The integral Hodge conjecture for threefolds

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Hodge decomposition: For X a smooth complex projective variety we have

$$H^i(X,\mathbb{C}) = \oplus_{j=0}^{i} H^{j,i-j}(X),$$

where $H^{j,i-j}(X) := H^{i-j}(X,\Omega_X^j)$.
Fact: For any complex subvariety $Y \subset X$ of codimension i the image of $[Y] \in H^{2i}(X,\mathbb{Z})$ in $H^{2i}(X,\mathbb{C})$ lies in $H^{i,i}(X)$.

Hodge conjecture (HC): For any $u \in H^{2i}(X,\mathbb{Q})$ whose image in $H^{2i}(X,\mathbb{C})$ is in $H^{i,i}(X)$, u is the class of an algebraic cycle, i.e., a $\mathbb{Q}$-linear combination of codimension i subvarieties of X.

Integral Hodge conjecture (IHC): same statement for $H^{2i}(X,\mathbb{Z})$ and algebraic cycles with $\mathbb{Z}$ coefficients.

The main evidence is the **Lefschetz** $(1,1)$ **theorem**, i.e., the IHC is true for $i=1$.

Corollary 1 *The Hodge conjecture (with $\mathbb{Q}$-coefficients) is true for 1-cycles, i.e., for $H^{2n-2}(X,\mathbb{Q})$, where $n=\dim(X)$.*

The first open case of the HC: 2-cycles on a 4-fold, even an abelian 4-fold. Weil described explicit Hodge classes in the middle cohomology of a special class of abelian 4-folds which are not known to be algebraic. [Bogomolov: And this is all one needs to prove in the case of abelian 4-folds. Totaro: That's right. For other classes of abelian 4-folds (not of Weil's type), there are not many Hodge classes, and the Hodge conjecture is known.]

Remark 1 *The Hodge decomposition of $H^{2i}(X, \mathbb{C})$ is not defined over $\mathbb{Q}$.*

For a family of smooth projective varieties $\mathcal{X} \to B$, the Hodge decompositions on $\mathcal{X}_t$ vary continuously for $t \in B$, and filtrations vary holomorphically. So the HC predicts that algebraic cycles 'jump up' on special varieties of the family, when $H^{i,i}(X)$ has larger than usual intersection with $H^{2i}(X, \mathbb{Q})$. The HC is often easy to check for *general* varieties in a family, as there are few Hodge cycles beyond those generated by divisors.

[Graber: Given the paucity of evidence, why do people believe the Hodge conjecture? Totaro: I cannot really answer this, but it is a key ingredient in a larger framework governing cycles in algebraic geometry.]

Integral Hodge conjecture for threefolds. This is false for very general hypersurfaces in $\mathbb{P}^4$ of degree d for certain d ($d \geq 48$); this is due to Kollár.

For any smooth hypersurface X in $\mathbb{P}^4$, we have

$$H^*(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & 0 & \mathbb{Z} & \mathbb{Z}^N & \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{cases}$$

and the IHC for X would predict that all of $H^4(X, \mathbb{Z}) \cong H_2(X, \mathbb{Z})$ is generated by curves in X. Note that

$$\deg : H^4(X, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z},$$

and thus the IHC is true for X if X contains a line (hence for $\deg(X) \leq 5$) and for some special hypersurfaces of any degree.

Conjecture 1 (Griffiths–Harris) *For a very general hypersurface X of degree $d \geq 6$ in $\mathbb{P}^4$, every curve in X has degree $\equiv 0 \pmod{d}$.*

Theorem 1 (Kollár) *For X a very general hypersurface of degree 48 in $\mathbb{P}^4$, every curve in X has even degree. In particular, IHC is false for X.*

Proof: Find a *singular* 3-fold $Y \subset \mathbb{P}^4$ of degree 48 such that every curve on Y has even degree. Then apply a degeneration argument, noting that any specialization of a curve of odd degree has at least one component of odd degree.

To produce the singular hypersurface Y, start with Z any smooth projective threefold and L a very ample line bundle with $L^3 = d$ (e.g., 48). Suppose that every curve $C \subset Z$ has $L \cdot C \equiv 0 \pmod{e}$, for e a positive integer not dividing 6 (e.g., 4). (For example, $Z = \mathbb{P}^3$, $L = \mathcal{O}(4)$, $d = 64$, $e = 4$. To give an example of degree 48, let S be a very general quartic surface S, $Z = S \times \mathbb{P}^1$, and $L = \pi_1^* O(1) \otimes \pi_2^* O(4)$ on Z; then $d = 48$ and every curve on Z has degree a multiple of 4 with respect to L.) Embed $Z \hookrightarrow \mathbb{P}^N$ using L and take a general linear projection to a hypersurface Y in $\mathbb{P}^4$. We might hope that every curve $C \subset Y$ has degree $\equiv 0 \pmod{e}$. However, the finite morphism $\pi : Z \to Y$ can be $2:1$ on a surface and $3:1$ on some curves (and show more complicated behavior over finitely many bad points). Thus every curve $C \subset Y$ is either

$$\pi_*(\text{curve}), \quad \frac{1}{2}\pi_*(\text{curve}), \quad \text{or } \frac{1}{3}\pi_*(\text{curve})$$

as a cycle. So every curve $C \subset Y$ has degree a multiple of $e/6$.

Hassett-Tschinkel construction of counterexamples to IHC defined over $\mathbb{Q}$:
Do the same over Spec($\mathbb{Z}$), i.e., choose a hypersurface with a reduction modulo p that is a general projection.

Positive results on IHC for threefolds:

Theorem 2 (Voisin) *Let X be a smooth projective threefold over $\mathbb{C}$ which is either uniruled or strongly Calabi-Yau ($K_X \simeq \mathcal{O}_X$ and $b_1(X) = 0$). Then the IHC is true for X, i.e., $H_2(X, \mathbb{Z})$ is generated by algebraic curves.*

Question 1 *Let X be a smooth projective threefold, uniruled or simply-connected Calabi-Yau. Is $H_2(X, \mathbb{Z})$ generated by* rational *curves?*

Question 2 (asked by Tschinkel) *Does every simply connected Calabi-Yau threefold X contain* any *rational curve?*

Question 2 has a positive answer for X of Picard number at least 14, by R. Heath-Brown and P.M.H. Wilson.

How about IHC for rationally connected varieties of higher dimension?

Theorem 3 (Colliot-Thélène, Voisin) *There is a rationally connected (RC) 6-fold (fibered over $\mathbb{P}^3$ with generic fiber a quadric) for which IHC fails for codimension-two cycles.*

This draws on previous work of Ojanguren and Colliot-Thélène on non-vanishing of unramified cohomology in this case. The key recent innovation is the development of links between unramified cohomology and the failure of the integral Hodge conjecture.

[Starr: How about blowing up something for which the IHC fails in projective space? Totaro: That works, but not for codimension two cycles.]

Despite Theorem 3, IHC for 1-cycles on a RC variety of any dimension looks plausible.

Theorem 4 (Voisin) *If the Tate conjecture (with $\mathbb{Q}_\ell$ coefficients) holds for 1-cycles on all smooth projective surfaces over $\mathbb{F}_q$, then the IHC holds for 1-cycles on RC varieties of any dimension over the complex numbers.*

Proof: The IHC for 1-cycles on RC varieties is deformation invariant. Indeed, can use very free rational curves to get curves with ample normal bundle, i.e., both

$$C + \text{very free curves} \quad \text{and} \quad \text{very free curves}$$

deform.

Voisin uses Chad Schoen's theorem: If the Tate conjecture (with $\mathbb{Q}_\ell$-coefficients) holds for all varieties [Colliot-Thélène: suffices for just surfaces] over $\mathbb{F}_q$ then it holds for 1-cycles *with $\mathbb{Z}_\ell$ coefficients* on all varieties over $\overline{\mathbb{F}_q}$.

Specialize your RC variety to a separably rationally connected variety over a finite field, then lift curve classes to characteristic zero. □

Theorem 5 (Totaro) *Can omit the assumption that $b_1(X) = 0$ for Calabi Yau 3-folds. In particular, IHC holds for 1-cycles on abelian 3-folds.*

How to prove IHC for CY 3-folds X or uniruled 3-folds: Look at the smooth surfaces S in $|dH|$ for H ample on X, $d \gg 0$. The Lefschetz hyperplane theorem says that

$$H_2(S, \mathbb{Z}) \to H_2(X, \mathbb{Z})$$

is onto. So let $u \in H_2(X, \mathbb{Z})$ be a Hodge class. If u is the image of a Hodge class on some surface S_t in the family then we win. (This reduces us to a question about Hodge theory.)

Idea: If K_X is "negative" or trivial then

$$h^0(S, N_{S/X}) \geq h^2(S, \mathcal{O}).$$

The former measures the dimension of the deformation space of S in X; the latter how complicated the Hodge structure on $H^2(S)$ is. Essentially, $h^2(S, \mathcal{O})$ measures the number of conditions that have to be satisfied for a given integral class to be Hodge.

That inequality suggests the possibility that every homology class in $H_2(X, \mathbb{Z})$ in some open cone near $H^{1,1}(X)$ may become a Hodge class on some surface S in the family near a given one. We justify that by an infinitesimal calculation to show that the Hodge structure on the family of surfaces S is varying "as much as possible".

[C. Xu: For RC varieties, is the effective cone generated by rational curves?

J. Li: How does the simple-connectivity come in? Totaro: For a Calabi-Yau threefold X with first Betti number X not zero, we also have $H^2(X, O_X) \neq 0$. So the IHC becomes a more complicated statement to prove: we have to prove that a specified subgroup of $H_2(X, \mathbb{Z})$ is generated by algebraic curves, rather than showing that all of $H_2(X, \mathbb{Z})$ is generated by algebraic curves.

Colliot-Thélène: Do you have a new proof of Voisin's result? Totaro: No, my goal was to extend her methods to a broader class of 3-folds.

McKernan: Does IHC hold for 3-folds of Kodaira dimensions 1 and 2? Totaro: There are counterexamples in those cases too, by Colliot-Thélène and Voisin. So the most one can hope would be that IHC might hold for all 3-folds of Kodaira dimension at most zero.]