PART 1. Given values $f_i$, $i = 0, \ldots, N$ for corresponding x-values $x_i$ (all $x_i$'s distinct), define the interpolating polynomial $p(x)$ as the polynomial of at most degree $N$ ($p(x) \in \Pi_n$), that obeys

$$p(x_i) = f_i, \ i = 0, \ldots, N.$$  

a) Show that $p(x)$ is unique.

b) We can find the interpolating polynomial, for example by determining the coefficients $a_0, \ldots, a_N$ in the ansatz

$$p_1(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_N x^N$$

or the coefficients $c_0, \ldots, c_N$ in Newton's ansatz

$$p_2(x) = c_0 + c_1 (x - x_0) + c_2 (x - x_0)(x - x_1) + \ldots c_N (x - x_0)(x - x_1) \ldots (x - x_{N-1}).$$

Let $x_i = x_0 + ih$, with $h = 1.0/N$, $i = 0, \ldots, N$ (such that $x_N = x_0 + 1$, i.e the length of the total interval is 1).

To compute the maximum error in the interpolation, we define $NL + 1$ control points $\xi_j = x_0 + jh_F$, $j = 0, \ldots, NL$ with $h_F = 1.0/NL$, with $NL = 200$.

i) Write down the two system of equations to be solved to determine the coefficients for $p_1(x)$ and $p_2(x)$ and comment on their structure.

ii) With $f(x) = \sin(2\pi x)$ such that $f_i = \sin(2\pi x_i)$, $i = 0, \ldots, N$ and with $x_0 = 0$, determine the coefficients in the ansatz for $p_1(x)$ and $p_2(x)$ for $N = 4$ and $N = 8$.

What is the maximum absolute error in the interpolation over the control points for $N = 4$ and $N = 8$? What is the maximum absolute difference between $p_1(x)$ and $p_2(x)$ at the same points? (In exact arithmetic, $p_1(x) \equiv p_2(x)$).

iii) Now, repeat the exercise in ii) with $x_0 = 1000$. What is the maximum absolute error in $p_1(x)$ compared to $f(x)$ over the control points for $N = 4$ and $N = 8$? And for $p_2(x)$? What is the maximum absolute difference between $p_1(x)$ and $p_2(x)$? Explain the results.

iv) Now, let $f(x) = \sin(10\pi x)$, and $x_0 = 0$. Compute the interpolating polynomial using Newton's ansatz for $N = 4, 8$ and 16. Comment on the results.
v) Let us now go back to the case in \( ii \), i.e. \( f(x) = \sin(2\pi x) \), and \( x_0 = 0 \). We want to see what the quality of the interpolating polynomial is outside of \( x \in [0, 1] \). Now, define \( NL + 1 \) control points in the interval \( [-0.25, 1.25] \). Plot \( f(x) \) and \( p_2(x) \) in this interval. What is the maximum difference? Comment on the results.

PART 2. The problem encountered in \( iv \) in part 1 illustrates the need of interpolation by a piecewise polynomial. That is, the interval \( [a, b] \) is divided into \( N \) subintervals, and different polynomials are used for interpolation on each subinterval. Here, let us consider a cubic spline interpolation.

We introduce a partitioning of the interval \( a = x_0 < x_1 < \ldots < x_N = b \). Let us here assume equidistant points, i.e. \( x_j = x_0 + jh, \) \( h = (b - a)/N, \) for \( j = 0, \ldots, N \). Given values \( f_j, j = 0, \ldots, N \), we require that the interpolating spline function obeys

\[ S(x_j) = f_j, \quad j = 0, \ldots, N. \]

On each interval, for \( x_j \leq x \leq x_{j+1}, j = 0, \ldots, N - 1 \), \( S(x) = S_j(x) \), where

\[ S_j(x) = \alpha_j + \beta_j(x - x_j) + \gamma_j(x - x_j)^2 + \delta_j(x - x_j)^3. \]

We require that \( S(x) \in C^2([a, b]) \), i.e. that

\[ S_j(x_{j+1}) = S_{j+1}(x_{j+1}), \quad S_j'(x_{j+1}) = S_{j+1}'(x_{j+1}), \quad S_j''(x_{j+1}) = S_{j+1}''(x_{j+1}), \]

\( j = 0, \ldots, N - 1 \).

In addition we assume that we have periodic boundary conditions, such that

\[ S_{N-1}(x_N) = S_0(x_0), \quad S'_{N-1}(x_N) = S'_0(x_0), \quad S''_{N-1}(x_N) = S''_0(x_0). \]

i) Given values \( f_j, j = 0, \ldots, N \) (with \( f_N = f_0 \)), describe how to compute the coefficients \( \alpha_j, \beta_j, \gamma_j \) and \( \delta_j \).

Remember: in doing this, it is convenient to define

\[ M_j = S''_j(x_j), \]

\( j = 0, \ldots, N - 1 \). and form the system of equations to solve for the \( M_j \)’s.

From the values of \( M_j \) and \( f_j, j = 0, \ldots, N - 1 \), one can then compute the coefficients \( \alpha_j, \beta_j, \gamma_j \) and \( \delta_j \) for all \( j \).

Write a program that computes the interpolating cubic spline.

ii) For \( f = \sin(10\pi x) \) on the interval \([0, 1]\) \((a = 0, b = 1 \) above\), compute the interpolating cubic spline for \( N = 2^k, k = 2, \ldots, 8 \).
Define \( NL + 1 \) control points \( \xi_j = jh_F, j = 0, \ldots, NL \) with \( h_F = 1.0/NL \) and \( NL = 2000 \). For each value of \( N \), compute the maximum error of the interpolation over the control points. Plot the error as a function of \( N \). With what factor does the error decrease for each doubling of \( N \)? What would you expect? Discuss the results.

**PART 3.** Introduce a partitioning of the interval \( a = x_0 < x_1 < \ldots < x_N = b \).

Let \( S(x) \) be the cubic spline function as above, but replace the periodic condition with the boundary conditions

\[
S'(x_0) = f'_0, \quad S'(x_N) = f'_N,
\]

for given values \( f'_0 \) and \( f'_N \).

Let \( g(x) \) be any twice continuously differentiable function on \([a, b]\) that obeys the same interpolating condition as \( S(x) \), i.e.

\[
g(x_j) = f_j, \quad j = 0, \ldots, N,
\]

and

\[
g'(x_0) = f'_0, \quad g'(x_N) = f'_N.
\]

Show that the spline function \( S(x) \) satisfies the optimality property

\[
\int_a^b |S''(x)|^2 \, dx \leq \int_a^b |g''(x)|^2 \, dx,
\]

i.e. \( S(x) \) "oscillates the least" of all smooth functions satisfying the interpolating condition. (Equality occurs only when \( g(x) \equiv S(x) \).)

Hint: Introduce the function \( k(x) = S(x) - g(x) \), expand the integral over \( g''(x) \) in terms of \( S''(x) \) and \( k''(x) \). Then use integration by parts, the interpolating conditions above, and the properties of the spline function.