Finite Element Approximation of Poisson’s equation.

PART 1.

Here, we consider the same Poisson problem as in Assignment 6, where it was discretized using a finite difference method. That is, we consider the following Poisson problem,

\[-\nabla^2 v = f \quad (x, y) \in R = (0, 1) \times (0, 1),\]
\[v(0, y) = g_1(y) \quad y \in [0, 1] \]
\[v(1, y) = g_1(y) \quad y \in [0, 1] \]
\[\frac{\partial v}{\partial y}(x, 0) = g_2(x) \quad x \in [0, 1] \]
\[v(x, 1) = g_3(x) \quad x \in [0, 1] \]

with

\[f(x, y) = \frac{2a^2}{\pi} (3-8a(x-1/2)^2 - 2a(y-1/2)^2) e^{-a(\frac{1}{2}+y)^2+(y-1/2)^2}\]

and

\[g_1(y) = \frac{a}{\pi} e^{-a(1/2+y-1/2)^2}, \quad g_2(x) = \frac{a^2}{\pi} e^{-a(2(x-1/2)^2+1/4)} \]
\[g_3(x) = \frac{a}{\pi} e^{-a(2(x-1/2)^2+1/4)} .\]

This equation has the exact solution

\[v(x, y) = \frac{a}{\pi} e^{-a(2(x-1/2)^2+(y-1/2)^2)} .\]

i) Derive the variational formulation of the problem. (Not the equivalent minimization problem).

ii) Consider a Galerkin method based on piecewise linear basis functions with local support, on a regular triangulation of the unit square, as in Figure 8.5 of Iserles.

Derive the linear system of equations at the level of integrals over basis functions and their gradients. Write it on the form

\[A\xi = b\]

and give the size of the system, and the entries \(A_{ij}\) and \(b_j\). Make sure it is clear what the entries \(\xi_i\) are, and how the discrete solution \(U\) is defined.

Also make sure to clearly introduce any notation that you are using, your numbering of the nodes etc.
iii) Since each basis function is piecewise linear, its gradient is piecewise constant. Because of that, all entries in the stiffness matrix, i.e. the matrix $A$ for the linear system of equations, can be evaluated analytically. And, since the triangulation is regular, many entries will be the same.

Compute all the entries in the stiffness matrix $A$ explicitly. Also compute the integrals in the contributions from the Dirichlet boundary conditions to the right hand side.

Give the entries $A_{ij}$ and $b_j$.

iv) There are still some integrals that have not been evaluated for the right hand side, coming both from the right hand side of the Poisson’s equation and the Neumann boundary condition.

For the integrals from the Neumann boundary conditions, apply the trapezoidal rule, and write out the result explicitly.

To evaluate a general integral over a triangle $K$, consider the simple quadrature rule

$$\int_K g(x)dx \sim \sum_{j=1}^{3} g(\eta_{K}^j)\frac{|K|}{3},$$

where $|K|$ is the area of triangle $K$ and $\eta_{K}^j$, $j = 1, 2, 3$ are the three vertices of the triangle.

Work out the result. Now you have evaluated everything that was written in terms of the basis functions explicitly, and your system should only contain known constants and function evaluations of functions given in the original problem formulation. Give the system. Compare it to what you had for the discretization using the finite difference method in Assignment 6.

Solve the problem for a set of different grid resolutions, plot the solution and compute the convergence rate.

PART 2.

Now, consider the Poisson equation with homogeneous Dirichlet boundary conditions,

$$-\nabla^2 v = f \quad (x, y) \in \Omega \subset \mathbb{R}^2,$$

$$v = 0 \quad \text{on } \partial \Omega.$$  \hspace{1cm} (6)

In the example above, we considered a regular triangulation on the unit square. The strength of the finite element method is however to compute in complex domains.
Here, assume that \( \Omega \) is a polygonal domain in \( \mathbb{R}^2 \), and that we are given a triangulation of this domain. The interior nodes are indexed from 1 to \( N_I \).

To handle the definition of basis functions and computations of integrals, any triangle in the mesh can be mapped to a reference triangle. Three piecewise linear basis functions are defined on the reference element, \( \varphi_{ref}^1 \), \( \varphi_{ref}^2 \) and \( \varphi_{ref}^3 \). The basis function \( \varphi_{ref}^j \) is one at node \( j \) and zero at the two other nodes, with the numbering node 1: \((0,0)\), node 2: \((1,0)\), node 3: \((0,1)\).

On the reference element (triangle) \( \tau \) to the right, a quadrature rule is given

\[
\int_{\tau} G(x) \, dx = \sum_{q=1}^{Q} w_q G(\eta_q)
\]

where \( \eta_q \) and \( w_q \) are the quadrature point and weights, respectively. All the quadrature points fall in the interior of the triangle.

In the process of deriving the discrete problem, it might be helpful to introduce the “tent” function

\[
\Psi_i(x) = \sum_{l, e} \varphi_e^l(x),
\]

s. t. \( g(e, l) = i \)

where \( i \) is the number of the global node, \( g(e, l) \) is an indexing scheme relating the element (or triangle) number \( e \) and the local node number \( l \) to the global node number of that node. \( \varphi_e^l \) is the basis function defined on element \( e \) such that it is 1 in local node \( l \) and zero at the two other nodes. This tent function will have support over all triangles with the common global node with number \( j \). Formally, it is not correct at the node since all the local basis functions are one at the node, but with a quadrature rule that is not using node values, we can use it as it is written.

i) Derive the variational formulation of the problem.

ii) Derive the discrete problem on the level of keeping continuous integrals over the reference element. Make sure to clearly define the mappings you are using, and to define any other notation that you introduce.

iii) Continue the derivation, explicitly using the quadrature rule that is given and analytical values of integrals computed on the reference element. Give the final discrete problem. Describe how you would implement an algorithm for solving this problem (not in complete detail).