ASSIGNMENT 8 for Numerical Methods II, Spring 2005.

Assigned Apr 18 -05, due May 2 -05.

Time dependent PDEs.

PART 1.

Consider the heat equation with periodic boundary conditions:

$$u_t = u_{xx} \qquad x \in [0, 2\pi), \tag{1}$$

$$u(0,t) = u(2\pi, t) \tag{2}$$

$$u(x,0) = f(x) = \sin(x) + \sin(20x) \tag{3}$$

i) Introduce the grid points

$$x_j = j\Delta x, \quad j = 0, \ldots, N,$$

with $\Delta x = 2\pi/N$, and a time step $\Delta t > 0$.

Implement the θ scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \theta \left[\frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{(\Delta x)^2} \right] + (1-\theta) \left[\frac{u_{j-1}^{n} - 2u_j^{n} + u_{j+1}^{n}}{(\Delta x)^2} \right]$$

to solve this equation. Note that, with $\theta = 0$, the scheme is explicit, otherwise, it is implicit.

ii) What is the order of the scheme for the choices $\theta = 0, 1/2$ and 1?

iii) Show that the scheme is unconditionally stable for $\theta \geq 1/2$. What is the stability condition for $\theta = 0$?

iv) Compute with N=100 up to $t = 0.5$ for $\theta = 0$, with a suitable Δt. Plot the exact solution and the error. Compute with N=100 and $\Delta t = 0.05$ for $\theta = 0.5$ and $\theta = 1.0$, and plot the error in both cases.

v) Carefully explain the error that occurs for the Crank-Nicholson scheme ($\theta = 0.5$) in *iv)*.

PART 2.

Consider the convection-diffusion equation

$$u_t + a u_x = \eta u_{xx}, \quad -\infty < x < \infty$$

with $u(x,0) = f(x)$.

Consider either the case with 2π periodic initial data, $f(x) = f(2\pi + x)$, and hence a 2π-periodic solution, or the Cauchy problem, whatever you prefer.

Assume a step size $\Delta x > 0$ and a time step $\Delta t > 0$. The following finite difference approximation is given:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{(2\Delta x)} = \eta\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{(\Delta x)^2}$$

i) What is the order in space and time of the proposed method?

ii) What is the stability condition for this method?

What is the stability condition for the cases $a = 0$ and $\eta = 0$?

iii) With $a = 1$, for what choices of $\eta > 0$ would you expect this method to work well, and produce an accurate solution? For what choices would you expect it to do worse? (Δt kept within stbility limits). Pick some initial conditions and perform numerical computations to illustrate your point. Compare to the solutions obtained when discretizing by

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = \eta\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{(\Delta x)^2}$$

Explain.

PART 3.

Consider the equation

$$u_t = u_{xx} + u,$$

with $u(x, 0) = f(x)$. Choose whatever interval and boundary condtions that you want.

Assume a step size $\Delta x > 0$ and a time step $\Delta t > 0$. The following finite difference approximation, based on the forward Euler method in time, is given:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{(\Delta x)^2} + u_j^n$$

Show that the stability condition $\mu = \Delta t/(\Delta x)^2 \leq 1/2$ that hold for Euler's method applied to the equation $u_t = u_{xx}$ is sufficient also in this case.

PART 4.

Do Exercise 14.4 in Iserles.