Construction of cubic spline function

We introduce a partitioning of the interval \( a = x_0 < x_1 < \ldots < x_N = b \).
Consider a set \( \{y_0, \ldots, y_N\} \) of \( N + 1 \) real numbers. We denote by \( S(x) \) an interpolating cubic spline function with
\[
S(x_j) = y_j \quad j = 0, \ldots, N. \quad (E1)
\]
On each interval, we have a cubic polynomial
\[
s_j(x) = a_j + b_j x + c_j x^2 + d_j x^3, \quad x_j \leq x \leq x_{j+1}, \quad j = 0, \ldots, N - 1,
\]
i.e. there are \( 4N \) coefficients \( \{a_j, b_j, c_j, d_j\} \).
In (E1), we have conditions at both end points for each \( s_j(x) \), i.e. 2\( N \) conditions.
From the conditions on continuity of first and second derivative,
\[
s_j'(x_{j+1}) = s_{j+1}'(x_{j+1}), \quad s_j''(x_{j+1}) = s_{j+1}''(x_{j+1}), \quad j = 0, \ldots, N - 2,
\]
we have 2\( N \) – 2 additional conditions.
We are lacking 2 conditions. The most common additional requirements to make the spline function unique are:

- \( S''(a) = S''(b) = 0. \)
- \( S^{(k)}(a) = S^{(k)}(b), \) for \( k = 1, 2 \), i.e. \( S(x) \) is periodic. (And of course, the \( y \)-values should reflect periodicity for \( S \) itself).
- \( S'(a) = y'_0, \) \( S'(b) = y'_N \), for given numbers \( y'_0 \) and \( y'_N \).

Let \( h_{j+1} = x_{j+1} - x_j, \ j = 0, \ldots, N - 1 \). Denote the values of the second derivatives at knots \( x_j \in [a, b] \),
\[
M_j = S''(x_j) \quad j = 0, \ldots, N.
\]
We call the \( M_j \)'s the moments of the spline function.
The second derivative \( S''(x) \) is a linear function in each interval \( [x_j, x_{j+1}] \), \( j = 0, \ldots, N - 1 \). We can write:
\[
s_j''(x) = M_j \frac{x_{j+1} - x}{h_{j+1}} + M_{j+1} \frac{x - x_j}{h_{j+1}}, \quad x \in [x_j, x_{j+1}].
\]
Integration yields
\[
s_j'(x) = -M_j \frac{(x_{j+1} - x)^2}{2h_{j+1}} + M_{j+1} \frac{(x - x_j)^2}{2h_{j+1}} + A_j, \quad (E2)
\]
\[
s_j(x) = M_j \frac{(x_{j+1} - x)^3}{6h_{j+1}} + M_{j+1} \frac{(x - x_j)^3}{6h_{j+1}} + A_j(x - x_j) + B_j.
\]
Using the conditions \( s_j(x_j) = y_j \) and \( s_j(x_{j+1}) = y_{j+1} \), we get

\[
B_j = y_j - M_j \frac{h_{j+1}^2}{6}, \quad A_j = \frac{y_{j+1} - y_j}{h_{j+1}} - \frac{h_{j+1}}{6}(M_{j+1} - M_j), \quad (E3)
\]

from where we can determine \( \{a_j, b_j, c_j, d_j\} \) in

\[
s_j(x) = a_j + b_j x + c_j x^2 + d_j x^3, \quad x_j \leq x \leq x_{j+1}, \quad j = 0, \ldots, N - 1,
\]

in terms of the moments \( M_j \) and \( M_{j+1} \).

Now, we need to solve for these moments. With \((E3)\) in \((E2)\),

\[
s_j'(x) = -M_j \frac{(x_{j+1} - x)^2}{2h_{j+1}} + M_{j+1} \frac{(x - x_j)^2}{2h_{j+1}} + \frac{y_{j+1} - y_j}{h_{j+1}} - \frac{h_{j+1}}{6}(M_{j+1} - M_j)
\]

The conditions we have not used are \( s_j'(x_{j+1}) = s_j'(x_{j+1}) \). Using this yields,

\[
\frac{h_j}{6} M_{j-1} + \frac{h_j + h_{j+1}}{3} M_j + \frac{h_{j+1}}{6} M_{j+1} = \frac{y_{j+1} - y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j} \quad (E4)
\]

for \( j = 1, 2, \ldots, N - 1 \).

Now, we need the two boundary conditions. Let us do the first case, i.e.

\( S''(a) = M_0 = 0 \) and \( S''(b) = M_N = 0 \). We then solve for the \( N - 1 \) unknowns \( M_1, \ldots, M_{N-1} \).

By multiplying \((E4)\) with \( 6/(h_j + h_{j+1}) \), and denoting

\[
\lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}}, \quad \mu_j = 1 - \lambda_j = \frac{h_j}{h_j + h_{j+1}},
\]

and

\[
d_j = \frac{6}{h_j + h_{j+1}} \left\{ \frac{y_{j+1} - y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j} \right\}
\]

we obtain the system

\[
\begin{align*}
2M_1 + \lambda_1 M_2 &= d_1 \\
\mu_2 M_1 + 2M_2 + \lambda_2 M_3 &= d_2 \\
&\vdots \\
&\vdots \\
\mu_{N-2} M_{N-3} + 2M_{N-2} + \lambda_{N-2} M_{N-1} &= d_{N-2} \\
\mu_{N-1} M_{N-2} + 2M_{N-1} &= d_{N-1}
\end{align*}
\]

This system of linear equations is nonsingular for any partition of \([a, b]\).