The Kobayashi pseudo-metric on K3 surfaces and hyperkähler manifolds

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March 25, 2015

This is joint with S. Lu and M. Verbitsky.

Definition 1 *A hyperkähler manifold M is a compact complex Kähler irreducible holomorphic symplectic manifold, i.e., $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}\sigma$ with σ non-degenerate.*

Examples:

- K3 surfaces
- $\mathrm{K3}^{[n]}$, i.e., Hilbert schemes of points on K3 surfaces
- generalized Kummer manifolds $K_n(A)$;
- O'Grady's sporadic examples.

Theorem 1 (Matsushita) *Consider a fibration $f \colon M \to B$ with $0 < \dim(B) < \dim(M)$. Then $\dim(B) = \frac{1}{2}\dim(M)$ and the generic fiber is a Lagrangian abelian variety.*

Theorem 2 (Huang) *If B is smooth then $B = \mathbb{P}^n$.*

Given M hyperkähler, there is the Beauville-Bogomolov-Fujiki form $(H^2(M, \mathbb{Z}), q)$ of signature $(3, b_2 - 3)$ satisfying the Fujiki relation

$$\int_M \alpha^{2n} = c \cdot q(\alpha)^n \qquad \text{for } \alpha \in H^2(M, \mathbb{Z}),$$

with $c > 0$ constant depending on the topological type of M. The form is non-degenerate, but in general it is not unimodular, although it is suspected to be even. It generalizes the intersection pairing on K3 surfaces.

Observation: Given $f\colon M \to \mathbb{P}^n$, h the hyperplane class on $\mathbb{P}^n$, and $\alpha = f^*h$, then α is nef and $q(\alpha) = 0$.

Conjecture 1 (SYZ) *If L is a nef line bundle on M with $q(L) = 0$ then L induces a Lagrangian fibration, as above.*

This conjecture is known for deformations of $\mathrm{K3}^{[n]}$ (Bayer–Macrì; Markman), and for deformations of $K_n(A)$ (Yoshioka).

Definition 2 *The* Kobayashi pseudometric *on M is the maximal pseudometric d_M such that all $f\colon (D,\rho) \to (M, d_M)$ are distance decreasing, where (D,ρ) is the disk with the Poincaré metric.*

A manifold is Kobayashi hyperbolic *if d_M is a metric.*

Theorem 3 (Brody) *Let M be a compact complex manifold. Then M fails to be Kobayashi hyperbolic if and only is there exists an entire curve $\mathbb{C} \to M$. The failure of Kobayashi hyperbolicity is preserved on taking limits.*

Conjecture 2 (Kobayashi)

1. *For S a K3 surface we have $d_S \equiv 0$.*
2. *For M hyperkähler we have $d_M \equiv 0$.*
3. *A hyperkähler manifold M is Kobayashi non-hyperbolic.*

Mori-Mukai '82: The first conjecture holds for projective K3 surfaces, using dominating families of (singular) elliptic curves.

K-Verbitsky '12: All known hyperkähler manifolds are Kobayashi non-hyperbolic. In 2013, Verbitsky extended this to all hyperkähler's.

Theorem 4 *Let S be a K3 surface. Then $d_S \equiv 0$.*

Theorem 5 *Let M be hyperkähler with $\rho < b_2 - 2$ and deformation equivalent to a Lagrangian fibration. Then $d_M \equiv 0$.*

Theorem 6 *Let M be hyperkähler with $\rho = b_2 - 2$ and $b_2(M) \geq 7$. Assume the SYZ conjecture holds for all deformations of M. Then $d_M \equiv 0$.*

Consider the Teichmüller space

$$\text{Teich} := \text{Complex structures}/\text{Diff}^0(M)$$

which admits an action by the mapping class group

$$\Gamma := \text{Diff}^+(M)/\text{Diff}^0(M).$$

The Teichmüller space is finite-dimensional for M Calabi-Yau. An element $I \in \text{Teich}$ is *ergodic* if the orbit $\Gamma \cdot I$ is dense in Teich, where

$$\Gamma \cdot I = \{I' \in \text{Teich} : (M, I) \sim (M, I')\}.$$

Theorem 7 (Verbitsky) *If M is hyperkähler and $I \in \text{Teich}$ then I is ergodic if and only if $\rho(M, I) < b_2 - 2$.*

Proposition 1 *Let (M, J) denote a complex manifold with $d_{(M,J)} \equiv 0$. Let $I \in \text{Teich}$ be deformation equivalent to J. Assume I is ergodic. Then $d_{(M,I)} \equiv 0$.*

Indeed, consider

$$\text{diam} : \text{Teich} \to \mathbb{R}_{\geq 0},$$

the maximal distance between two points. This is upper semi-continuous. Then

$$0 \leq \text{diam}(I) \leq \text{diam}(J) = 0.$$

Proof of Theorem 4

Case I: $\rho = b_2 - 2$

Then S is projective and $d_S \equiv 0$.

Case II: $\rho(S, I) < b_2 - 2$

Then I is ergodic and we deform (S, I) to a projective (S, J) whence $d_{(S,I)} \equiv 0$.

We turn to Theorem 5. This will use:

Theorem 8 *Let M be hyperkähler, admitting two Lagrangian fibrations associated to non-proportional nef parabolic classes. Then $d_M \equiv 0$.*

Proof: Suppose we have

$$\pi_i : M \to X_i, i = 1, 2$$

with h_i ample on X_i and α_i its pull back to M. Then $q(\alpha_i) = 0$ and the lattice $\langle \alpha_1, \alpha_2 \rangle$ has signature $(1,1)$. Since $q(\alpha_1, \alpha_2) \neq 0$ we can compute as follows: Let F_i denote a fiber of π_i, i.e., $[F_i] = \alpha_i^n$. Then

$$[F_1][F_2] = \int_M \alpha_1^n \wedge \alpha_2^n = cq\,(\alpha_1, \alpha_2)^n \neq 0.$$

Note pseudodistances of points *in* a fiber is zero. Use this to connect arbitrary pairs of points in M using the two fibration structures.

Idea of Theorem 5: The locus of Teich consisting of Lagrangian fibrations self-intersects. In the intersection one can choose a deformation with two Lagrangian fibrations as in Theorem 8, hence $d_M \equiv 0$. Since the original complex structure is ergodic ($\rho < b_2 - 2$) and deformation equivalent to one with vanishing Kobayashi pseudometric, we use Proposition 1 in order to complete the proof.

Idea of Theorem 6: If $\rho = b_2 - 2 \geq 5$ then there exists $z \in \mathrm{Pic}(M)$ with $q(z) = 0, z \neq 0$ (by Meyer's theorem for indefinite latices of rank at least 5). The SYZ conjecture says z gives rise to a Lagrangian fibration. Consider

$$\gamma \in \Gamma_1 := \mathrm{Aut}(\mathrm{Pic}(M))$$

with $z' = \gamma(z) \neq z$. This way we get a *second* Lagrangian fibration. Apply Theorem 8.