Curves and cycles on K3 surfaces

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1 Introduction

Let X be a K3 surface over a field k. We can write

$$\mathrm{CH}^*(X) = \mathrm{CH}^0(X) \oplus \mathrm{CH}^1(X) \oplus \mathrm{CH}^2(X) \simeq \mathbb{Z} \oplus \mathbb{Z}^{\rho(X)} \oplus \mathrm{CH}^2(X).$$

Theorem 1 (Roitman-Bloch-Milne-Voisin) *Let k be algebraically closed. Then*

$$\mathrm{CH}^2(X)_0 := \ker(\deg \mathrm{CH}^2(X) \to \mathbb{Z})$$

is divisible and torsion free.

Examples:

1. When $k = \bar{\mathbb{F}}_p$, we have $\mathrm{CH}^2(X)_0 = 0$, as $\mathrm{Pic}^0(C)$ is torsion for a curve over a finite field.

2. Let $\mathcal{X} \subset |\mathcal{O}(4)| \times \mathbb{P}^3$ be the universal quartic with generic fiber $\mathcal{X}_\eta$. Then $\mathrm{CH}^2(\mathcal{X}_\eta) = \mathbb{Z}$. This uses the localization exact sequence for $U \subset |\mathcal{O}(4)|$, which gives

$$\mathrm{CH}^2(\mathcal{X}_\eta) = \lim \mathrm{CH}^2(\mathcal{X}_U).$$

   We don't expect $\mathrm{CH}^2(\mathcal{X}_{\bar\eta}) = \mathbb{Z}$.

Question 1 *What about the universal K3 surface $\mathcal{X} \to \mathcal{K}_g$ over the moduli space of K3 surfaces of degree $2g-2$?*

This should be compared to the Franchetta conjecture, where

$$\mathrm{Pic}(\mathcal{C}/\mathcal{M}_g) = \mathbb{Z}\omega_{\mathcal{C}/\mathcal{M}_g}.$$

More examples:

3. Take a general Lefschetz pencil of quartics

$$\begin{array}{ccc} \mathcal{X} & \subset & \mathbb{P}^1 \times \mathbb{P}^3 \\ \downarrow & & \\ \mathbb{P}^1 & & \end{array}$$

so that $\mathcal{X} = \mathrm{Bl}_C(\mathbb{P}^3)$ where C is a complete intersection curve and $\mathrm{CH}^2(\mathcal{X}) = \mathbb{Z} \oplus \mathrm{Pic}(C)$. For all $U = \mathbb{P}^1 \setminus \{t_1, \ldots, t_m\}$ we get

$$\oplus_j \mathrm{CH}^1(\mathcal{X}_{t_j}) \to \mathrm{CH}^2(\mathcal{X}) \to \mathrm{CH}^2(\mathcal{X}_U) \to 0$$

and thus over $\mathbb{C}$

$$\mathrm{CH}^2(\mathcal{X}_\eta)_0 \neq 0.$$

What about countable fields?

2 Background and conjectures

Recall the following:

- Mumford in 1968 showed that $\dim \mathrm{CH}^2(X) = \infty$ over $k = \mathbb{C}$, i.e., there is no curve C such that $\mathrm{Pic}(C) \twoheadrightarrow \mathrm{CH}^2(X)$.
- Bloch-Beilinson conjecture: For $k = \overline{\mathbb{F}_p(t)}, \overline{\mathbb{Q}}$, we expect $\mathrm{CH}^2(X) = \mathbb{Z}$. Not a single example is known other than isotrivial families over $\overline{\mathbb{F}_p(t)}$.
- Bloch: For X/k and an extension of fields $k \subset K$ we have

  $$\mathrm{CH}^2(X) \to \mathrm{CH}^2(X_K)$$

  with torison kernel. The diagonal $\Delta \subset X \times X$ yields $\eta \in X \times k(\eta)$ and

  $$[\eta] \notin \mathrm{Im}(\mathrm{CH}^2(X) \to \mathrm{CH}^2(X \times k(\eta))$$

  if X is not supersingular.

For all $k \subset K = \overline{K}$, with transcendence degree ≥ 2, X not supersingular, the map

$$\mathrm{CH}^2(X) \otimes \mathbb{Q} \to \mathrm{CH}^2(X_K) \otimes \mathbb{Q}$$

fails to be surjective. Green-Griffiths-Paranjape in characteristic zero get the same result for extensions of transcendence degree one. The idea is to use the generic point of some curve $C \subset X$.

Remark 1 *For all k algebraically closed, if $\mathrm{CH}^2(X) \neq \mathbb{Z}$ then the map*

$$\mathrm{CH}^2(X) \to \mathrm{CH}^2(X \times K)$$

is not surjective for algebraically closed extensions of positive transcendence degree.

Conjecture 1 *(Easier than Bloch-Beilinson?) Given X/k with k algebraically closed of characteristic zero and positive transcendence degree, then*

$$\mathrm{CH}^2(X)_0 \neq 0.$$

The *Beauville-Voisin* ring is relevant: there exists $c_X \in \mathrm{CH}^2(X)$ such that decomposable cycles in the image of the map

$$\mathrm{CH}^1(X) \to \mathrm{CH}^1(X) \to \mathrm{CH}^2(X),$$

are proportional to c_X over algebraically closed fields. The resulting ring $R(X)$ injects into cohomology.

3 Curves

Given X/k and a mapping $f : C \to X$ from a curve, the kernel of the map

$$\mathrm{Pic}^0(C) \to \mathrm{CH}^2(X)_0$$

is a union of translates A_i of abelian varieties. What can we say about the dimensions of A_i? For general C, we expect $\dim A_i = 0$. The most special case will correspond to the constant cycle curves, which we know define.

Definition 1 *A curve $C \hookrightarrow X$ is a constant cycle curve (CCC)*

- *over $\mathbb{C}$: if the image of $\mathrm{Pic}^0(C) \to \mathrm{CH}^2(X)$ is trivial.*

- *For arbitrary fields k: if there exists $d > 0$ and $x_0 \in X$ such that*

$$d[\Delta_C - C \times x_0]$$

is supported in $Z \subset X \times C$ with $\pi(Z) \neq C$.

By Bloch-Srinivas the two definitions are equivalent over $\mathbb{C}$.

There is a localization exact sequence

$$\bigoplus_{t_i \in C} \mathrm{CH}^1(X \times t_i) \to \mathrm{CH}^2(X \times C) \to \mathrm{CH}^2(X \times k(C)) \to 0,$$

and the class of $\Delta_C - C \times x_0$ in $\mathrm{CH}^2(X \times C)$ maps to a d-torsion class. The *order* of C is defined as the order of the image in $\mathrm{CH}^2(X \times k(C))$.

Remark 2 *The two definitions differ over small fields like $\overline{\mathbb{Q}}$. While $\mathrm{CH}^2(X)_0$ is expected to be trivial, most curves are not constant cycle curves.*

Conjecture 2 (cf. Bloch-Beilinson, Bogomolov) *Given $X/\overline{\mathbb{Q}}$ and $x \in X(\overline{\mathbb{Q}})$, there exists a curve $x \in C \subset X$ that is CCC.*

4 Theorems and examples

Remark 3 *Let X be a K3 surface over $k = \overline{\mathbb{F}}_p$. Then $\mathrm{CH}^2(X \times \overline{\mathbb{F}_p(t)}) = 0$ (cf. Bloch-Beilinson) if and only if* every *$C \hookrightarrow X$ is CCC.*

Over finite fields, every point is contained in a CCC.

Remark 4 *Rational curves $C \hookrightarrow X$ are CCC of order one.*

There exist examples of genus one curves that are CCC of order one.

There exist examples of genus 201 curves that are CCC of order at most four.

Let $X \to \mathbb{P}^2$ be a double cover ramified over a sextic curve. Then the ramification curve is CCC of order at most two.

Theorem 2 *Let X be defined over an algebraically closed field k of characteristic zero. Let L be ample and $d > 0$. Then the number of CCC curves in $|L|$ of order d is finite.*

Can they be enumerated? How do they behave under deformations?

Theorem 3 (Voisin) *There exist a non-empty Zariski open $U \subset \mathcal{K}_g$ and $d > 0$ such that*

$$\bigcup_{\text{CCC of order } d} C \subset X$$

is dense for all $X \in U$.

Distribution of points: Fix $\alpha \in \mathrm{CH}^2(X)$ and set

$$X_\alpha := \{x \in X : [x] = \alpha\}$$

over k algebraically closed. Then X_α is dense if non-empty (McLean).

- $\alpha \neq c_X$: $X_\alpha = \bigcup_{i=1}^{\infty} x_i$
- $\alpha = c_X$: $X_\alpha = \bigcup_{i=1}^{\infty} C_i \cup (\bigcup_{i=1}^{\infty} x_i)$ where the C_i are CCC and the last piece can be taken to be empty by Conjecture 1.