Moduli of Enriques surfaces

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This is joint work with V. Gritsenko. We work mostly over $\mathbb{C}$ but there will be a few exceptions at the end.

1 Introduction

Let S be an Enriques surface over $\mathbb{C}$, i.e., $\omega_S \neq \mathcal{O}_S$ but $\omega_S^{\otimes 2} = \mathcal{O}_S$ and $q(S) = 0$. Let $X \to S$ denote the étale double cover by a K3 surface. We have $\rho(S) = \frac{1}{2}e(X) = 12$ and $b_2(S) = 10$. The Néron-Severi group of S is

$$\mathrm{NS}(S) \simeq U \oplus E_8(-1) \oplus \mathbb{Z}/2\mathbb{Z}$$

where U is a hyperbolic plane and E_8 is associated with the corresponding root system. The $\mathbb{Z}/2\mathbb{Z}$ is generated by the class of ω_S.

Consider a marking of S

$$\varphi : H^2(S,\mathbb{Z})/\text{torsion} \simeq U \oplus E_8(-1).$$

Recall

$$H^2(X,\mathbb{Z}) \simeq 3U \oplus 2E_8(-1) =: L_{K3}$$

with involution

$$\begin{array}{rcl} \varrho : L_{K3} & \to & L_{K3} \\ (x,y,z,u,v) & \mapsto & (z,-y,x,v,u) \end{array}$$

admitting eigenspaces

$$\mathrm{Eig}(\varrho)^+ = \{(x,0,x,u,u), x \in U, u \in E_8(-1)\} = U(2) \oplus E_8(-2) =: M$$

and

$$\mathrm{Eig}(\varrho)^- = \{(x,y,-x,u,-u)\} \simeq U \oplus U(2) \oplus E_8(-2) =: N.$$

The signature of N is $(2, 10)$.

Fact: Given φ we can find a marking

$$\tilde{\varphi} : H^2(X, \mathbb{Z}) \to L_{K3}$$

such that $\varrho \circ \tilde{\varphi} = \tilde{\varphi} \circ \sigma^*$ where σ is the involution on X. Note that $\tilde{\varphi}(p^* H^2(S, \mathbb{Z})) \simeq \mathrm{Eig}(\varrho)^+ = M$. Since σ acts on $H^{2,0}(X)$ by -1 we find

$$\tilde{\varphi}(\omega_X) \in N = \mathrm{Eig}(\varrho)^-.$$

Then we can define a period domain

$$\Omega_N = \{[x] \in \mathbb{P}(N \otimes \mathbb{C}) : (x, x) = 0, (x, \bar{x}) > 0\} = D_N \sqcup D'_N$$

and we may assume $\tilde{\varphi}(\omega_X) \in D_N$.

Let $O(N)$ denote the integral orthogonal group and $O^+(N)$ those isometries preserving D_N. We take

$$M_{En} = O^+(N) \backslash D_N,$$

which is quasi-projective of dimension ten. For each root $\delta \in N$ (with $\delta^2 = -2$) we have a hyperplane

$$H_\delta := \{[x] : (x, \delta) = 0\} \to \Delta_{-2} \subset M_{En}.$$

Theorem 1 (Horikawa, Namikawa) *Associating to an Enriques surface a period point as above gives a bijection between*

$$M^0_{En} := M_{En} \setminus \Delta_{-2}$$

and Enriques surfaces up to isomorphism.

Remark 1 *This is not a moduli space in the sense of representing a functor of Enriques surfaces.*

Liedtke has studied technical issues with defining this as a stack. A polarization or other additional data is needed.

Theorem 2 (Kondo) M_{En} *is rational.*

2 Moduli spaces

Definition 1 *A* (semi-)polarized Enriques surface *is a pair* $(S, \mathcal{L})$ *where* $\mathcal{L}$ *is a (semi-)ample line bundle.*

A numerically polarized Enriques surface *is a pair* $(S, \overline{\mathcal{L}})$ *where* $\mathcal{L}$ *is ample and* $\overline{\mathcal{L}} \in \mathrm{Num}(S)$.

Viehweg has contructed quasi-projective coarse moduli spaces $\mathcal{M}^a_{En,h}$ of polarized Enriques surfaces with a given type of polarization.

Definition 2 *The* type *of a polarization is an* $O(M) = O(M(\frac{1}{2}))$*-orbit of a primitive vector* $h \in U \oplus E_8(-1) = M(\frac{1}{2})$.

Notation. We use $\mathcal{M}$ to denote moduli spaces, and M to denote orthogonal modular varieties.

We have an involution

$$\begin{array}{rcl} \iota : \mathcal{M}^a_{En,h} & \to & \mathcal{M}^a_{En,h} \\ (S, \mathcal{L}) & \mapsto & (S, \mathcal{L} \otimes \omega_S) \end{array}$$

and set

$$\mathcal{M}^{num}_{En,h} := \mathcal{M}^a_{En,h} / \langle \iota \rangle \, .$$

3 Groups and orthogonal modular varieties

Given a lattice L, let $D(L)$ denote the discriminant group with its standard quadratic form. Note that $D(M) = D(N) = \mathbb{F}_2^{10}$ and

$$O(D(M)) = O(D(N)) = O^+(\mathbb{F}_2^{10})$$

is the orthogonal group of even type which has order $2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$.

Fix $h \in U \oplus E_8(-1) = M(\frac{1}{2})$ primtive and

$$O(M, h) = \{g \in O(M), g(h) = h\}.$$

Set

$$\tilde{O}(N) \subset \Gamma_h = \pi_N^{-1}(\pi_M(O(M,h)) \subset O(N)$$

so that $\Gamma_h^+ = \Gamma_h \cap O^+(N)$ acts on D_N. Here $\pi_M : O(M) \to O(D(M))$ and similarly for π_N.

Let

$$\begin{array}{rcl} M_{En,h} = \Gamma_h^+ \backslash D_N & \to & M_{En} \\ \cup & & \cup \\ \Delta_{-2,h} & \to & \Delta_{-2} \end{array}$$

where the typersurface $\Delta_{-2,h}$ is the inverse image of Δ_{-2}. It is irreducible.

An element $v \in N, v^2 = -4$ belongs to one of two $O^+(N)$ orbits. It is even or odd respectively depending on whether (v, N) is $2\mathbb{Z}$ or $\mathbb{Z}$. Even (-4)-vectors are of the form $(\delta', -\delta)$ for $\delta' \in U \oplus E_8(-1)$ when thinking of the embedding of $U \oplus E_8(-1)$ into $\mathrm{Eig}(\rho)^-$.

The (-4)-vectors define a hypersurface

$$\Delta_{-4}^{ev} \subset M_{En},$$

the nodal Enriques surfaces, with preimage

$$\Delta_{-4,h}^{ev} \subset M_{En,h}.$$

Let

$$\Delta_{-4,h^\perp}^{ev} \subset \Delta_{-4,h}^{ev}$$

consist of those irreducible components where $\delta' \cdot h = 0$. We have spaces

$$M_{En,h}^{num} = M_{En,h} \setminus (\Delta_{-2,h} \cup \Delta_{-4,h^\perp}^{ev}).$$

Theorem 3 *There is a degree* 2 *étale double cover*

$$\mathcal{M}_{Eh,h}^a \to M_{En,h}^{num}$$

which identifies $M_{En,h}^{num}$ with $\mathcal{M}_{En,h}^{num}$.

Remark 2 *1. The open part $M_{En,h}^{num}$ which is given by removing all of $\Delta_{-4,h}^{ev}$ corrresponds to non-nodal polarized Enriques surfaces.*

2. The points on $\Delta_{-4,h^\perp}^{ev} \setminus \Delta_{-2}$ can be interpreted as semi-polarized Enriques surfaces.

Question 1 *1. Is $\mathcal{M}_{En,h}^a$ always connected?*

2. Is $\mathcal{M}_{En,h}^a$ an orthogonal modular variety?

Corollary 1 *There are only finitely many isomorphism classes of moduli spaces of (primitively) polarized Enriques surfaces.*

Ingredients of the proof:

1. There are only finitely many groups

$$\tilde{O}(N) \subset \Gamma_h \subset O(N).$$

2. $\Delta^{ev}_{-4,h}$ has finitely many components.

3. $M^{num}_{En,h}$ has only finitely many étale double covers since this is a finite CW complex and thus $H^1(M^{num}_{En,h}, \mathbb{Z}/2\mathbb{Z})$ is finite.

Corollary 2 *The Kodaira dimension of the moduli space of numerically polarized Enriques surfaces is negative for* $h^2 < 32$.

Corollary 3 *There are moduli space of numerically polarized Enriques surface birational to the moduli space with* 2*-level structure*

$$M_{En}(2) = \tilde{O}^+(N)\backslash D_N$$

which is of general type (Gritsenko, in preparation).

4 Arithmetic questions

Question 2 *Over which fields can we construct these moduli spaces?*

For the Cossec-Verra presentation, Liedtke showed the moduli space is defined over $\mathbb{Q}$. Also, Kondo has constructed a projective model of $M_{En}(2) \subset \mathbb{P}^{185}$ given by $2^2 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$ quartic equations. This embedding is given via automorphic forms and the relation can be understood on terms of representation theory.

Question 3 *Over which field is* $M_{En}(2)$ *defined?*

This is a projective variety of general type which might be of arithmetic interest.