Assigned March 21-05, due April 4-05.

This assignment is concerned with the issue of adaptive time step control in the solution of ordinary differential equations.

i) Design and implement an adaptive Runge-Kutta method, based on the standard fourth order Runge-Kutta method. The error estimate should be based on comparing the result from one Runge-Kutta step with step size $h$, and two Runge-Kutta steps with step size $h/2$.

The program should take a user specified parameter TOL, that can be chosen by the user to set either the allowed error per time step, or the allowed error per “unit” time.

Explain and justify your method. How is the error estimated? When is a result accepted and when is it rejected? How is the step size adjusted?

Make sure that the last time step is such that the end time is the one intended.

ii) Use your program to solve the following problem:

$$x_1'' = x_1 + 2x_2' - \mu \frac{x_1 + \mu}{N_1} - \mu \frac{x_1 - \hat{\mu}}{N_2},$$

$$x_2'' = x_2 - 2x_1' - \mu \frac{x_2}{N_1} - \mu \frac{x_2}{N_2},$$

with initial values

$$x_1(0) = 0.994, \quad x_1'(0) = 0, \quad x_2(0) = 0, \quad x_2'(0) = -2.001585106,$$

where

$$N_1 = ((x_1 + \mu)^2 + x_2^2)^{3/2}, \quad N_2 = ((x_1 - \hat{\mu})^2 + x_2^2)^{3/2},$$

and $\mu = 0.012277471$ and $\hat{\mu} = 1.0 - \mu$.

This set of equations describes the motion of a satellite under the gravitational influence of the Earth and the Moon. The exact solution is periodic with period $T = 17.0652166$. One measure of the error can hence be obtained by comparing how close we are to the initial position at $t = T$.

Note that this system of two second order equations should be rewritten as a system of four first order equations before you solve them numerically.

iii) Plot the trajectory. Which is the most difficult part to compute?

Try some different choices of the error tolerance TOL, both as the allowed error per time step and the allowed error per “unit” time. Comment on the error tolerances used and the actual error that is obtained. Is the result sensitive to the choice of the size of the first time step?
Keep a record of how many time steps are accepted and rejected, and plot the step size as a function of time.

Is it possible to solve this equation with the classical fourth order Runge-Kutta method and a fixed step size? If so, how many time steps are needed?