Exceptional isogenies of Elliptic Curves and Frobenius Distribution

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1 Introduction–Tate conjecture and Picard jumping

Consider first a field $k \subset \mathbb{C}$. Given a K3 surface X, the Néron-Severi group is given by the Lefschetz $(1,1)$ Theorem:

$$\mathrm{NS}(X) = H^2(X, \mathbb{Z}(1)) \cap H^{1,1}(X).$$

This is a free abelian group of rank ρ with

$$0 \leq \rho \leq 20 = \dim H^{1,1}(X).$$

Now let $k = \mathbb{F}_q$ and let ℓ be a prime not dividing q. Then we have

$$\mathrm{NS}(X) \otimes \mathbb{Z}_\ell \xrightarrow{c_1} H^2(X_{\bar{k}}, \mathbb{Z}_\ell(1))^{\mathrm{Gal}(\bar{k}/k)}.$$

Theorem 1 (Maulik,C-,Madapusi-Pera) *If the characteristic ≥ 3 then c_1 is onto.*

Sketch assuming $\rho(X) \geq 2$: This is equivalent to finiteness of the Brauer group, the proof is a variant of Artin and Swinnerton-Dyer's proof for elliptic surfaces.

The group $\mathrm{Br}(X)$ is finite if and only if $\mathrm{Br}(X)[\ell^\infty]$ is finite. We can get elements in $\mathrm{Br}(X)$ in the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X, \mathbb{Z}_\ell(1))$. Using

$$H^2(X, \mathbb{Z}_\ell(1)) \to H^2(X, \mu_{\ell^n}) \twoheadrightarrow \mathrm{Br}(X)[\ell^n],$$

an $\alpha \in H^2(X, \mathbb{Z}_\ell(1))$ yields $\alpha_n \in \mathrm{Br}(X)[\ell^n]$. Chern classes of α_n-twisted sheaves lie in

$$N^{\alpha_n} = \{(r\ell^n, D + r\alpha, c\omega)\} \subset H^0(X_{\bar{k}}, \mathbb{Z}_\ell) \oplus H^2(X_{\bar{k}}, \mathbb{Z}_\ell(1)) \oplus H^4(X_{\bar{k}}, \mathbb{Z}_\ell(2))$$

with $D \in \mathrm{NS}(X)$ and $r, c \in \mathbb{Z}$.

Starting with $v \in N^{\alpha_n}$, consider moduli M_v of twisted sheaves with Mukai vector v. Then, assuming some conditions on v (cf. Yoshioka's work): If $v^2 = 0$ then M_v is a K3 surface and

$$\mathrm{NS}(M_v) = v^\perp / \mathbb{Z}v$$

up to some p-groups.

Since $\rho(X) \geq 2$, we can pick two divisors D_1, D_2 on X with $D_1^2 > 0, D_2^2 < 0$, and $D_1 D_2 = 0$. If the Tate conjecture is false for X, find ℓ, α such that $\alpha^2 = -D_2^2$ and $\alpha \cdot \mathrm{NS}(X) = 0$. Set

$$v = (\ell^n, \alpha + D_2, 0), \quad v^2 = 0,$$

then M_v is a K3 surface and $(0, D_1, 0) \in \mathrm{NS}(M_v)$. Results of Saint Donat imply there are finitely many possibilities for M_v but the discriminant $\operatorname{disc} \mathrm{NS}(M_v) \to \infty$ as $n \to \infty$.

Remark 1 *Assuming the Tate conjecture, the rank of* $\mathrm{NS}(X_{\bar{k}})$ *is even* ≥ 2. *Indeed, for surfaces Shioda shows*

$$\mathrm{rank}(\mathrm{NS}(X_{\bar{k}})) = b_2(X) \pmod 2.$$

2 K3 surfaces over one-parameter base

Question 1 *Consider* $\mathcal{X} \xrightarrow{\pi} S$ *a smooth projective family of K3 surfaces over one-dimensional base. As* s *varies through the geometric points of* S*, what can we say about* $\rho(\mathcal{X}_s)$*?*

Suppose $S = \Delta$ is the complex unit disk and π is not isotrivial. For s very general $\rho(\mathcal{X}_s)$ is constant. The Noether-Lefschetz locus

$$\mathrm{NL}(\pi) = \{s : \rho(\mathcal{X}_s) > \rho_{gen}\}$$

is dense in Δ. This is due to Green and Oguiso; and also Borcherds, Katzarkov, Pantev, Shepherd-Barron using arguments with automorphic forms. The latter is more complicated and delicate but will be more useful for our purposes.

Now take $S \subset \operatorname{Spec} \mathcal{O}_K$ where K is a number field (some primes should be inverted to get a smooth π). There is a parity issue: if $\rho(\mathcal{X}_{\bar{\eta}})$ is odd then all closed points of S belong to the Noether Lefschetz locus of π. We can work out what happens for 'most' closed points, e.g., a set of density one.

Question 2 *What about the 'Noether-Lefschetz locus', i.e., $s \in S$ such that $\rho(\mathcal{X}_{\bar{s}})$ is bigger than usual?*

Expectation: There are infinitely many s with $\rho(\mathcal{X}_{\bar{s}}) > \rho(\mathcal{X}_{\bar{\eta}})$. This is known if $\rho(\mathcal{X}_{\bar{\eta}})$ is odd by Remark 1 and the Tate conjecture.

3 Statement of the main result

Theorem 2 *Let E_1 and E_2 be two elliptic curves over a number field K. Then the set of $s \in \operatorname{Spec}(\mathcal{O}_K)$ such that $E_{1,\bar{s}}$ and $E_{2,\bar{s}}$ are isogenous is infinite.*

The corresponding K3 surface X is the Kummer surface associated with $E_1 \times E_2$.

Remark 2 (On the heuristic) *Suppose that $K = \mathbb{Q}$ and E_1, E_2 are elliptic curves over $\mathbb{Q}$ that are not CM or isogenous. Here s corresponds to a prime p. The isogeny condition is equivalent to equality of the traces* Tr *of Frobenius on*

$$H^1(E_{1,\bar{s}}, \mathbb{Z}_\ell) \text{ and } H^1(E_{2,\bar{s}}, \mathbb{Z}_\ell).$$

Since $|\mathrm{Tr}| \leq 2\sqrt{p}$ by the Weil conjectures, the probability of having an isogeny at p is $\simeq 1/\sqrt{p}$, assuming some equidistribution. This sum diverges.

Corollary 1 *Let K be a number field and E/K an elliptic curve. Then one of the following holds*

1. *E has infinitely many supersingular primes;*
2. *for all k quadratic imaginary field, E has infinitely many primes with complex multiplication by k.*

This should be compared with Elkies' Theorem. Note that his method gives quite few supersingular primes, far fewer than the Lang-Trotter heuristic would predict.

Idea of the proof. The curves E_1, E_2 yield $\mathcal{O}_K$-points of the modular curve $X(1)$, say Y, Z. Interpret these as two arithmetic curves on an arithmetic surface. We have the Hecke correspondences

$$T_N \hookrightarrow X(1) \times_{\mathbb{Z}} X(1)$$

where

$$T_N = \{E \to E' \text{ cyclic isogeny of degree } N\}.$$

Our goal is to show that $\bigcup_N (Y \cap T_{N*}Z)$ is infinite, *as a set*. This splits into two statements:

1. consider the zero-cycle $Y \cdot T_{N*}Z$: we want $\widehat{\deg}(Y \cdot T_{N*}Z)$ to be big
2. local degrees, i.e., multiplicies of intersections, are not too large.

The first estimate concerns te height of the Hecke correspondences – it corresponds to the result of Borcherds mentioned above. In our context, one uses Hecke equidistribution results of Clozel and Ullmo.

Here are the precise estimates: First the height of T_N is on the order of $N \log(N)$. Second, the local degrees should be $\ll N \log N$.

Example: Consider the Archimedean term, which is controlled by

$$\sum_{\text{cyclic subgroup } C \subset E_2, |C|=N} \log |j(E_1) - j(E_2/C)|.$$

This is large if E_1 and E_2/C are 'close' in the upper half plane. Look at the structure of the corresponding Hecke orbit. Note that the sum 'looks like' the Riemann sum for an integral, which actually converges giving the estimate.

To complete the argument, we need to do some Diophantine approximation.

Question: Is there a plausible Sato-Tate hypothesis that would give results of this kind for a large class of situations. Answer: M. Harris has proven Sato-Tate equidistribution results for products of elliptic curves but these do not yield our theorem as a corollary.