattaneo F., Vainshtein S. I., 1991, ApJ, 376, L21
Cattaneo F., Kim E., Proctor M. R. E., Tao L., 1995, submitted to Phys. Rev. Lett.
Chun E., Rosner R., 1993, ApJ, 408, 678
Constantin P., 1990, Commun. Math. Phys., 129, 241
Constantin P., 1994, SIAM Rev., 36, 73
Constantin P., Procaccia I., 1992, Phys. Rev. A, 46, 4736
Constantin P., Procaccia I., Sreenivasan K. R., 1991, Phys. Rev. Lett., 67, 1739
Du Y., Ott E., 1993, Physica D, 67, 387
Edge A. C., Stewart G. C., Fabian A. C., 1992, MNRAS, 258, 177
Evans L. C., Garipsy R. F., 1992, Measure Theory and Fine Properties of Functions. CRC Press, Boca Raton
Fabian A. C., 1992, in Fabian A. C., ed., Clusters and Superclusters of Galaxies. Kluwer, Dordrecht, 369
Fabian A. C., 1994, ARA&A, 32, 277
Fabian A. C., Nulsen P. E. J., Canizares C. R., 1984, Nat, 310, 733
Federer H., 1969, Geometric Measure Theory. Springer-Verlag, New York
Gray D. R., Kilkenny J. D., 1980, Plasma Phys., 22, 81
Hollweg J. V., Jokipii J. R., 1972, J. Geophys. Res., 77, 3311
Jokipii J. R., 1966, ApJ, 146, 480
Jokipii J. R., 1971, Rev. Geophys. Space Phys., 9, 27
Jokipii J. R., Parker E. N., 1969, ApJ, 155, 777
Kim K.-T., Tribble P., Kronberg P. P., 1991, ApJ, 379, 80
Malagoli A., Rosner R., Bodo G., 1987, ApJ, 319, 632
Meiksin A., 1988, ApJ, 334, 59
Meneguzzi M., Frisch U., Pouquet A., 1981, Phys. Rev. Lett., 47, 1060
Moffatt H. K., 1969, J. Fluid Mech., 35, 117
Morgan F., 1988, Geometric Measure Theory: A Beginner's Guide. Academic Press, New York
Procaccia I., Cing E., Constantin P., Kadanoff L., Libchaber A., Wu X. Z., 1991, Phys. Rev. A, 44, 8091
Proctor M. R. E., Weiss N. O., 1982, Rep. Prog. Phys., 45, 1317
Rosner R., Tucker W. H., 1989, ApJ, 338, 761
Rosner R., Low B. C., Tsinganos K., Berger M. A., 1989, Geophys. Astrophys. Fluid Dyn., 48, 251
Salmon R., 1988, Ann. Rev. Fluid Mech., 20, 225
Sarazin C. L., 1988, X-ray Emissions from Clusters of Galaxies. Cambridge Univ. Press, Cambridge, 303
Soker N., Sarazin C. L., 1990, ApJ, 348, 73
Spiter L., Härm R., 1953, Phys. Rev., 89, 977
Stewart G. C., Canizares C. R., Fabian A. C., Nulsen P. E. J., 1984, ApJ, 278, 536
Tao L., Cattaneo F., Vainshtein S. I., 1993, in Proctor M. R. E., Matthews P. C., Rucklidge A. M., eds, Solar and Planetary Dynamos. Cambridge Univ. Press, Cambridge, 303
Tribble P. C., 1989, MNRAS, 238, 1247
Tucker W. H., Rosner R., 1983, ApJ, 267, 547
Vainshtein S. I., Cattaneo F., 1992, ApJ, 393, 165
Vainshtein S. I., Rosner R., 1991, ApJ, 376, 199
Vainshtein S. I., Tao L., Cattaneo F., Rosner R., 1993, in Proctor M. R. E., Matthews P. C., Rucklidge A. M., eds, Solar and Planetary Dynamos. Cambridge Univ. Press, Cambridge, 311
Zel'dovich Ya. B., 1957, Soviet Phys. JETP, 4, 460

APPENDIX A: THE GEOMETRY OF MAGNETIC FIELD LINES

Recent work in turbulence has progressed towards the understanding of small-scale structures using the notion of fractals, focusing in particular on the structure of isoscalar surfaces in turbulence (e.g. Constantin 1990; Constantin, Procaccia & Sreenivasan 1991; Procaccia et al. 1991; Constantin & Procaccia 1992; Constantin 1994, and references therein). In view of this work, our mathematical strategy in this section is to represent magnetic field lines as isoscalar structures in various guises, and, in this way, to obtain upper bounds for the average length of field lines by applying existing theoretical results directly to the mathematical representations of the magnetic field.

In the following, we discuss the two-dimensional and the three-dimensional cases separately. The difference between representations of the magnetic field in two and in three dimensions will become apparent. In each case, we show that a normalized square root of the magnetic energy density is an upper bound for the average length of magnetic field lines. The equation central to the mathematical analysis in this section is the co-area formula of geometric measure theory (cf. Federer 1969; Morgan 1988; Evans & Garipey 1992):

$$\int_{\nu} \mathscr{J}_{\theta}(x)\,\mathrm{d}^n x = \int \mathrm{d}\beta \int_{\mathscr{A}} \mathrm{d}H^{(n-p)}(x), \tag{A1}$$

where θ is any smooth transformation that maps n-dimensional Euclidean space to p-dimensional Euclidean space, and $\mathscr{J}_{\theta}(x)$ is the Jacobian of the transformation, $\theta(x)$. ν is the n-dimensional domain of interest and, in our case, we take it to be the periodic domain of our numerical simulations. $H^{(n-p)}$ is the $(n-p)$-dimensional Hausdorff measure [i.e., $\mathrm{d}H^{(n-p)}$ is a differential element of $(n-p)$-dimensional surfaces]; we perform the integral over the set $\mathscr{A}=\{x\in\nu;\ |\theta(x)|=\beta\}$. This formula states that integrals over the level sets of scalar functions can be computed by integrating the appropriate Jacobian. In other words, equation (A1) provides us with a way of relating integrals over isoscalar surfaces to integrals in physical space.

A1 The two-dimensional case

To apply the co-area formula to two-dimensional magnetohydrodynamics, we use the fact that the Jacobian of any scalar transformation is the gradient of the scalar to rewrite equation (A1) as

$$\int_{\nu} |\nabla\theta(x)|\,\mathrm{d}^n x = \int \mathrm{d}\beta \int_{\mathscr{A}} \mathrm{d}H^{(n-1)}(x), \tag{A2}$$

where $\mathscr{A}=\{x\in\nu;\,|\theta(x)|=\beta\}$ is an isoscalar surface.

We set $n=2$, then identify the magnetic flux function with θ and the strength of the magnetic field with $|\nabla\theta|$ (cf. equation 9); immediately we have

$$\int_{\nu} |\boldsymbol{B}(x)|\,\mathrm{d}^2 x = \int \mathrm{d}\beta \int_{\mathscr{A}} \mathrm{d}H^{(1)}(x), \tag{A3}$$

where $\mathscr{A}=\{x\in\nu;\ |A(x)|=\beta\}$ is an equipotential line, i.e., precisely a field line. The right-hand side is proportional to the average length of field lines:

$$\left\langle \frac{\Delta s}{\Delta z} \right\rangle_{v} = l^{-1}(\beta_2-\beta_1)^{-1} \int_{\beta=\beta_1}^{\beta_2} \mathrm{d}\beta \int_{\mathscr{A}} \mathrm{d}H^{(1)}(x), \tag{A4}$$

where we denote the average length of field lines by $\langle \Delta s/\Delta z \rangle$ to make the connection with our discussion in Section 3 (cf. equations 2–4). l is the length of the initial field line (also identical to the length of the system). β_2 and β_1 are the maximum and minimum values attained by the flux function, A, within the domain of interest, v.

We can bound the left-hand side of equation (A3) from above by applying the Cauchy–Schwarz inequality,[8]

$$\mathrm{Vol}(v) \left(\int_v B^2 \, \mathrm{d}^2x \right) \geq \left(\int_v |\boldsymbol{B}(x)| \, \mathrm{d}^2x \right)^2, \tag{A5}$$

where Vol(v) is the volume of v, and the equality is only attained when $|\boldsymbol{B}|$ is a constant function of space.

Substituting equations (A4) and (A5) in equation (A3), we have

$$\left\langle \frac{\Delta s}{\Delta z} \right\rangle_{v} \leq \frac{[\mathrm{Vol}(v)]^{1/2}}{l(\beta_2-\beta_1)} \left(\int_v B^2 \, \mathrm{d}^2x \right)^{1/2}. \tag{A6}$$

In other words, the length of field lines is bounded from above by a linear function of the square root of the average magnetic energy density. For our model system with an imposed uniform field, we can rewrite equation (A6) purely in terms of volume-averaged magnetic quantities by noting that $\mathrm{Vol}(v)=(2\pi)^2$ and $l=\beta_2-\beta_1=2\pi$ for our periodic domain:

$$\left\langle \frac{\Delta s}{\Delta z} \right\rangle \leq \left[\frac{\langle B^2 \rangle}{|\langle \boldsymbol{B} \rangle|^2} \right]^{1/2}, \tag{A7}$$

which is exactly equation (7) in the main text.

A2 The three-dimensional case

In contrast to the two-dimensional case, field lines can be knotted in three dimensions; furthermore, the magnetic field is no longer the gradient of a scalar (Rosner et al. 1989; Rosner et al. 1995, in preparation). Hence it is not obvious that our result (7) applies in three dimensions. We now proceed to show that equation (7) can be recovered for three dimensions as well.

An important measure of the twistedness and knottedness of the field is given by the magnetic helicity (Moffatt 1969; Berger & Field 1985; Arnold 1986),

$$H = \int_v \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}^3x, \tag{A8}$$

[8]The Cauchy–Schwarz inequality states that for any f and g, square-integrable over v,

$$\left(\int_v f^2 \, \mathrm{d}^nx \right) \left(\int_v g^2 \, \mathrm{d}^nx \right) \geq \left(\int_v fg \, \mathrm{d}^nx \right)^2.$$

Letting $f=1$ and $g=|\boldsymbol{B}|$, we obtain the inequality stated in the text.

where $\boldsymbol{A}$ is the magnetic flux potential and $\boldsymbol{A}\cdot\boldsymbol{B}$ is the helicity density. Magnetic helicity is an invariant in the ideal (zero-diffusivity) case, and the conservation of helicity may be understood as the preservation of the topological properties of the field when the field lines are frozen into the fluid. Conversely, the reconnection of field lines in the non-ideal case is the only way to change the topology of the magnetic field.

Locally we may represent magnetic fields by Clebsch variables (also known as Clebsch or Euler potentials; see Salmon 1988; Rosner et al. 1989; Rosner et al. 1995, in preparation, and references therein):

$$\boldsymbol{B}(x) = \nabla\phi(x) \times \nabla\psi(x), \tag{A9}$$

where ϕ and ψ are scalar fields. The intersections of the level surfaces of ϕ and ψ locally define curves which are integral curves of the vector field $\boldsymbol{B}$. Therefore the gradients, $\nabla\phi$ and $\nabla\psi$, along with the magnetic field, $\boldsymbol{B}$, define a local coordinate system (as illustrated in Fig. A1). For fields with identically zero-helicity density, this representation is *global*. We note, however, that there is no unique way of defining the Clebsch potentials, locally or globally.

Consider the map $\theta(x)=[\phi(x), \psi(x)]$ given by the direct product of the Clebsch potentials. This map is a transformation with the domain in three-dimensional space (the physical space) and range in two-dimensional space (spanned by the ranges of the Clebsch potentials). The Jacobian of θ is exactly

$$\mathscr{J}_\theta(x) = |\nabla\phi(x) \times \nabla\psi(x)|. \tag{A10}$$

Specializing the co-area formula (equation A1) to $n=3$ and $p=2$, we have

$$\int_v |\nabla\phi(x) \times \nabla\psi(x)| \, \mathrm{d}^3x = \int \mathrm{d}\beta \int \mathrm{d}\alpha \int_{\mathscr{A}} \mathrm{d}H^{(1)}(x), \tag{A11}$$

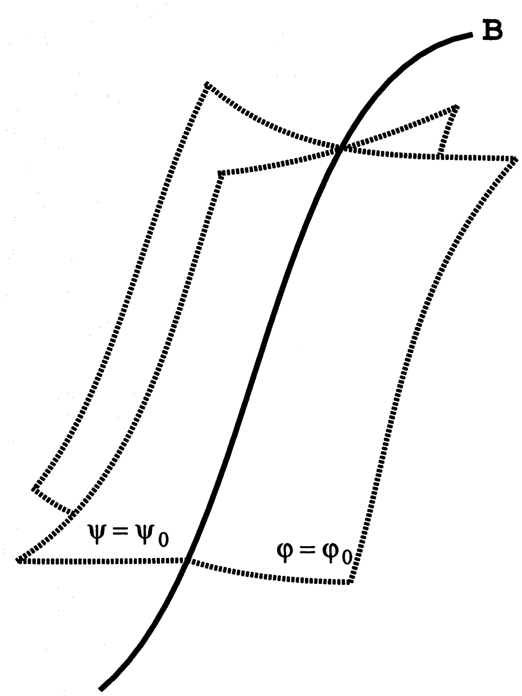

Figure A1. Local Clebsch representation of the magnetic field: the magnetic field (solid) and the isopotential planes (dotted) form a local coordinate system.

where $H^{(1)}$ is the one-dimensional Hausdorff measure, v again is the domain of interest, and $\mathscr{A}=\{x\in v;\ |\phi(x)|=\beta, |\psi(x)|=\alpha\}$. Equation (A11) provides a relation between the length of isoscalar lines, which are identically the magnetic field lines in the Clebsch representation, and the magnetic flux (cf. equation A9). Immediately the average length of field lines is given by

$$\left\langle\frac{\Delta s}{\Delta z}\right\rangle_v=\left(\frac{1}{l}\right)\frac{1}{(\beta_2-\beta_1)}\frac{1}{(\alpha_2-\alpha_1)}\int_v|\boldsymbol{B}(x)|\,\mathrm{d}^3x, \qquad (A12)$$

where the subscripts 1 and 2 on α and β indicate minimum and maximum values, respectively.

As in the two-dimensional case, we can apply the Cauchy–Schwarz inequality (cf. equation A5) to bound equation (A12) from above by a linear function of the rms magnetic field:

$$\left\langle\frac{\Delta s}{\Delta z}\right\rangle_v\leq\frac{[\mathrm{Vol}(v)]^{1/2}}{l(\beta_2-\beta_1)(\alpha_2-\alpha_1)}\left(\int_v B^2\mathrm{d}^3x\right)^{1/2}. \qquad (A13)$$

For our model system, flux conservation gives a lower bound to the product $(\beta_2-\beta_1)(\alpha_2-\alpha_1)$. Any stretching of field lines will increase the range of values attained by Clebsch potentials. Therefore, by rewriting equation (A13) purely in terms volume-averaged magnetic quantities for our model system, we recover equation (A7):

$$\left\langle\frac{\Delta s}{Dz}\right\rangle\leq\left[\frac{\langle B^2\rangle}{|\langle\boldsymbol{B}\rangle|^2}\right]^{1/2}.$$

For field configurations with non-vanishing helicity density, it is no longer possible to construct global Clebsch potentials. However, it is possible to divide the domain into local patches in which we may construct Clebsch representations *locally* (Rosner et al. 1989). The global average of the length of field lines is then a weighted sum of the local averages, each of which is bounded from the above by local versions of equation (A7). It is easy to see that our formula for the average length of field lines still holds for helical fields.