A uniformization of the moduli space of abelian varieties of dimension six

Gavril Farkas

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This is joint with Alexeev, Donagi, Izadi, and Ortega.

1 Classical Prym constructions

Theorem 1 (Wirtinger, 1895) *A general principally polarized abelian variety (ppva) $[A, \Theta] \in \mathcal{A}_g$ is a Prym variety for $g \leq 5$.*

Let $\mathcal{R}_g$ denote the moduli space of pairs

$$\{(C, \eta) : C \text{ curve of genus } g, \eta^2 = \mathcal{O}_C, \eta \neq \mathcal{O}_C\}$$

and

$$P_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$$

the Prym mapping. The former has dimension $3g-3$; the latter has dimension $\frac{g(g+1)}{2}$. The morphism P_g is dominant for $g \leq 6$, which explains the Wirtinger Theorem.

In the special case $g = 6$ both spaces have dimension 15 and the mapping P_6 has degree 27 with Galois group $W(E_6)$ by work of Donagi and Smith.

Since $\mathcal{R}_6$ is unirational for $g \leq 6$, $\mathcal{A}_g$ is unirational for $g \leq 5$. On the other hand we have the following result.

Theorem 2 (Mumford, Tai, Freitag) *$\overline{\mathcal{A}}_g$ is of general type for $g \geq 7$.*

But what about $\mathcal{A}_6$? Its status remains open.

2 A Prym-Tyurin construction

This is inspred by work of Kanev and Tyurin.

Let $X \subset \mathbb{P}^4$ denote a cubic threefold. Consider a pencil of cubic surfaces

$$\{X_\lambda = X \cap H_\lambda\}_{\lambda \in \mathbb{P}^1}.$$

We have a curve of lines

$$C := \{(\lambda, \ell) : \ell \subset X_\lambda \text{ line }\}.$$

Since there are 24 singular fibers the genus of C is 46.

Consider the symmetric incidence correspondence

$$\Sigma = \{((\lambda, \ell), (\lambda', \ell')) \in C \times C : \ell \cap \ell' \neq \emptyset\}$$

with projections of degree 10

$$\pi_1, \pi_2 : \Sigma \to C.$$

This yields an endomorphism

$$\gamma\colon \operatorname{Jac} C \to \operatorname{Jac} C, \quad \gamma = \pi_{2*}{\pi_1}^*.$$

Kanev showed that for all $x \in \Sigma$

$$\Sigma(\Sigma(x)) + 4\Sigma(x) - 5x = 5g^1_{27},$$

whence $(\gamma - 1)(\gamma + 5) = 0 \in \operatorname{End}(\operatorname{Jac} C)$. Consider the Prym-Tyurin variety

$$\mathrm{PT}(C, \Sigma) = \operatorname{Im}(1 - \gamma) \subset \operatorname{Jac} C$$

with

$$\Theta|_{\mathrm{PT}} = 6\Xi,$$

where Ξ is a principal polarization.

The Prym-Tyurin variety associated to the curve of lines described above has dimension six and can be expressed as

$$\mathrm{PT}(C, \Sigma) \simeq \mathcal{J}X \times E$$

where $\mathcal{J}X$ is the intermediate Jacobian and E is the base locus of the pencil. Thus we obtain an element of $\mathcal{A}_5 \times \mathcal{A}_1$. This assignment is far from dominant on $\mathcal{A}_6$. But note there are 24 degenerate fibers, depending on 21 parameters, and $\dim \mathcal{A}_6 = 21$ as well.

3 Root systems and cubic surfaces

Let S be a cubic surface and write

$$E_6 = K_S^{\perp} \simeq \mathbb{Z}^6$$

with roots α satisfying $\alpha^2 = -2$ and associated reflections $r_\alpha : E_6 \to E_6$. The Weyl group

$$W(E_6) = \langle r_\alpha \rangle$$

has order $51,840$ and admits a realization $W(E_6) \subset S_{27}$ via its action on the 27 lines.

Let $f : C \to \mathbb{P}^1$ denote a covering of degree 27 with monodromy group $W(E_6)$. We have the Hurwitz space $\mathcal{H}_{E_6}$ parametrizing such covers branched over $p_1, \ldots, p_{24} \in \mathbb{P}^1$ and local monodromy given by $r_\alpha \in W(E_6)$.

All such Hurwitz schemes associated with Weyl groups are irreducible. There is a mapping

$$\mathcal{H}_{E_6} \to \mathcal{M}_{0,24}/S_{24}.$$

To give a point of $\mathcal{H}_{E_6}$ we need to specify the 24 points in $\mathbb{P}^1$, the 24 roots multiplying to 1 and generating $W(E_6)$. Let $\overline{\mathcal{H}}_{E_6}$ denote the normalization of the compactification by admissible covers and

$$br : \overline{\mathcal{H}}_{E_6} \to \widetilde{\mathcal{M}}_{0,24} = \overline{\mathcal{M}}_{0,24}/S_{24}.$$

Theorem 3 (A-D-F-I-O) *The Prym-Tjurin map $PT : \mathcal{H}_{E_6} \to \mathcal{A}_6$ is generically finite, i.e., a general ppav of dimension 6 is an E_6-Prym-Tjurin variety of exponent 6.*

Question: What is the degree? Answer: Don't know but it's at least 2^{12}.

$$\begin{array}{ccc} \overline{\mathcal{H}}_{E_6} & \overset{PT}{\dashrightarrow} & \mathcal{A}_6 \\ \downarrow & & \\ \widetilde{\mathcal{M}}_{0,24} & & \end{array}$$

Unfortunately, this parametrization is not decisively helpful concerning the Kodaira dimension of $\mathcal{A}_6$:

Theorem 4 *$\mathcal{H}_{E_6}$ is of general type.*

4 Proof of the second theorem

Let B_j denote the boundary divisor on $\widetilde{\mathcal{M}}_{0,24}$ corresponding to j points on one component and $24 - j$ on the other. Recall

$$K_{\overline{\mathcal{M}}_{0,24}} = \sum_{j=2}^{22} \left(\frac{j(24-j)}{23} - 2 \right) B_j$$

with negative coefficient on B_2 and positive on the others. We analyze the Hurwitz formula for br, using the correspondence between ramification divisors and roots of E_6.

When two roots coincide (ramification points collide to produce a node) the remaining roots $r_1, \ldots, r_{22}$ determine a root system L of type A_5 or D_5. These correspond to divisors

$$D_{A_5}, D_{D_5} \subset \overline{\mathcal{H}}_{E_6}.$$

The divisor D_{D_5} gives rise to a tower

$$C_1 \stackrel{2:1}{\to} Y \stackrel{5:1}{\to} \mathbb{P}^1$$

and

$$\mathrm{PT}(C, \Sigma) = \mathrm{Prym}(C_1/Y).$$

Note that $D_{D_5} \to \mathcal{R}_7$ has two-dimensional fibers.

When two roots are distinct then we get *syzygetic* and *asyzygetic* configurations, which were studied extensively in the 19th century.

This analysis makes possible computations of the ramification:

$$br^* B_2 = \sum_{L \subset E_6} D_L + 3D_{\mathrm{asyz}} + 2D_{\mathrm{syz}}.$$

5 Discussion of first theorem

To prove domination we ‘tropicalize’, producing a graph extracted from the root system corresponding to a ‘total degeneration’ of the original fibration.